GRAVITATIONAL STABILIZATION OF SCALAR POTENTIALS ⋆

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Using the formalism of Boucher we derive the conditions under which gravity can stabilize otherwise metastable or unstable scalar field configurations. For the case of a double-welled potential we rederive the stability criteria first obtained by Coleman and DeLuccia. Our derivation does not require the assumption of O(4) symmetry or the thin-walled approximation used in their work. We also give the conditions for which gravity can stabilize the symmetric points of general quartic potentials and of a Coleman–Weinberg potential.

Under certain circumstances gravitational effects can stabilize a scalar field configuration which would be metastable or unstable in the absence of gravity [1–4]. Consider for example, a potential with two local minima, one at $\phi = \phi_+$ and one at $\phi = \phi_-$, with an energy splitting $V(\phi_+) - V(\phi_-) = \epsilon$. Normally, the state $\phi = \phi_+$ would be metastable. However, Coleman and DeLuccia [1] have shown that for sufficiently small ϵ if $V(\phi_+) \leqslant 0$, the tunnelling rate between the states $\phi = \phi_+$ and $\phi = \phi_-$ vanishes when gravity is included in the calculation. More recently, Boucher [4] has provided a general formalism for analysing the stability of various field theories based on Witten's proof of the positive energy theorem [5] applied to the energy as it is defined in the flat [6] or anti-de Sitter [7] space cases. Here, we apply this formalism to the situation considered by Coleman and DeLuccia rederiving their stability conditions [1,8]. Our approach allows this to be done without requiring the assumption of O(4) invariance and without using the thin-walled approximation. It also gives additional information which constrains the general form of the potential. In addition we derive the conditions for which gravity can stabilize the symmetric points of general quartic potentials and of a Coleman–Weinberg potential [9].

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Boucher's formalism is remarkably simple to describe and implement. For any potential $V(\phi)$ with an extremum $V'(\phi_+) = 0$, a positive energy theorem for the state $\phi = \phi_+$ can be proven using the approach of Witten provided that there exists a real function $f(\phi)$ satisfying

$$f(\phi_+) = [-V(\phi_+)/3\kappa]^{1/2} ,$$

$$(f')^2 - \tfrac{3}{2}\kappa f^2 \leqslant \tfrac{1}{2} V(\phi) , \qquad (1)$$

where $\kappa = 8\pi G$. Clearly the first equation above requires that $V(\phi_+) \leqslant 0$. Because of the resulting positive energy theorem, the existence of such a function is sufficient to prove that the state $\phi = \phi_+$ is both classically and semi-classically stable.

We consider first the asymmetric double-welled potential. In this case we can write $V(\phi)$ in the form

$$V(\phi) = V_0(\phi) - \delta V(\phi) , \qquad (2)$$

where V_0 is a potential with $V_0(\phi_+) = V_0(\phi_-)$ and δV introduces the splitting $V(\phi_+) - V(\phi_-) = \epsilon$. We also choose δV to satisfy

$$\delta V(\phi_+) = 0 ,$$

and

$$\delta V'(\phi_+) = \delta V'(\phi_-) = 0 \qquad (3)$$

so that the minima are not shifted. If we write

$$f(\phi) = g(\phi) + [-V(\phi_+)/3\kappa]^{1/2} , \qquad (4)$$

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then eq. (1) is equivalent to

$$g(\phi_+) = 0 ,$$

and

$$(g')^2 - g[-3\kappa V(\phi_+)]^{1/2} - \tfrac{3}{2}\kappa g^2 \leqslant \tfrac{1}{2}[V_0(\phi) - V_0(\phi_+)] - \tfrac{1}{2}\delta V(\phi) . \quad (5)$$

To solve these equations we choose g to satisfy

$$(g')^2 = \tfrac{1}{2}[V_0(\phi) - V_0(\phi_+)] . \quad (6)$$

With the boundary condition $g(\phi_+) = 0$, this gives

$$g = \pm \int_{\phi_+}^{\phi} d\phi' \{\tfrac{1}{2}[V_0(\phi') - V_0(\phi_+)]\}^{1/2} . \quad (7)$$

