Massless particles with continuous spin indices*

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Massless representations of the Poincaré group include states with continuous spin indices. Free field operators corresponding to these states are constructed and are shown to obey noncausal commutation or anticommutation relations.

Massless representations of the Poincaré group include states with continuous spin indices as well as the more familiar integer and half-integer helicity states. Wigner has argued that the existence of physical particles corresponding to these states would give the vacuum an infinite heat capacity.[1] Here we discuss massless particles with continuous spin indices within the context of field theory. Free field operators connecting one-particle states to the vacuum are constructed. These fields do not obey causal commutation or anticommutation relations.

To construct massless representations of the Poincaré group[1-3] we first form standard states with the momentum

$$\vec{k} = \kappa\hat{z}$$

and energy

$$\omega = \kappa,$$

which transform according to a representation of the little group formed from elements of the Lorentz group which leave this standard momentum invariant. Letting $\vec{J}$ be the rotation generators and $\vec{K}$ the boost generators, the generators of the little group are J_3, L_1, and L_2 with

$$L_1 = K_1 - J_2, \tag{1a}$$

$$L_2 = K_2 + J_1, \tag{1b}$$

which satisfy

$$[L_1, L_2] = 0, \tag{2a}$$

$$[J_3, L_1] = iL_2, \tag{2b}$$

$$[J_3, L_2] = -iL_1. \tag{2c}$$

The standard states are characterized by a fixed parameter l and a variable angle ϕ with

$$L_1|\kappa\hat{z}, l, \phi\rangle = l\cos\phi|\kappa\hat{z}, l, \phi\rangle, \tag{3a}$$

$$L_2|\kappa\hat{z}, l, \phi\rangle = l\sin\phi|\kappa\hat{z}, l, \phi\rangle, \tag{3b}$$

$$e^{-i\theta J_3}|\kappa\hat{z}, l, \phi\rangle = |\kappa\hat{z}, l, \phi+\theta\rangle. \tag{3c}$$

Since any representation of the Poincaré group can be made either single-valued or double-valued[1,2] we take for $n = 1$ or 2

$$|\kappa\hat{z}, l, \phi + 2\pi n\rangle = |\kappa\hat{z}, l, \phi\rangle. \tag{4}$$

In the limit $l \to 0$ states with different values of ϕ become degenerate up to a phase,

$$|\kappa\hat{z}, 0, \phi+\theta\rangle = e^{-i\theta\lambda}|\kappa\hat{z}, 0, \phi\rangle,$$

and according to Eq. (4), λ must be an integer or half-integer. It is of course the usual helicity quantum number.

States with momentum $\vec{k}$ characterized by azimuthal angle α and polar angle β and a parameter

$$\gamma = \ln\left(\frac{|\vec{k}|}{\kappa}\right)$$

are obtained from the standard states by appropriate Lorentz transformation. Using ordinary δ-function normalization,

$$|\vec{k}, l, \phi\rangle = \left(\frac{\kappa}{|\vec{k}|}\right)^{1/2} e^{-i\alpha J_3}e^{-i\beta J_2}e^{-i\gamma K_3}|\kappa\hat{z}, l, \phi\rangle. \tag{5}$$

Because of the continuous variation of the angle ϕ, massless particles with $l \neq 0$ cannot be created or destroyed by any field operator $\Phi_\sigma(x^\mu)$ with a finite number of components. Instead we introduce a field $\Phi(x^\mu, \theta_n)$ depending on a set of continuous variables θ_n. Under an arbitrary Lorentz transformation $U(\Lambda)$,

$$U(\Lambda)\Phi(x^\mu, \theta_n)U^{-1}(\Lambda) = \Phi(\Lambda^\mu{}_\nu x^\nu, \Gamma_{mn}(\Lambda^{-1})\theta_m), \tag{6}$$

where a sum over repeated indices is assumed throughout. The reason for this particular transformation law will be given below. The matrices Γ_{mn} form a representation of the Lorentz group since

$$\Gamma_{nl}(\Lambda_2{}^{-1})\Gamma_{mn}(\Lambda_1{}^{-1}) = \Gamma_{ml}((\Lambda_2\Lambda_1)^{-1}) = \Gamma_{ml}(\Lambda_1{}^{-1}\Lambda_2{}^{-1}).$$

