ANISOTROPY OF THE MICROWAVE BACKGROUND IN THE INFLATIONARY COSMOLOGY

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Using the Harrison–Zel dovich energy-density fluctuation spectrum predicted by the inflationary cosmology, we compute multipole moments of the cosmic blackbody temperature. Our results are independent of the details of non-linear galactic evolution

The inflationary cosmology [1] makes a prediction for the spectrum of energy-density fluctuations in the early universe [2]. The result is the scale-invariant Harrison Zel'dovich spectrum [3]: fluctuations of wavelength L at a time $t \approx L/c$ (the time of horizon crossing) are predicted to have an amplitude $\epsilon_{\rm H}$ which is independent of L. The scale-invariant spectrum is a model-independent prediction of inflation, while the value of the constant $\epsilon_{\rm H}$ depends on parameters of the particular theory being considered. A determination of the validity of the Harrison–Zel'dovich spectrum would test the inflationary hypothesis and an accurate measurement of $\epsilon_{\rm H}$ would have a direct impact on model building.

Unfortunately, our present knowledge of galactic structure can provide neither of these. Fluctuations on the galactic scale have become nonlinear and have clearly gone through an extremely complex evolution to become present-day galaxies and galactic clusters. However, the inflationary cosmology also predicts that fluctuations of much longer wavelength exist, and have amplitude $\epsilon_{\rm H}$ when they enter the horizon. Fluctuations which entered the horizon since the time of recombination are still evolving linearly today, if $\epsilon_{\rm H}$ is of order 10^{-4}. These long-wavelength fluctuations can be analyzed without any of the complexities and uncertainties of nonlinear dynamics. In this letter, we show that they produce observable anisotropies in the microwave background radiation. Using the scale-invariant spectrum we predict moments of the background blackbody temperature. Because the lower multipole moments are only sensitive to large scale fluctuations our predictions for these moments are insensitive to the nonlinear behavior of shorter-wavelength fluctuations.

In the absence of fluctuations spacetime is described by a spatially flat Robertson–Walker metric,

$$\mathrm{d}s^2 = S^2(\tau)[-\mathrm{d}\tau^2 + \mathrm{d}\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x}], \tag{1}$$

where τ is conformal time. Since we are concerned with phenomena which occurred after recombination we assume the universe is matter dominated. Then τ is proportional to the ordinary time t to the 1/3 power.

We begin by considering fluctuations characterized by a single plane wave. Later we superimpose these plane waves by integrating over wave-vectors. Fluctuations of wave-vector, $\boldsymbol{k}$, produce variations in the metric and energy momentum tensor given by [‡1]

$$g_{00} = -S^2(\tau)[1 + 2A(\tau)\,Q(\boldsymbol{x})], \tag{2a}$$

[‡1] Here we adopt the notation of ref. [4]

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$$g_{0i} = -S^2(\tau)[B(\tau)\, Q_i(\boldsymbol{x})], \tag{2b}$$

$$g_{ij} = S^2(\tau)\{[1 + 2H_{\mathrm{L}}(\tau)\, Q(\boldsymbol{x})]\, \delta_{ij} + 2H_{\mathrm{T}}(\tau)\, Q_{ij}(\boldsymbol{x})\}, \tag{2c}$$

$$T_0^0 = -\rho(\tau)\,[1 + \delta(\tau)\, Q(\boldsymbol{x})], \tag{2d}$$

$$T_0^i = -[\rho(\tau) + p(\tau)]\, v(\tau)\, Q^i(\boldsymbol{x}), \tag{2e}$$

$$T_j^i = p(\tau)\{[1 + \pi_{\mathrm{L}}(\tau)\, Q(\boldsymbol{x})]\, \delta_j^i + \pi_{\mathrm{T}}(\tau)\, Q_j^i(\boldsymbol{x})\}, \tag{2f}$$

where $Q(\boldsymbol{x})$ is the plane wave $\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x})$ and

$$Q_i = -k^{-1}\partial_i Q, \qquad Q_{ij} = k^{-2}\partial_i\partial_j Q + \tfrac{1}{3}\delta_{ij} Q. \tag{3a, b}$$

Indices on Q_i and Q_{ij} are raised and lowered with the Kronecker delta. The fluctuations of eqs. (2) affect the observed blackbody background temperature in two ways[‡2].

