Bound states of the two-dimensional O(N) model*

L. F. Abbott
Department of Physics, Brandeis University, Waltham, Massachusetts 02154
(Received 8 March 1976)

Using the semiclassical methods of Dashen, Hasslacher, and Neveu we compute the bound-state spectrum of a two-dimensional model of scalar bosons with O(N) symmetry and ϕ^4 coupling.

I. INTRODUCTION

In a series of papers,[1-3] Dashen, Hasslacher, and Neveu (DHN) have pioneered the use of semiclassical and WKB methods in quantum field theory. In particular, they have employed semiclassical and inverse scattering techniques to calculate the bound-state spectrum of the Gross-Neveu model.[3] Here, we apply these methods to an O(N)-symmetric theory of scalar bosons with ϕ^4 coupling in two spacetime dimensions. Our computation is to leading order in an expansion by powers of $1/N$, and therefore corresponds to the many-field limit of the theory.

In Sec. II we introduce the two-dimensional O(N) model and discuss some of its relevant features. The semiclassical methods of DHN are applied in Sec. III, and bound-state masses and form factors are calculated in Sec. IV. A last section is devoted to summary and conclusions.

II. THE O(N) MODEL

The O(N) model with ϕ^4 coupling is given by the (unrenormalized) Lagrangian

$$\mathcal{L}=\tfrac{1}{2}\partial_\mu\phi_a\partial^\mu\phi_a-\tfrac{1}{2}\mu_0^2\phi_a\phi_a-\frac{\lambda}{4!N}(\phi_a\phi_a)^2, \tag{2.1}$$

where a sum over repeated index a from 1 to N is implied throughout. In two dimensions we require only a mass renormalization of μ_0. A more convenient form for the Lagrangian is

$$\mathcal{L}=\tfrac{1}{2}\partial_\mu\phi_a\partial^\mu\phi_a-\tfrac{1}{2}\chi\phi_a\phi_a+\frac{3N}{2\lambda}\chi^2-\frac{3N}{\lambda}\mu_0^2\chi\,. \tag{2.2}$$

By solving the field equation for χ we see that this Lagrangian is equivalent to (2.1).

The effective potential for this model has been computed to leading order in $1/N$ elsewhere.[4,5] It determines the vacuum values of the fields, which we denote by $\bar{\phi}_a$ and $\bar{\chi}$. In accordance with Coleman's theorem,[6] the continuous O(N) symmetry cannot be spontaneously broken in two dimensions, so

$$\bar{\phi}_a=0. \tag{2.3}$$

To this order, the square of the physical meson mass is given by

$$\bar{\chi}=m^2 \tag{2.4}$$

and $\bar{\chi}$ is determined by the gap equation

$$\bar{\chi}=\mu^2-\frac{\lambda}{24\pi}\ln\left(\frac{\bar{\chi}}{M^2}\right), \tag{2.5}$$

where M is an arbitrary renormalization mass. The parameter μ (an intermediate renormalized mass) depends on the value of M chosen to renormalize μ_0, while λ is independent of M since it is not renormalized. A convenient renormalization-invariant mass parameter is

$$\chi_0=M^2\exp\left(\frac{24\pi\mu^2}{\lambda}\right). \tag{2.6}$$

Then, writing

$$\bar{\chi}=\rho\chi_0, \tag{2.7}$$

Eq. (2.5) becomes

$$\ln\rho=-\left(\frac{24\pi\chi_0}{\lambda}\right)\rho. \tag{2.8}$$

An examination of Fig. 1 will indicate the key features of this equation. For

$$\frac{24\pi\chi_0}{\lambda}\geq 0 \tag{2.9}$$

there is a unique solution in the range

$$0<\rho\leq 1. \tag{2.10}$$

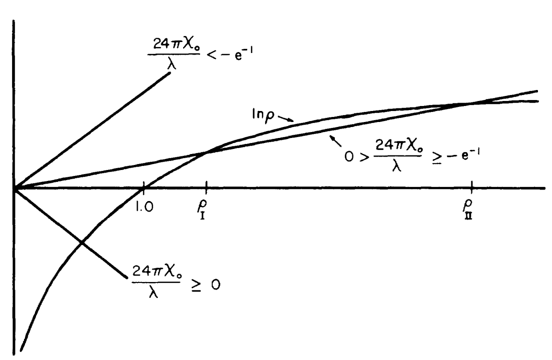

FIG. 1. Graphical solutions to Eq. (2.8) for the three ranges discussed in the text.