If the θ_n transform according to a representation of the Lorentz group designated by (A, B), the rotation and boost generators for the Γ_{mn} are given by[3]

$$\vec{\mathcal{J}} = \vec{\mathcal{J}}^{(A)} + \vec{\mathcal{J}}^{(B)}, \quad \vec{\mathcal{K}} = -i(\vec{\mathcal{J}}^{(A)} - \vec{\mathcal{J}}^{(B)}),$$

where the $\vec{\mathcal{J}}^{(A)}$ and $\vec{\mathcal{J}}^{(B)}$ can be written simply in terms of ordinary spin matrices for the spins A and B. Recalling Eqs. (1), the matrix generators corresponding to the L_1 and L_2 operators are

$$\mathcal{L}_1 = \mathcal{K}_1 - \mathcal{J}_2 = -i(\mathcal{J}_-^{(A)} - \mathcal{J}_+^{(B)}), \tag{7a}$$

$$\mathcal{L}_2 = \mathcal{K}_2 + \mathcal{J}_1 = (\mathcal{J}_-^{(A)} + \mathcal{J}_+^{(B)}), \tag{7b}$$

where as usual

$$\mathcal{J}_\pm = \mathcal{J}_1 \pm i\mathcal{J}_2 .$$

We take $\Phi(x, \theta_n)$ to be a free field operator creating or destroying only single particles. The one-particle matrix elements can be written as

$$\langle 0|\Phi(x,\theta_n)|\vec{k}, l, \phi\rangle = \frac{1}{(2\pi)^{3/2}} \frac{e^{-ikx}}{(2|\vec{k}|)^{1/2}} \times F(\vec{k}, l, \phi; \theta_n). \tag{8}$$

These are related to the standard state matrix elements by a Lorentz transformation

$$U(\Lambda(\vec{k})) = e^{-i\alpha J_3} e^{-i\beta J_2} e^{-i\gamma K_3}.$$

Thus from Eq. (5)

$$\langle 0|\Phi(x,\theta_n)|\vec{k}, l, \phi\rangle = \left(\frac{\kappa}{|\vec{k}|}\right)^{1/2} \langle 0|U^{-1}(\Lambda(\vec{k}))\,\Phi(x,\theta_n)U(\Lambda(\vec{k}))|\kappa\hat{z}, l, \phi\rangle .$$

Then using (5), (6), and (8)

$$F(\vec{k}, l, \phi; \theta_n) = F(\kappa\hat{z}, l, \phi; \Gamma_{mn}(\Lambda(\vec{k}))\theta_m), \tag{9}$$

with

$$\Gamma_{mn}(\Lambda(\vec{k})) = (e^{-i\alpha\mathcal{J}_3} e^{-i\beta\mathcal{J}_2} e^{-i\gamma\mathcal{K}_3})_{mn} .$$

The function $F(\kappa\hat{z}, l, \phi; \theta_n)$ can be obtained by considering Eqs. (3a) and (3b) and writing for infinitesimal ϵ_1 and ϵ_2

$$\langle 0|(1 - i\epsilon_1 L_1 - i\epsilon_2 L_2)\Phi(x,\theta_n)(1 + i\epsilon_1 L_1 + i\epsilon_2 L_2)|\kappa\hat{z}, l, \phi\rangle = (1 + i\epsilon_1 l\cos\phi + i\epsilon_2 l\sin\phi)\langle 0|\Phi(x,\theta_n)|\kappa\hat{z}, l, \phi\rangle .$$

Using (6) and (8) this gives

$$\left\{[\mathcal{L}_1]_{mn}\,\theta_m \frac{\partial}{\partial\theta_n} - l\cos\phi\right\} F(\kappa\hat{z}, l, \phi; \theta_n) = 0,$$

$$\left\{[\mathcal{L}_2]_{mn}\,\theta_m \frac{\partial}{\partial\theta_n} - l\sin\phi\right\} F(\kappa\hat{z}, l, \phi; \theta_n) = 0,$$

or from the explicit forms of Eqs. (7) after adding and subtracting

$$\left\{[\mathcal{J}_-^{(A)}]_{mn}\theta_m \frac{\partial}{\partial\theta_n} - \tfrac{1}{2}ile^{-i\phi}\right\} F(\kappa\hat{z}, l, \phi; \theta_n) = 0,$$

$$\left\{[\mathcal{J}_+^{(B)}]_{mn}\theta_m \frac{\partial}{\partial\theta_n} + \tfrac{1}{2}ile^{i\phi}\right\} F(\kappa\hat{z}, l, \phi; \theta_n) = 0.$$