First, they produce fluctuations in the temperature of the plasma which emitted the radiation at the time of recombination τ_{E}. These are given by [‡3]

$$\delta T_{\mathrm{E}}/T_{\mathrm{E}} = \tfrac{1}{3}\delta(\tau_{\mathrm{E}})\, Q(\boldsymbol{x}). \tag{4}$$

Second, fluctuations affect the amount of redshift which the radiation undergoes before being received today at conformal time τ_0. The observed blackbody temperature, $T_0 + \delta T_0$, is related to the temperature at emission by [‡4]

$$T_0 + \delta T_0 = (T_{\mathrm{E}} + \delta T_{\mathrm{E}})/(1+z) \tag{5}$$

If we observe the blackbody radiation along a direction specified by the unit vector $\boldsymbol{e}$, then the factor $1+z$ is given according to the Sachs–Wolf formula [6] by

$$1 + z = \frac{S(\tau_0)}{S(\tau_{\mathrm{E}})}\Big(1 - \dot{B}(\tau)\, Q(\boldsymbol{x})/k\,\Big|_{y=0}^{y=\tau_0-\tau_{\mathrm{E}}} + \int_0^{\tau_0-\tau_{\mathrm{E}}} \mathrm{d}y\, Q(\boldsymbol{x})\,[\dot{H}_{\mathrm{L}}(\tau) + \tfrac{1}{3}\dot{H}_{\mathrm{T}}(\tau) - \ddot{B}(\tau)/k - \dot{H}_{\mathrm{T}}(\tau)\,(\boldsymbol{e}\cdot\boldsymbol{k})^2/k^2]\Big), \tag{6}$$

where a dot denotes differentiation with respect to τ and

$$\boldsymbol{x} = \boldsymbol{e}y, \qquad \tau = \tau_0 - y. \tag{7a, b}$$

Eqs (4) (5) and (6) can be combined to give a general, gauge-invariant expression for the observed blackbody temperature. This expression will be discussed in more detail in a further publication [7]. For the present purposes it is sufficient to specialize to the case of a perfect fluid in a matter-dominated universe. Then, the Einstein equations and the energy and momentum conservation equations can be used to simplify the result yielding

$$T_0 + \delta T_0 = \frac{T_{\mathrm{E}} S(\tau_{\mathrm{E}})}{S(\tau_0)}\Big(1 - \dot{B}(\tau)\, Q(\boldsymbol{x})/k\,|_{y=0} + \frac{1}{k^2}\int_0^{\tau_0-\tau_{\mathrm{E}}} \mathrm{d}y\, Q(\boldsymbol{x})\, \dot{\epsilon}(\tau)\, (\boldsymbol{e}\cdot\boldsymbol{k})^2\Big), \tag{8}$$

with $\boldsymbol{x}$ and τ given by eqs. (7). The function ϵ is Bardeen's [4] gauge-invariant generalization of the energy-density fluctuation

$$\epsilon = \delta + 3\,[(\rho + p)/\rho]\; k^{-1}(\dot{S}/S)\,(v - B). \tag{9}$$

Eq. (8) is gauge invariant because gauge dependence of the time of reception cancels against the gauge dependence of the term $\dot{B}Q/k\,|_{y=0}$. The factor $\dot{B}Q/k\,|_{y=0}$ produces an overall renormalization of the observed temperature due to fluctuations evaluated at the observer's location. Since it is independent of the direction vector $\boldsymbol{e}$ it does not contribute to the anisotropy and we absorb it into our definition of T_0. Then

$$\frac{\delta T_0}{T_0} = \frac{1}{k^2}\int_0^{\tau_0-\tau_{\mathrm{E}}} \mathrm{d}y\, Q(\boldsymbol{x})\, \dot{\epsilon}(\tau)(\boldsymbol{e}\cdot\boldsymbol{k})^2. \tag{10}$$

In the matter-dominated era it is well known that ϵ is proportional to $t^{2/3}$ or equivalently to τ^2. The inflationary cosmology equates ϵ at the time of horizon

‡2 The variables $A, B, H_{\mathrm{L}}, H_{\mathrm{T}}, \pi_{\mathrm{L}}, \delta$ and v are not gauge invariant. We work in a gauge where $A = v = 0$ although our final result is gauge invariant.

‡3 Equivalently one can think of recombination occurring at a fixed temperature but at different times at different points in space. We assume the fluctuations are adiabatic.

