PHYSICAL REVIEW D VOLUME 14, NUMBER 8 15 OCTOBER 1976

Semiclassical bound-state methods in four-dimensional field theory: Trace identities, mode sums, and renormalization for scalar theories*

L. F. Abbott and Howard J. Schnitzer

Department of Physics, Brandeis University, Waltham, Massachusetts 02154

(Received 10 May 1976)

Semiclassical and inverse scattering methods, previously restricted to two-dimensional models, are extended to four-dimensional scalar field theories. The necessary trace identities are derived, mode sums are carried out, and renormalization is explicitly demonstrated. The four-dimensional O(N) model is used to illustrate these techniques. A renormalized expression for bound-state energies of this model, parametrized by the scattering data of a three-dimensional Schrödinger equation, is obtained. This expression contains nontrivial quantum fluctuation effects, and corresponds to the leading-order term of a $1/N$ expansion. In this model, stable n-body bound states do not seem to appear, even though a two-body bound state has been found in a $1/N$ expansion of the Green's functions. Possible reasons for this failure to bind are discussed.

I. INTRODUCTION

Semiclassical methods, introduced into quantum field theory by Dashen, Hasslacher, and Neveu (DHN),[1] provide a powerful formalism for computing bound-state spectra in field theory. Such computations have previously been performed for several models in two-dimensional space-time[1-3] where the simplicity of the sum over normal modes[1] and the existence of trace identities[4] allow one to express the bound-state energies in terms of the scattering data of a one-dimensional Schrödinger equation. Until now, the extension of the DHN formalism to more realistic models has been hampered by the absence of analogous technical tools for four-dimensional theories.[5]

In this paper, we derive the trace identities and perform the mode sums needed to apply semiclassical methods to four-dimensional scalar field theories. These results are presented within the context of the O(N) model for which the semiclassical method corresponds to a leading order in $1/N$ or Hartree approximation. We obtain a renormalized expression for the bound-state energies parametrized by the scattering data of a three-dimensional Schrödinger equation. This expression contains quantum fluctuation effects which correspond to an infinite set of Feynman diagrams of ϕ^4 theory and demonstrates the renormalization program in a nontrivial way. The bound-state energies must be minimized with respect to the independent Schrödinger equation scattering data to determine the bound-state spectrum. Here we meet with failure, finding no stable n-body bound states in the four-dimensional O(N) model even though a two-body bound state appears, to leading order in $1/N$, in the Green's functions of the theory.[6] In spite of this, we present our results as a demonstration of techniques required for the extention of semiclassical methods to four-dimensional quantum field theories. It is hoped that this will provide a basis for further insight into this subject.

The O(N) model, evaluated in the Hartree approximation, has an interesting ground-state structure[6] which is reflected throughout our calculations. The effective potential has two local minima, one corresponding to the true stable ground state and the other representing an unstable vacuum of higher energy. A remarkable feature of this model is that the stable Hartree vacuum does not permit spontaneous symmetry breaking to occur. This double-valued structure of the effective potential is also relevant to the bound-state spectrum of the model as has already been demonstrated for the two-dimensional O(N) model.[2]

We begin with a presentation of the DHN formalism as applied to the four-dimensional O(N) model. Renormalization is then discussed, followed by a review of the structure of the vacuum state for this model. In Sec. V, we derive the trace identities and mode sums needed to obtain a renormalized expression for the bound-state energies, which is given in Sec. VI. We analyze the bound-state spectrum in Sec. VII, and conclude with a discussion of our results. Various technical calculations are left to the Appendixes.

II. UNRENORMALIZED BOUND-STATE ENERGIES

The O(N) model is given by the Lagrangian

$$\mathcal{L}=\tfrac{1}{2}\,\partial_\mu\phi_a\partial^\mu\phi_a-\tfrac{1}{2}\,\mu_0^{\,2}\,\phi_a\phi_a-\frac{\lambda_0}{4!N}\,(\phi_a\phi_a)^2\,, \quad (2.1)$$

where a sum over the repeated index a from 1 to N is implied throughout. For our purposes a more convenient Lagrangian to consider is

$$\mathcal{L}=\tfrac{1}{2}\partial_\mu\phi_a\partial^\mu\phi_a-\tfrac{1}{2}\chi\phi_a\phi_a+\frac{3N}{2\lambda_0}\chi^2-\frac{3N}{\lambda_0}\mu_0^2\chi\,. \tag{2.2}$$

By solving the field equation for χ we see that this Lagrangian is equivalent to (2.1).