These equations cannot be solved for $l \neq 0$ if either A or B is zero. This rules out scalar or two-component spinor transformation laws for the θ_n. However, if θ is a four-component spinor

$$\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix}$$

the equations read

$$\left(\theta_2 \frac{\partial}{\partial\theta_1} - \tfrac{1}{2}ile^{-i\phi}\right) F(\kappa\hat{z}, l, \phi; \theta) = 0,$$

$$\left(\theta_3 \frac{\partial}{\partial\theta_4} + \tfrac{1}{2}ile^{i\phi}\right) F(\kappa\hat{z}, l, \phi; \theta) = 0,$$

with the solution

$$F(\kappa\hat{z}, l, \phi; \theta) = f(\theta_2, \theta_3)\exp\left[\tfrac{1}{2}il\left(\frac{\theta_1}{\theta_2}e^{-i\phi} - \frac{\theta_4}{\theta_3}e^{i\phi}\right)\right]. \tag{10}$$

The function $f(\theta_2, \theta_3)$ is determined by specifying the $l \to 0$ limit of the particular field theory being discussed. If for $l = 0$ the continuous-spin states go over into a state of helicity λ, $f(\theta_2, \theta_3)$ must be proportional to appropriately coupled factors of

$$\theta_2^p \theta_3^q ,$$

with

$$\lambda = (q - p)/2 .$$

This serves to reproduce the conventional $l = 0$ helicity state "wave functions" of familiar massless field theory, multiplied by factors of θ. As a simple example, we consider

$$f(\theta_2, \theta_3) = \theta_2 = \delta_{n2}\theta_n .$$

Then, performing a Lorentz transformation as in Eq. (9),

$$F(\vec{k}, 0, \phi; \theta) = \delta_{n2}\Gamma_{mn}(\Lambda(\vec{k}))\theta_m = u_m(k)\theta_m,$$

where $u_m(k)$ is just the usual $-\frac{1}{2}$ helicity spinor.[3] This explains the particular form taken in Eq. (6) and indicates the connection between $l=0$ and $l\neq 0$ cases. The properties peculiar to continuous-spin states are entirely contained in the exponential factor of Eq. (10). In fact, we can eliminate the extra factors of θ in $f(\theta_2, \theta_3)$ by introducing the multicomponent field $\Phi_\sigma(x, \theta)$ and writing

$$\langle 0|\Phi_\sigma(x,\theta)|\vec{k}, l, \phi\rangle = \frac{1}{(2\pi)^{3/2}}\frac{e^{-ik\cdot x}}{(2|\vec{k}|)^{1/2}} \times u_\sigma(k) g(\vec{k}, l, \phi; \theta), \tag{11}$$

where $u_\sigma(k)$ is the conventional "wave function" corresponding to the particular helicity state obtained from the continuous-spin states in the $l\to 0$ limit. The function $g(\vec{k}, l, \phi; \theta)$ is just the exponential term in (10) Lorentz-transformed according to Eq. (9). Explicitly,

$$g(\vec{k}, l, \phi; \theta) = \exp\{\tfrac{1}{2}il[G_1(\vec{k},\theta)e^{-i(\alpha+\phi)} + G_2(\vec{k},\theta)e^{i(\alpha+\phi)}]\}, \tag{12a}$$

where

$$G_1(\vec{k},\theta) = \begin{pmatrix} \left(\frac{\kappa}{|\vec{k}|}\right)^{1/2}\theta_1\cos\frac{1}{2}\beta + \left(\frac{|\vec{k}|}{\kappa}\right)^{1/2}\theta_2\sin\frac{1}{2}\beta \\ \hline \left(\frac{|\vec{k}|}{\kappa}\right)^{1/2}\theta_2\cos\frac{1}{2}\beta - \left(\frac{\kappa}{|\vec{k}|}\right)^{1/2}\theta_1\sin\frac{1}{2}\beta \end{pmatrix}, \tag{12b}$$

$$G_2(\vec{k},\theta) = -\begin{pmatrix} \left(\frac{\kappa}{|\vec{k}|}\right)^{1/2}\theta_4\cos\frac{1}{2}\beta - \left(\frac{|\vec{k}|}{\kappa}\right)^{1/2}\theta_3\sin\frac{1}{2}\beta \\ \hline \left(\frac{|\vec{k}|}{\kappa}\right)^{1/2}\theta_3\cos\frac{1}{2}\beta + \left(\frac{\kappa}{|\vec{k}|}\right)^{1/2}\theta_4\sin\frac{1}{2}\beta \end{pmatrix}, \tag{12c}$$

with α and β as before, the azimuthal and polar angles of $\vec{k}$.