If

$$0 > \frac{24\pi\chi_0}{\lambda} \geq -e^{-1} \tag{2.11}$$

Eq. (2.8) has two solutions:

$$e > \rho_{\rm I} > 1, \tag{2.12}$$

$$\rho_{\rm II} \geq e. \tag{2.13}$$

The larger of these two solutions, $\rho_{\rm II}$, will always give a lower vacuum energy and hence will determine $\bar{\chi}$ through (2.7). Finally, when

$$\frac{24\pi\chi_0}{\lambda} < -e^{-1} \tag{2.14}$$

there are no solutions at all for ρ and the theory has no vacuum state. Using (2.13) with (2.4) and (2.7), this means that the physical parameters of the model must satisfy

$$\frac{\lambda}{24\pi m^2} > -1 \tag{2.15}$$

in order for a vacuum to exist. This corresponds to the allowed domain found by Schnitzer[5] in order to avoid tachyons in the two-point Green's functions.

If $\lambda < 0$, the effective potential, $V(\phi_a\phi_a)$, becomes complex for sufficiently large values of $\phi_a\phi_a$, and furthermore

$$\mathrm{Re}[V(\phi_a\phi_a)] \xrightarrow[\phi_a\phi_a\to\infty]{} -\infty\ . \tag{2.16}$$

In Sec. IV we will require $\lambda<0$ to bind mesons. Equation (2.16) would suggest stability problems with such a scheme. However, very large $\phi_a\phi_a$ lies outside the domain of applicability of the $1/N$ expansion, so Eq. (2.16) may not be a definite property of the model. Also, it has been shown[5,7] that, to leading order in $1/N$, the model is consistent and possesses no tachyons in the negative range of λ allowed by (2.15). Schnitzer[5] has found a two-body bound state in the negatively coupled theory when (2.15) is satisfied.

We will therefore accept the consistency of the model in the entire domain of (2.15) and leave the ultimate problem of stability to be settled by higher-order calculations or different methods. One strong motivation for this approach is the striking similarity between the negatively coupled two-dimensional theory and its four-dimensional analog for either sign of λ, when evaluated in the many-field limit.[7,8] It is therefore hoped that our calculations might suggest the type of behavior to be found in the four-dimensional O(N) model.

III. THE SEMICLASSICAL METHOD

In the DHN approach,[1-3] we identify bound-state energies by considering

$$\mathrm{Tr}(e^{-iHT}) = \oint \delta[\phi_a]\delta[\chi]\exp\left[\,i\int_0^T dt\int_{-\infty}^{\infty}dx\left(\tfrac{1}{2}\partial_\mu\phi_a\partial^\mu\phi_a - \tfrac{1}{2}\chi\phi_a\phi_a + \frac{3N}{2\lambda}\chi^2 - \frac{3N}{\lambda}\mu_0^2\chi\right)\right], \tag{3.1}$$

where the integration runs over all periodic fields,

$$\phi_a(x,0) = \phi_a(x,T), \tag{3.2}$$

$$\chi(x,0) = \chi(x,T). \tag{3.3}$$

The semiclassical method involves evaluating the χ-field integration by stationary-phase approximation, and ignoring the Gaussian corrections which would be computed in a complete WKB calculation. Here we will not consider all of the stationary-phase points of the integrand (3.1), but only those involving time-independent χ fields which automatically satisfy (3.3). This will not allow us to compute the complete semiclassical particle spectrum, but will nevertheless provide interesting bound-state results.