In the DHN approach,[1] we identify bound-state energies by considering

$$\mathrm{Tr}(e^{-iHT})=\oint\delta[\phi_a]\delta[\chi]\exp\left[i\int_0^T dt\int d^3r\left(\tfrac{1}{2}\partial_\mu\phi_a\partial^\mu\phi_a-\tfrac{1}{2}\chi\phi_a\phi_a+\frac{3N}{2\lambda_0}\chi^2-\frac{3N}{\lambda_0}\mu_0^2\chi\right)\right], \tag{2.3}$$

where the integration is over all periodic fields

$$\phi_a(\vec{r},0)=\phi_a(\vec{r},T)\,, \tag{2.4}$$

$$\chi(\vec{r},0)=\chi(\vec{r},T)\,. \tag{2.5}$$

Since the ϕ_a fields enter Eq. (2.3) only quadratically, the ϕ functional integral can be carried out exactly. In the semiclassical method we then evaluate the χ functional integral by stationary-phase approximation. Here we will only consider those contributions to the trace (2.3) which come from stationary-phase points involving time-independent spherically symmetric χ fields:

$$\chi(\vec{r},t)=\chi(r) \tag{2.6}$$

with $r=|\vec{r}|$. In this case we introduce eigenfunctions and eigenvalues of the three-dimensional Schrödinger equation

$$[-\nabla^2+\chi(r)]\Psi_{\alpha l m}(\vec{r})=\omega_{\alpha l}^2[\chi]\Psi_{\alpha l m}(\vec{r})\,, \tag{2.7}$$

where we have explicitly displayed the functional dependence of the $\omega_{\alpha l}$ on the field $\chi(r)$. The quadratic functional integral can easily be evaluated by expanding the ϕ_a in the normal modes $\Psi_{\alpha l m}$. Then, applying the stationary-phase approximation, we find[2]

$$\mathrm{Tr}(e^{-iHT})=\sum_{\{n\}}C[\{n\}]e^{-i\tilde{E}[\{n\}]T}+\cdots\,, \tag{2.8}$$

where the sum is over all sets of non-negative integers $n_{\alpha l m}$, and

$$C[\{n\}]=\prod_{\alpha,l,m}\left[\frac{(N+n_{\alpha l m}-1)!}{(N-1)!\,n_{\alpha l m}!}\right], \tag{2.9}$$

$$\tilde{E}[\{n\}]=N\left[\int d^3r\left(-\frac{3}{2\lambda_0}\chi^2+\frac{3}{\lambda_0}\mu_0^2\chi\right)+\frac{1}{N}\sum_{\alpha,l,m}n_{\alpha l m}\omega_{\alpha l}[\chi]+\tfrac{1}{2}\sum_{\alpha,l}(2l+1)\omega_{\alpha l}[\chi]\right], \tag{2.10}$$

with $\chi=\chi(r)$ determined by the stationary-phase condition

$$\frac{\delta\tilde{E}[\{n\}]}{\delta\chi(r)}=0\,. \tag{2.11}$$

In order to obtain the bound-state energies from (2.10) and (2.11) we must eliminate all unbound excitations by setting all $n_{\alpha l m}=0$ unless

$$\omega_{\alpha l}[\chi]<m\,, \tag{2.12}$$

where m is the physical mass of the elementary meson, and we must subtract out the vacuum energy which is given by

$$E_0=N\left[\int d^3r\left(-\frac{3}{2\lambda_0}\bar{\chi}^2+\frac{3}{\lambda_0}\mu_0^2\bar{\chi}\right)+\tfrac{1}{2}\sum_{\alpha,l}(2l+1)\omega_{\alpha l}[\bar{\chi}]\right], \tag{2.13}$$

where $\bar{\chi}$ is the vacuum expectation value of the field χ. The physical meson mass and the vacuum value of χ are related by[6]

$$\bar{\chi}=m^2\,. \tag{2.14}$$

In all of the following, we will indicate with a prime restricted sums over those bound excitations satisfying (2.12). Using this notation, the unrenormalized bound-state energies are

$$E[\{n\}]=N\left\{\int d^3r\left[-\frac{3}{2\lambda_0}(\chi-\bar{\chi})^2+\frac{3}{\lambda_0}(\mu_0^2-\bar{\chi})(\chi-\bar{\chi})\right]+\tfrac{1}{2}\sum_{\alpha,l}(2l+1)(\omega_{\alpha l}[\chi]-\omega_{\alpha l}[\bar{\chi}])+\frac{1}{N}\sum_{\alpha,l,m}{}'\,n_{\alpha l m}\omega_{\alpha l}[\chi]\right\}, \tag{2.15}$$