‡4 We assume that there is not a significant density of intergalactic ionized hydrogen, so that the surface of last scattering is near recombination. If intergalactic hydrogen is ionized by energy released during the gravitational contraction of galaxies and clusters of galaxies at $z \sim 20$ (see ref [5]) then we require the density of ionized hydrogen be less than about 10^{-8} cm^{-3}.

crossing, $\tau = 2/k$, to a random variable $\hat{a}(k)$ satisfying

$$\langle \hat{a}^*(k)\,\hat{a}(k')\rangle = (\epsilon_H^2/k^3)\,\delta^3(k-k'). \tag{11}$$

This definition assures that the expectation value of ϵ^2 at the time of horizon crossing

$$\langle \epsilon^2\rangle \equiv k^3 \int d^3q\,\langle \hat{a}^*(k)\,\hat{a}(q)\rangle, \tag{12}$$

takes the predicted value ϵ_H^2. Combining these results we find that the inflationary prediction for ϵ is

$$\epsilon(\tau) = \tfrac{1}{4}\,\hat{a}k^2\tau^2, \tag{13}$$

with $\hat{a}$ satisfying eq. (11).

The quantity $\delta T_0/T_0$ can be expanded in multipoles

$$\frac{\delta T_0}{T_0} = \sum_{l,m} a_{lm}\,Y_{lm}(e). \tag{14}$$

We will predict expectation values for the rotationally invariant quantities [‡5]

$$a_l^2 \equiv \sum_{m=-l}^{l} |a_{lm}|^2, \tag{15}$$

by projecting out the appropriate multipole

$$a_{lm} = \int d\Omega_e\,Y_{lm}^*(e)\,(\delta T_0/T_0) \tag{16}$$

with $\delta T_0/T_0$ given by eq. (10). The expectation value of a_l^2 is then obtained by substituting eq. (13) into eq. (16), integrating over wave-vectors k, squaring, using eq. (11) to evaluate the expectation value and summing over m. The result for $l \geqslant 2$ is

$$\langle a_l^2\rangle = \frac{4\pi^2}{2l+1}\,\epsilon_H^2 \int_0^{\omega_{max}} \frac{d\omega}{\omega}\{(2l+1)\,j_l(\omega) + [\tau_E/(\tau_0-\tau_E)]\,\omega[l\,j_{l-1}(\omega)-(l+1)\,j_{l+1}(\omega)]\}^2, \tag{17}$$

where j_l is a spherical Bessel function and we have expressed the integral over k in terms of the dimensionless variable $\omega = k(\tau_0-\tau_E)$. The k integration is cut off at the value $\omega_{max} \approx 80$, which restricts it to fluctuations that entered the horizon since the time of recombination. We have evaluated eq. (17) numerically for $l = 2-15$ and find that it is cut off independent, at the level of twenty percent, for a factor of two change in ω_{max}. This cutoff independence indicates that for these low moments our results are insensitive to the shorter wavelength fluctuations that have gone nonlinear, and to the details of the process of recombination [8].

The numerical results for (17) are well approximated by a simple analytic result. In eq. (17) $\tau_E/\tau_0 \simeq 0.03$ so the first term in the integrand dominates over the last two. Also, since the resulting integral is essentially cutoff independent we can extend ω_{max} to infinity. We then find that

$$\langle a_l^2\rangle \simeq 2\pi^2\,\epsilon_H^2\,[(2l+1)/l(l+1)]. \tag{18}$$

Eq. (18) is our main result. If observations of the anisotropy of the microwave background confirm this l-dependence, then the spectrum of fluctuations predicted by the inflationary cosmology is correct [‡6]. The present limit on the quadrupole moment [10], $a_2^2 < 5 \times 10^{-8}$, gives, from eq. (18), the bound

$$\epsilon_H < 6 \times 10^{-5}. \tag{19}$$

The inflationary cosmology also predicts the variance of a_l^2. Since ϵ is generated by the quantum fluctuations of a free scalar field in de Sitter space $\hat{a}(k)$ is a random variable with a gaussian probability distribution. Therefore,

$$[\langle (a_l^2)^2 - \langle a_l^2\rangle^2\rangle]^{1/2} = [2/(2l+1)]^{1/2}\langle a_l^2\rangle. \tag{20}$$

From eq. (10) we can get the contribution of large scale fluctuations to the dipole moment of the microwave background. We find that

$$\langle a_1^2\rangle = \tfrac{2}{3}\pi^2\,\omega_{max}^2\,\epsilon_H^2. \tag{21}$$

If ϵ_H is of order 10^{-5} this contributes significantly to the total dipole moment.

After completion of this paper it was brought to our attention that a similar computation was performed in ref. [11]. We thank C.J. Hogan for informing us of this work.

‡5 The expectation values of the $|a_{lm}|^2$ for a given l are the same and equal to $[1/(2l+1)]\langle a_l^2\rangle$.

‡6 There are other scenarios for generating fluctuations which can give a Harrison–Zel'dovich spectrum. See, for example, ref. [9].

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