For time-independent χ fields we introduce the eigenvalues and eigenfunctions of the Schrödinger-type equation

$$\left[-\frac{d^2}{dx^2} + \chi(x)\right]\psi_i(x) = \omega_i^2[\chi]\psi_i(x)\ , \tag{3.4}$$

where we have explicitly displayed the ω_i as functionals of χ. We can use this complete set of eigenfunctions (appropriately normalized) to write

$$\phi_a(x,t) = \sum_i A_a^i(t)\psi_i(x). \tag{3.5}$$

This reduces the integrations over the fields ϕ_a to functional integrals of simple harmonic-oscillator coordinates. Then, using well-known results,[9] we find for time-independent χ

$$\begin{aligned}\oint \delta[\phi_a]\exp\left[\,i\int_0^T dt\int_{-\infty}^{\infty}dx\,(\tfrac{1}{2}\partial_\mu\phi_a\partial^\mu\phi_a - \tfrac{1}{2}\chi\phi_a\phi_a)\right] &= \left[\prod_i \frac{e^{-i(\omega_i/2)T}}{1-e^{-i\omega_i T}}\right]^N \\ &= \exp\left(-\frac{iN}{2}T\sum_i\omega_i\right)\sum_{\{n\}}C(\{n\})\exp\left(-iT\sum_i n_i\omega_i\right),\end{aligned} \tag{3.6}$$

where the sum is over all sets of non-negative integers n_i and

$$C(\{n\}) = \prod_i \frac{(N+n_i-1)!}{(N-1)!\,n_i!}\ . \tag{3.7}$$

Thus, we can write out the contribution to the trace (3.1) from time-independent stationary-phase points explicitly, obtaining

$$\mathrm{Tr}(e^{-iHT}) = \sum_{\{n\}} C(\{n\}) e^{-i\tilde{E}(\{n\})T} + \cdots, \tag{3.8}$$

where

$$\tilde{E}(\{n\}) = N\left[\int_{-\infty}^{\infty} dx\left(-\frac{3}{2\lambda}\chi^2 + \frac{3}{\lambda}\mu_0^2\chi\right) + \tfrac{1}{2}\sum_i \omega_i[\chi] + \frac{1}{N}\sum_i n_i\omega_i[\chi]\right] \tag{3.9}$$

with $\chi = \chi(x)$ given by the stationary-phase condition

$$\frac{\delta\tilde{E}(\{n\})}{\delta\chi} = 0\ . \tag{3.10}$$

The various terms in Eqs. (3.8) and (3.9) have direct physical interpretations. $C(\{n\})$ is the number of degenerate states with quantum numbers $\{n\}$ and energy $\tilde{E}(\{n\})$. $\tilde{E}(\{n\})$ consists of the classical energy of the χ field, a contribution from quantum fluctuations, and the energy of n_i particles occupying states with energies ω_i. Since we are interested in bound states, we will only consider those terms of (3.8) in which all $n_i = 0$ except those corresponding to bound excitations and thus satisfying

$$\omega_i < m\ . \tag{3.11}$$

In Sec. IV we will see that only one of the eigenvalues of Eq. (3.4) satisfies (3.11) and we will label it by ω_0 with the corresponding bound-state eigenfunction $\psi_0(x)$. Thus, we will consider bound states labeled by a single quantum number n_0, with the multiplicity of degenerate states given by

$$C(n_0) = \frac{(N+n_0-1)!}{(N-1)!\,n_0!}\ . \tag{3.12}$$

If we want the bound-state energies normalized to a vacuum energy of zero we must subtract the vacuum energy

$$E_0 = N\left[\int_{-\infty}^{\infty} dx\left(-\frac{3}{2\lambda}\bar{\chi}^2 + \frac{3}{\lambda}\mu_0^2\bar{\chi}\right) + \tfrac{1}{2}\sum_i \omega_i[\bar{\chi}]\right] \tag{3.13}$$

from (3.9) so that

$$E(n_0) = \tilde{E}(n_0) - E_0. \tag{3.14}$$

The $\omega_i^2[\bar{\chi}]$ are just the eigenvalues of Eq. (3.4) with $\chi(x) = \bar{\chi}$. Thus, from (3.9) and (3.13) we have

$$E(n_0) = N\left\{\int_{-\infty}^{\infty} dx\left[-\frac{3}{2\lambda}(\chi-\bar{\chi})^2 + \frac{3}{\lambda}(\mu_0^2-\bar{\chi})(\chi-\bar{\chi})\right] + \tfrac{1}{2}\left(\sum_i \omega_i[\chi] - \sum_i \omega_i[\bar{\chi}]\right) + \frac{n_0}{N}\omega_0[\chi]\right\}, \tag{3.15}$$

with $\chi = \chi(x)$ determined by

$$\frac{\delta E(n_0)}{\delta\chi} = 0. \tag{3.16}$$

Our problem is to solve (3.16). This is done in Sec. IV.