with $\chi=\chi(r)$ determined by

$$\frac{\delta E[\{n\}]}{\delta\chi(r)}=0\,. \tag{2.16}$$

The combinatoric factor $C[\{n\}]$, which in this case is

$$C[\{n\}]=\prod_{\alpha,l,m}{}'\left[\frac{(N+n_{\alpha l m}-1)!}{(N-1)!\,n_{\alpha l m}!}\right], \tag{2.17}$$

gives the number of degenerate bound states with quantum numbers $n_{\alpha l m}$. This degeneracy is due to the O(N) group structure of the bound-state multiplets. There is a further degeneracy due to the various orbital angular momentum states of the bound excitations. This arises because the bound-state energies (2.15) do not depend on all of the quantum numbers $n_{\alpha l m}$, but only on the partial sums, $\sum_{m=-l}^{l} n_{\alpha l m}$. The complete bound-state multiplet structure can be determined from a knowledge of these degeneracies taking into account both the O(N) group and the rotation group representations which are "occupied."

It is convenient at this point to introduce the following two Schrödinger equations:

$$\{-\nabla^2\}\Psi_{\alpha l m}=k_{\alpha l}{}^2\Psi_{\alpha l m}\,, \tag{2.18}$$

$$\{-\nabla^2+\chi(r)-\bar{\chi}\}\,\Psi'_{\alpha l m}=k'_{\alpha l}{}^2\Psi'_{\alpha l m}\,. \tag{2.19}$$

Comparison with (2.7), using (2.14), shows that

$$\omega_{\alpha l}[\chi]=(k'_{\alpha l}{}^2+m^2)^{1/2} \tag{2.20}$$

and

$$\omega_{\alpha l}[\bar{\chi}]=(k_{\alpha l}{}^2+m^2)^{1/2}\,. \tag{2.21}$$

The bound excitations of (2.7) satisfying (2.12) correspond to solutions of (2.19) with negative $k'_{\alpha l}{}^2$, which we will denote by

$$k'_{\alpha l}=i\kappa_{\alpha l}\,,\quad \kappa_{\alpha l}>0 \tag{2.22}$$

so that for bound excitations

$$\omega_{\alpha l}[\chi]=(m^2-\kappa_{\alpha l}{}^2)^{1/2}\,. \tag{2.23}$$

Finally, we introduce the notation

$$I_1=\int d^3r[\chi(r)-\bar{\chi}] \tag{2.24}$$

and

$$I_2=\int d^3r[\chi(r)-\bar{\chi}]^2\,. \tag{2.25}$$

The sum over zero-point energies in Eq. (2.15) can be split into a sum over bound excitations plus a contribution from the continuum sum, which we denote by a superscript c. Then, using the notation of (2.20), (2.21), and (2.23), we have[1]

$$\tfrac{1}{2}\sum_{\alpha,l}(2l+1)(\omega_{\alpha l}[\chi]-\omega_{\alpha l}[\bar{\chi}])=\tfrac{1}{2}\sum_{\alpha,l}{}'(2l+1)[(m^2-\kappa_{\alpha l}{}^2)^{1/2}-m]+\tfrac{1}{2}\sum_{\alpha,l}{}^{c}(2l+1)[(k'_{\alpha l}{}^2+m^2)^{1/2}-(k_{\alpha l}{}^2+m^2)^{1/2}]\,. \tag{2.26}$$

Combining this result with (2.24) and (2.25) we can write the bound-state energies (2.15) as

$$E[\{n\}]=N\Big\{-\frac{3}{2\lambda_0}I_2+\frac{3}{\lambda_0}(\mu_0{}^2-\bar{\chi})I_1+\tfrac{1}{2}\sum_{\alpha,l}{}'(2l+1)\,[(m^2-\kappa_{\alpha l}{}^2)^{1/2}-m]$$
$$+\tfrac{1}{2}\sum_{\alpha,l}{}^{c}(2l+1)[(k'_{\alpha l}{}^2+m^2)^{1/2}-(k_{\alpha l}{}^2+m^2)^{1/2}]+\frac{1}{N}\sum_{\alpha,l,m}{}'n_{\alpha l m}(m^2-\kappa_{\alpha l}{}^2)^{1/2}\Big\}\,. \tag{2.27}$$