The above derivations can be repeated for the antiparticle states created by $\Phi_\sigma(x,\theta)$ for which

$$\langle\vec{k}, l, \phi|\Phi_\sigma(x,\theta)|0\rangle = \frac{1}{(2\pi)^{3/2}}\frac{e^{ik\cdot x}}{(2|\vec{k}|)^{1/2}} v_\sigma(k) \times g(\vec{k}, l, \phi+\pi; \theta), \tag{13}$$

where $v_\sigma(k)$ is the conventional antiparticle "wave function."

The vacuum expectation value of the free field commutator or anticommutator is obtained by inserting complete sets of particle and antiparticle states and using Eqs. (11) and (13). Thus,

$$\langle 0|[\Phi_\sigma(x,\theta), \Phi^\dagger_{\sigma'}(x',\theta')]_\pm|0\rangle = \frac{1}{(2\pi)^3}\int\frac{d^3k}{2|\vec{k}|}\int d\phi\{[e^{ik\cdot(x'-x)}u_\sigma(k)u^*_{\sigma'}(k)g(\vec{k}, l, \phi; \theta)g^*(\vec{k}, l, \phi; \theta')] \pm [e^{-ik\cdot(x'-x)}v_\sigma(k)v^*_{\sigma'}(k)g(\vec{k}, l, \phi; \theta)g^*(\vec{k}, l, \phi; \theta')]\}.$$

The factor $u_\sigma(k)u^*_{\sigma'}(k)$ is a polynomial in the four-vector k and satisfies[3]

$$u_\sigma(k)u^*_{\sigma'}(k) = (-1)^{2\lambda}v_\sigma(-k)v^*_{\sigma'}(-k).$$

Then causality in the $l\to 0$ limit gives us the usual spin and statistics relationship and requires that

$$\langle 0|[\Phi_\sigma(x,\theta), \Phi^\dagger_{\sigma'}(x',\theta')]_\pm|0\rangle = u_\sigma(i\partial_x)u^*_{\sigma'}(i\partial_x)\frac{1}{(2\pi)^3}\int\frac{d^3k}{2|\vec{k}|}[(e^{ik\cdot(x'-x)} - e^{-ik\cdot(x'-x)})C(\vec{k}, l; \theta, \theta')], \tag{14}$$

with

$$C(\vec{k}, l; \theta, \theta') = \int d\phi\, g(\vec{k}, l, \phi; \theta)g^*(\vec{k}, l, \phi; \theta').$$

Using Eqs. (12) we can evaluate the continuous-spin integral to find, after eliminating an irrelevant constant,

$$C(\vec{k}, l; \theta, \theta') = \sum_{n=0}^{\infty}\left[(\tfrac{1}{2}il)^n\frac{1}{n!}\right]^2\{[G_1(\vec{k},\theta) - G_2^*(\vec{k},\theta')][G_2(\vec{k},\theta) - G_1^*(\vec{k},\theta')]\}^n. \tag{15}$$

For example, in the special case

$$\theta = \theta' = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

we have

$$C(\vec{k}, l; \theta, \theta') = \sum_{n=0}^{\infty} \left[(\tfrac{1}{2} i l)^n \frac{1}{n!} \right]^2 \left[-\left(\frac{4|\vec{k}|}{\left(\frac{|\vec{k}|^2}{\kappa} - \kappa\right) + \left(\frac{|\vec{k}|^2}{\kappa} + \kappa\right) \cos\beta} \right)^2 \right]^n .$$

$C(\vec{k}, l; \theta, \theta')$ is clearly not in general a polynomial of finite order in k, nor is it symmetric,

$$C(\vec{k}, l; \theta, \theta') \neq C(-\vec{k}, l; \theta, \theta') .$$

In order for our field theory of massless particles with continuous spin indices to be causal, the integral in Eq. (14) must vanish at equal times, $x_0' = x_0$. Changing a $\vec{k}$ to $-\vec{k}$ the integral in question is

$$\int \frac{d^3k}{2|\vec{k}|} e^{-i\vec{k}\cdot(\vec{x}'-\vec{x})} [C(\vec{k}, l; \theta, \theta') - C(-\vec{k}, l; \theta, \theta')] ,$$

which must vanish for all $\vec{x}' - \vec{x} \neq 0$. This requires that the factor multiplying the exponential in the integrand be a polynomial of finite order in $\vec{k}$.[4] From Eq. (15) we see that this would be possible only if $C(\vec{k}, l; \theta, \theta')$ were symmetric in $\vec{k}$. Since this is not the case, the field commutator or anticommutator does not vanish at equal times except in the familiar case $l = 0$. Thus we have concluded a field-theoretic argument against the existence of massless particles with continuous spin indices in nature.

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