Equation (3.12) tells us that the bound states of energy $E(n_0)$ form a degenerate reducible representation of the O(N) group consisting of a symmetric tensor of rank n_0. In terms of irreducible representations, the supermultiplet of energy $E(n_0)$ is formed from multiplets which are symmetric traceless O(N) tensors of rank n_0', where

$$n_0' = 0, 2, 4, \ldots, n_0 \quad \text{for } n_0 \text{ even}, \tag{3.17}$$

$$n_0' = 1, 3, 5, \ldots, n_0 \quad \text{for } n_0 \text{ odd}. \tag{3.18}$$

This degeneracy in the spectrum is much greater than what would be required by O(N) symmetry alone.

IV. THE BOUND-STATE CALCULATION

It is convenient to use the vacuum energy (3.13) to determine the renormalization of μ_0. All divergent momentum integrals will be regulated with a cutoff Λ. When $\chi(x) = \bar{\chi}$ Eq. (3.4) becomes a free-particle equation, so, recalling (2.4),

$$\sum_i \omega_i[\bar{\chi}] = \int_0^\Lambda \frac{dk}{\pi}(k^2+m^2)^{1/2}\int_{-\infty}^{\infty} dx\ . \tag{4.1}$$

The condition

$$\frac{\delta E_0}{\delta\bar{\chi}} = 0\ . \tag{4.2}$$

determines the mass renormalization,

$$\bar{\chi} = m^2 = \mu_0^2 + \frac{\lambda}{6}\int_0^\Lambda \frac{dk}{2\pi}\left(\frac{1}{k^2+m^2}\right)^{1/2}\ . \tag{4.3}$$

Now consider the Schrödinger-type equation

$$\left[-\frac{d^2}{dx^2} + (\chi - \bar{\chi})\right]\psi_i = k_i^2\psi_i\ . \tag{4.4}$$

Comparison with Eq. (3.4) using (2.4) gives

$$k_i^2 = \omega_i^2 - m^2\ . \tag{4.5}$$

The program for solving Eq. (3.16) is to rewrite the bound-state energy (3.15) in terms of the reflection coefficient and bound-state energies of Eq. (4.4) using the trace identities of the one-dimensional Schrödinger equation.[3, 10] To simplify our calculations we will use two results of DHN.[3] They have shown how the functional derivative with respect to the reflection coefficient implied in (3.16) leads to functions $[\chi(x) - \bar{\chi}]$ which act as reflectionless potentials in Eq. (4.4). Furthermore, since the bound-state energies of (4.4) enter additively into (3.15) we need only consider a single bound state of (4.4). Thus, we will take our solution $\chi=\chi(x)$ to form a reflectionless potential in Eq. (4.4) with a single bound state given by

$$k_0 = i\kappa \tag{4.6}$$

or

$$\omega_0 = (m^2-\kappa^2)^{1/2}. \tag{4.7}$$

Under these restrictions we have the simple trace identities

$$\int_{-\infty}^{\infty} dx(\chi-\bar{\chi}) = -4\kappa, \tag{4.8}$$

$$\int_{-\infty}^{\infty} dx(\chi-\bar{\chi})^2 = \tfrac{16}{3}\kappa^3 . \tag{4.9}$$