Of course, the free Schrödinger Eq. (2.18) has no bound excitations, so the sum over zero-point energies for the total energy of the vacuum, (2.13), only has a continuum contribution. Then, from (2.21) and (2.13), we can write the vacuum energy as

$$E_0=N\Big[\int d^3r\Big(-\frac{3}{2\lambda_0}\bar{\chi}^2+\frac{3}{\lambda_0}\mu_0{}^2\bar{\chi}\Big)+\tfrac{1}{2}\sum_{\alpha,l}{}^{c}(2l+1)(k_{\alpha l}{}^2+m^2)^{1/2}\Big]. \tag{2.28}$$

Equation (2.28) will be useful for purposes of renormalization. Our program is to investigate (2.27) using properties of the Schrödinger equation (2.19) in an attempt to find the minima described by (2.16), and thus determine the semiclassical bound-state spectrum.

III. RENORMALIZATION

The density of states for the free Schrödinger equation (2.18) is well known so we may write immediately

$$\tfrac{1}{2}\sum_{\alpha,l}{}^{c}(2l+1)(k_{\alpha l}{}^2+m^2)^{1/2}=\tfrac{1}{2}\int d^3r\int\frac{d^3k}{(2\pi)^3}(k^2+m^2)^{1/2}\,. \tag{3.1}$$

This allows us to express the vacuum energy (2.28) in terms of an effective potential V, defined by

$$E_0=V\int d^3r\,, \tag{3.2}$$

with

$$V=-\frac{3}{2\lambda_0}\bar{\chi}^2+\frac{3}{\lambda_0}\mu_0{}^2\bar{\chi}+\tfrac{1}{2}\int\frac{d^3k}{(2\pi)^3}(k^2+m^2)^{1/2}\,. \tag{3.3}$$

$\bar{\chi}$ is determined by the condition

$$\frac{\partial V}{\partial\bar{\chi}}=0\,. \tag{3.4}$$

Recalling that $\bar{\chi}=m^2$ we get

$$\frac{3}{\lambda_0}(\mu_0{}^2-\bar{\chi})=-\frac{1}{8\pi^2}\int_0^\Lambda\frac{k^2dk}{(k^2+m^2)^{1/2}}. \tag{3.5}$$

All divergent momentum integrals will be regulated, as above, with a cutoff Λ. We renormalize by defining

$$\frac{\mu_0{}^2}{\lambda_0}=\frac{\mu^2}{g}-\frac{\Lambda^2}{48\pi^2}, \tag{3.6}$$

$$\frac{1}{\lambda_0}=\frac{1}{g}+\frac{1}{48\pi^2}\ln\left(\frac{M}{2\Lambda}\right)+\frac{1}{96\pi^2}, \tag{3.7}$$

where M is an arbitrary renormalization mass. Then (3.5) becomes the renormalized gap equation,

$$\bar{\chi}=\mu^2+\frac{g\bar{\chi}}{96\pi^2}\ln\left(\frac{\bar{\chi}}{M^2}\right). \tag{3.8}$$

IV. THE VACUUM

We now review some result from a previous analysis of the vacuum state of the four-dimensional O(N) model.[6] First, as stated in the Introduction, there can be no spontaneous symmetry breaking, and all the fields ϕ_a have zero vacuum expectation values. The vacuum value of the χ field is then determined by the gap equation, (3.8). This equation has been studied by defining a renormalization-invariant parameter with dimensions of mass squared,

$$\chi_0=M^2\exp\left(\frac{96\pi^2}{g}\right), \tag{4.1}$$

and a dimensionless parameter ρ by

$$\bar{\chi}=\rho\chi_0, \tag{4.2}$$

so that the gap equation (3.8) becomes[6]

$$\rho\ln\rho=-\frac{96\pi^2}{\chi_0}\frac{\mu^2}{g}. \tag{4.3}$$

A stable vacuum exists for

$$\left(\frac{96\pi^2}{\chi_0}\right)\frac{\mu^2}{g}<e^{-1}, \tag{4.4}$$

and requires that

$$\rho_{\text{II}}>e^{-1}. \tag{4.5}$$

There is also an unstable vacuum given by the other solution to (4.3) satisfying

$$\rho_{\text{I}}<e^{-1}. \tag{4.6}$$

These determine

$$\bar{\chi}=\bar{\chi}_{\text{II}}=\rho_{\text{II}}\chi_0>\bar{\chi}_{\text{I}}=\rho_{\text{I}}\chi_0. \tag{4.7}$$

Finally, the χ propagator exhibits a two-body bound state *and* a resonance when[6]

$$-1<\ln\rho_{\text{II}}<1, \tag{4.8}$$

with the binding energy increasing as $\ln\rho_{\text{II}}$ decreases to -1. (If $\ln\rho_{\text{II}}<-1$ a tachyon appears.) This led to the hope that, as in the two-dimensional case,[2,7] a single bound state in the χ propagator would signal the presence of a rich bound-state spectrum. However, the existence of an accompanying resonance and the two-valued vacuum structure may be relevant features (as discussed in Sec. VIII) for the failure of this expected spectrum to appear in the semiclassical approximation.