We choose the sign in eq. (7) so that g is always positive. Then in order for that state $\phi = \phi_+$ to be stable δV must satisfy what is left of eq. (5),

$$\delta V \leqslant 3\kappa g^2 + 2g[-3\kappa V(\phi_+)]^{1/2} . \quad (8)$$

This restricts δV in a way we discuss more fully below. However, whatever δV is, at the point $\phi = \phi_-$ where by definition $\delta V = \epsilon$ it must satisfy eq. (8). Following Coleman and DeLuccia [1] we define

$$S_1 = \int_{\phi_+}^{\phi_-} d\phi' \{2[V_0(\phi') - V_0(\phi_+)]\}^{1/2} , \quad (9)$$

and then eq. (8) gives us the condition for stability,

$$\epsilon \leqslant \tfrac{3}{4}\kappa S_1^2 + [-3\kappa V(\phi_+)]^{1/2} S_1 . \quad (10)$$

This agrees exactly with the results [1,8] obtained by assuming O(4) symmetry and demanding that the tunnelling rate vanishes in the thin-walled approximation. This agreement supports the assumption that O(4) invariant solutions dominate the semi-classical tunnelling rate. For $V(\phi_+) = 0$ we recover from eq. (10) the original result of Coleman and DeLuccia [1] that the state $\phi = \phi_+$ with $V(\phi_+) = 0$ is stable if

$$\epsilon \leqslant \tfrac{3}{4}\kappa S_1^2 . \quad (11)$$

Eq. (8) gives a general constraint on the form of δV. For example, if

$$V_0(\phi) = \lambda(\phi^2 - b^2)^2 , \quad (12)$$

and we choose $\phi_+ = -b$, then stability is assured if

$$\delta V \leqslant \delta V_{max} \quad (13)$$

where

$$\delta V_{max} = \tfrac{1}{6}\lambda\kappa(\phi^3 - 3b^2\phi - 2b^3)^2 \quad \phi \leqslant b ,$$
$$= \tfrac{1}{6}\lambda\kappa(\phi^3 - 3b^2\phi + 6b^3)^2 \quad \phi > b . \quad (14)$$

It is interesting to apply the Boucher [4] formalism to some other simple cases. For example, by considering the form $f = a + b\phi^2$ we find that the symmetric point $\phi = 0$ of the potential

$$V = \tfrac{1}{2}m^2\phi^2 + \tfrac{1}{4}\lambda\phi^4 - \Lambda \quad (15)$$

is stable, provided that

$$m^2 \geqslant -\tfrac{3}{4}\kappa\Lambda ,$$

and

$$\lambda \geqslant -\tfrac{3}{8}\kappa[2m^2 + 3\kappa\Lambda + \kappa\Lambda(9 + 12m^2/\kappa\Lambda)^{1/2}] . \quad (16)$$

Note that this includes the possibility of gravitational stabilization of the state $\phi = 0$ for a potential with negative quadratic or negative quartic terms or even with both of these.

Another interesting case is the Coleman–Weinberg [9] type of potential

$$V = B\phi^4[\ln(\phi^2/\sigma^2) - \tfrac{1}{2}] - \Lambda , \quad (17)$$

arising from radiative corrections to a purely quartic potential. Here B is a positive constant. Once again we consider the stability of the state $\phi = 0$ which is unstable in the absence of gravity. This is clearly a more complicated case than the simple quartic potential. The stability criterion for the potential (17) must give stability for $B < B_{critical}$ with $B_{critical}$ to be determined. In the limiting case, the inequality in eq. (1) is saturated and thus to find $B_{critical}$ we can rewrite eq. (1) as

$$f(0) = (\Lambda/3\kappa)^{1/2} , \quad f' = (\tfrac{1}{2}V + \tfrac{3}{2}\kappa f^2)^{1/2} . \quad (18)$$

For small values of ϕ eq. (18) requires that

$$f = (\Lambda/3\kappa)^{1/2} + \tfrac{1}{4}\sqrt{3\nu\Lambda}\,\phi^2 + O(\phi^4) . \quad (19)$$

To determine $B_{critical}$ we use (19) as initial data and numerically integrate (18) to larger values of ϕ. The stability condition is then that the parameters Λ and B be chosen so that the square root in eq. (18) never becomes imaginary. We find for $\kappa\sigma^2 < 0.1$ that the state $\phi = 0$ of the potential (17) is stable provided that

$$B \leqslant 1.25\kappa\Lambda/\sigma^2 . \quad (20)$$

The Boucher method can obviously be applied to many other types of potentials (see for example ref. [10]). However, for most purposes the results of eqs. (10), (16) and (20) should be sufficiently general.

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