These permit us to write the first two terms of (3.15) solely in terms of the unknown κ. Using the DHN results for summing over modes,[1-3] we find

$$\sum_i (\omega_i[\chi]-\omega_i[\bar{\chi}]) = \omega_0[\chi]-m-\int_0^{\Lambda}\frac{k\,dk}{\pi}\left[\frac{\delta(k)}{(k^2+m^2)^{1/2}}\right], \tag{4.10}$$

where $\delta(k)$ is the phase shift of Eq. (4.4) given by

$$\delta(k) = 2\tan^{-1}(\kappa/k). \tag{4.11}$$

Finally,

$$\int_0^{\Lambda}\frac{k\,dk}{\pi}\,\frac{2\tan^{-1}(\kappa/k)}{(k^2+m^2)^{1/2}} = \frac{2}{\pi}\kappa - m + \frac{2}{\pi}(m^2-\kappa^2)^{1/2}\tan^{-1}\left[\frac{(m^2-\kappa^2)^{1/2}}{\kappa}\right] + 4\kappa\int_0^{\Lambda}\frac{dk}{2\pi}\left(\frac{1}{k^2+m^2}\right)^{1/2} . \tag{4.12}$$

Then, combining (4.7), (4.8), (4.9), (4.10), and (4.12) into (3.15) we can write the bound-state energy as a function of κ,

$$E(n_0) = N\left\{-\frac{8}{\lambda}\kappa^3 - \frac{12}{\lambda}(\mu_0^{\,2}-m^2)\kappa + \tfrac{1}{2}(m^2-\kappa^2)^{1/2} - \frac{1}{\pi}\kappa + \frac{n_0}{N}(m^2-\kappa^2)^{1/2} - \frac{(m^2-\kappa^2)^{1/2}}{\pi}\tan^{-1}\left[\frac{(m^2-\kappa^2)^{1/2}}{\kappa}\right] - 2\kappa\int_0^{\Lambda}\frac{dk}{2\pi}\left(\frac{1}{k^2+m^2}\right)^{1/2}\right\}. \tag{4.13a}$$

Recalling (4.3) we see that the divergent integrals cancel, and the result is, after renormalization,

$$E(n_0) = N\left\{-\frac{8}{\lambda}\kappa^3 + \left(\frac{n_0}{N}+\frac{1}{2}\right)(m^2-\kappa^2)^{1/2} - \frac{\kappa}{\pi} - \frac{(m^2-\kappa^2)^{1/2}}{\pi}\tan^{-1}\left[\frac{(m^2-\kappa^2)^{1/2}}{\kappa}\right]\right\}. \tag{4.13b}$$

It is convenient to write

$$\kappa = m\sin\theta \tag{4.14}$$

for

$$0 \le \theta \le \frac{\pi}{2} . \tag{4.15}$$

Then, in terms of θ,

$$E(n_0) = Nm\left[-\frac{8m^2}{\lambda}\sin^3\theta + \frac{n_0}{N}\cos\theta + \frac{1}{\pi}(\theta\cos\theta - \sin\theta)\right] \tag{4.16}$$

and Eq. (3.16) becomes simply

$$\frac{dE(n_0)}{d\theta} = 0, \tag{4.17}$$

giving

$$\sin\theta\left[\sin\theta\cos\theta + \frac{\lambda}{24\pi m^2}\left(\theta+\frac{n_0}{N}\pi\right)\right] = 0. \tag{4.18}$$

The solution $\sin\theta=0$ corresponds to the vacuum state $\chi=\bar{\chi}$. Nontrivial solutions must satisfy

$$\sin 2\theta = -2\left(\frac{\lambda}{24\pi m^2}\right)\left(\theta+\frac{n_0}{N}\pi\right). \tag{4.19}$$

For $n_0=0$ we have the solution $\theta=0$, which again gives the vacuum state. If $\lambda>0$ there are no solutions to (4.19) in the allowed range (4.15) for nonzero n_0. Thus, we must take λ negative, but in the allowed range of (2.15), to get bound states. For negative λ we define

$$g = \frac{|\lambda|}{24\pi m^2} < 1. \tag{4.20}$$

Then, (4.19) becomes for $\lambda<0$

$$\sin 2\theta = 2g\left(\theta+\frac{n_0}{N}\pi\right), \tag{4.21}$$

and subject to (4.21) the bound-state energy (4.16) is

$$E(n_0) = \frac{Nm}{g\pi}\sin\theta(1-g-\tfrac{2}{3}\sin^2\theta). \tag{4.22}$$

For stability we must require

$$\frac{d^2E(n_0)}{d\theta^2} > 0, \tag{4.23}$$

which gives the condition

$$\cos 2\theta > g. \tag{4.24}$$

Combined with (4.15) this means that we must solve (4.21) in the range

$$0 \le \theta < \theta_{\max} < \frac{\pi}{4}, \tag{4.25}$$

where

$$\cos 2\theta_{max} = g. \tag{4.26}$$

In the range (4.25), (4.22) is never negative.

When bound states are present, Eq. (4.21) will have two solutions and condition (4.25) tells us always to take the smaller one. These features are displayed in Fig. 2, where we have taken

$$\frac{n_0}{N} = 0.1, \tag{4.27}$$

and we find bound states for $g=0.1$ and $g=0.5$ but none for $g=0.9$.

For

$$n_0 g\pi \ll N(1-g) \tag{4.28}$$

we have the approximate solution to (4.21)

$$\theta = \frac{n_0 g\pi}{N(1-g)} + \frac{2}{3(1-g)}\left[\frac{n_0 g\pi}{N(1-g)}\right]^3 + O\left(\left[\frac{n_0 g\pi}{N(1-g)}\right]^5\right) \tag{4.29}$$

with the energy

$$E(n_0) = n_0 m - \tfrac{1}{6} n_0 m \left[\frac{n_0 g\pi}{N(1-g)}\right]^2 + O\left(\left[\frac{n_0 g\pi}{N(1-g)}\right]^4\right). \tag{4.30}$$

This clearly shows the presence of an n-body bound state.

Once κ has been computed from Eqs. (4.21) and (4.14) we can reconstruct the functions $\chi(x)$ and $\psi_0(x)$ by inverse scattering methods.[10,11] We form the kernel

$$F(x,y) = c_0 e^{\kappa x} e^{\kappa y} \tag{4.31}$$

and solve the Gel'fand-Levitan equation

$$K(x,y) + F(x,y) + \int_{-\infty}^{x} dz\, K(x,z) F(z,y) = 0, \tag{4.32}$$

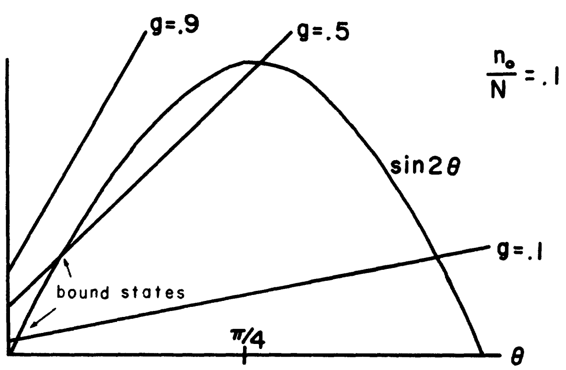

FIG. 2. Graphical solutions to Eq. (4.21) for $n_0/N = 0.1$ and $g=0.1$, 0.5, and 0.9.

obtaining

$$K(x,y) = \frac{-c_0 e^{\kappa x} e^{\kappa y}}{1 + (c_0/2\kappa)e^{2\kappa x}}. \tag{4.33}$$

Now[11]

$$K(x,y) \propto \psi_0(x) e^{\kappa y}, \tag{4.34}$$

and this determines the bound-state normalization factor

$$c_0 = 2\kappa \tag{4.35}$$

and the bound-state eigenfunction

$$\psi_0(x) = \frac{(\kappa/2)^{1/2}}{\cosh(\kappa x)}. \tag{4.36}$$

$\chi(x)$ is determined by

$$\chi(x) = m^2 + 2\frac{d}{dx}K(x,x), \tag{4.37}$$

giving

$$\chi(x) = m^2 - 2\kappa^2 + 2\kappa^2 \tanh^2(\kappa x). \tag{4.38}$$

These functions are sketched in Fig. 3.

V. SUMMARY AND CONCLUSIONS

We have found a rich bound-state spectrum in the two-dimensional O(N) model with negative ϕ^4 coupling. It consists of supermultiplets which are reducible symmetric tensor representations of the O(N) group. These are ordinary n-body bound states and not what is referred to in the physics literature as solitons. A soliton would have appeared in our model as a solution to Eq. (4.21) with $\theta = \pi/2$, which is forbidden by Eq. (4.25). The absence of such a solution is no doubt connected to the absence of spontaneous symmetry breaking in our model. We note therefore that semiclassical bound states and solitons are not necessarily related, although computational methods may be similar for the two.

VI. NOTE ADDED: THE STABILITY PROBLEM

For the reader who is justifiably concerned about the stability problem for negatively coupled ϕ^4 theory (discussed in Sec. II) we make the following comments. Consider the general O(N)-sym-

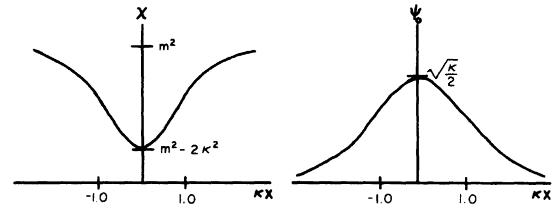

FIG. 3. Sketches of $\chi(x)$ and $\psi_0(x)$ versus κx.

metric Lagrangian

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu \phi_a \partial^\mu \phi_a - NV\left(\frac{\phi_a\phi_a}{N}\right). \qquad (6.1)$$

Townsend[12] and Schnitzer[5] have shown that a substitute Lagrangian equivalent to (6.1) to leading order in $1/N$ is

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu \phi_a \partial^\mu \phi_a - N\left[V(\chi) - \left(\chi - \frac{\phi_a\phi_a}{N}\right)V^{(1)}(\chi)\right], \qquad (6.2)$$

where we use the notation

$$V^{(n)}(\chi) = \frac{d^n V(\chi)}{d\chi^n} \quad . \qquad (6.3)$$

Using this substitute Lagrangian we can proceed, as in Sec. III, to write down the bound-state energies in terms of $\chi(x)$ and the eigenvalues of a one-dimensional Schrödinger equation [the generalization of (3.4)],

$$\left(-\frac{d^2}{dx^2} + 2V^{(1)}(\chi(x))\right)\psi_i(x) = \omega_i^2[\chi]\psi_i(x) . \qquad (6.4)$$

In the general case, the elementary meson mass is[5]

$$m^2 = 2V^{(1)}(\bar{\chi}), \qquad (6.5)$$

with $\bar{\chi}$ determined by the gap equation[5] [the generalization of (2.5)]

$$\bar{\chi} = -\frac{1}{4\pi}\ln\left(\frac{2V^{(1)}(\bar{\chi})}{M^2}\right). \qquad (6.6)$$

The analysis of Sec. IV cannot be extended to the general O(N)-symmetric interaction because of the lack of appropriate trace identities. However, if the *effective* ϕ^4 coupling constant,[5] which determines the behavior of the meson four-point function,

$$\lambda_{\text{eff}} = 12V^{(2)}(\bar{\chi}) , \qquad (6.7)$$

is negative, it is easy to see that the results of Sec. IV are just the *approximate* bound-state energies calculated to zeroth order in the coupling constants appropriate to the interactions ϕ^{2n} with $n \geq 3$.

We can therefore add a term like

$$\frac{-\eta}{6!N^2}(\phi_a\phi_a)^3, \text{ with } \eta > 0,$$

to the Lagrangian (2.1), which guarantees a stable ground state in our model [as established in Eq. (5.2) of Ref. 5] even for small η. Then, provided that

$$\lambda_{\text{eff}} = \lambda + \tfrac{1}{10}\eta\bar{\chi} < 0, \qquad (6.8)$$

the bound-state spectrum is given to zeroth order in η by the results of Sec. IV, Eqs. (4.21) and (4.22). Thus, our computation can be viewed as an approximation to a stable theory, with $\eta\phi^6$ coupling, for small but positive η.

ACKNOWLEDGMENTS

I am very grateful to Howard J. Schnitzer for his guidance and supervision for this project. I also acknowledge helpful conversations with Hugh Pendleton and Parthasarathi Majumdar.

*Research supported by ERDA under Contract No. E(11-1)3230.

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