V. TRACE IDENTITY AND MODE SUM

The Schrödinger equation (2.19) defines a scattering problem which we now discuss. The outgoing Green's function for the three-dimensional Schrödinger equation (2.18) is

$$G(\vec{r}_1-\vec{r}_2,k)=\int\frac{d^3p}{(2\pi)^3}\left[\frac{e^{i\vec{p}\cdot(\vec{r}_1-\vec{r}_2)}}{k^2-p^2+i\epsilon}\right]$$
$$=-\left(\frac{1}{4\pi}\right)\left(\frac{e^{ik|\vec{r}_1-\vec{r}_2|}}{|\vec{r}_1-\vec{r}_2|}\right). \tag{5.1}$$

We define the kernel

$$K_{ij}(k)=G(\vec{r}_i-\vec{r}_j,k)[\chi(r_j)-\bar{\chi}]. \tag{5.2}$$

Then the *modified* Fredholm determinant for the scattering problem described by Eq. (2.19) is[8]

$$D(k)=\det(1-K)e^{\text{Tr}K}$$
$$=1+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\int d^3r_1\cdots d^3r_n\begin{vmatrix}0&K_{12}&K_{13}&\cdot&\cdot&\cdot&K_{1n}\\K_{21}&0&&&&&\\K_{31}&&0&&&&\\\cdot&&&\cdot&&&\\\cdot&&&&\cdot&&\\\cdot&&&&&\cdot&\\K_{n1}&&&&&&0\end{vmatrix}. \tag{5.3}$$

One must consider a modified Fredholm determinant to avoid ill-defined expressions. This corresponds to separating the first Born approximation from the S matrix.

The scattering operator, $S(k)$, for fixed k, is an operator on the unit sphere in momentum space with matrix elements expressed in terms of the partial-wave phase shifts $\delta_l(k)$ by

$$\langle \hat{k}' | S(k) | \hat{k} \rangle = \frac{1}{4\pi} \sum_{l=0}^{\infty} (2l+1) P_l(\hat{k}'\cdot\hat{k})\, e^{2i\delta_l(k)} . \tag{5.4}$$

The S operator, taken between angular momentum eigenstates, gives

$$\langle l'm' | S(k) | lm \rangle = \delta_{l',l}\delta_{m',m} e^{2i\delta_l(k)} . \tag{5.5}$$

As a result of (5.5) we find

$$\ln \det S(k) = \mathrm{Tr}\ln S(k) = 2i \sum_{l=0}^{\infty} (2l+1)\delta_l(k) . \tag{5.6}$$

The modified Fredholm determinant and the S operator are related by[9]

$$\frac{D^*(k)}{D(k)} = [\det S(k)] e^{i(k/2\pi)I_1} , \tag{5.7}$$

with I_1 defined as in (2.24). Thus,

$$\arg D(k) = \tfrac{1}{2} i \ln \det S(k) - \frac{k}{4\pi} I_1 . \tag{5.8}$$

Equations (5.7) and (5.8) take into account the well-known result that the Born approximation gives the high-energy limit of the scattering amplitude for potential scattering. That is,

$$\ln \det S(k) \underset{k\to\infty}{\sim} \frac{k}{2\pi i} I_1 \tag{5.9}$$

since

$$\lim_{|k|\to\infty} D(k) = 1 \tag{5.10}$$

and

$$\lim_{|k|\to\infty} \arg D(k) = 0 . \tag{5.11}$$

Thus, we find the trace identity

$$I_1 = \lim_{k\to\infty} \left[\frac{2\pi i}{k} \ln \det S(k) \right] = \lim_{k\to\infty} \left[\frac{-4\pi}{k} \sum_{l=0}^{\infty} (2l+1)\delta_l(k) \right] . \tag{5.12}$$

It will also be recognized that I_1 is given directly in terms of the Born scattering amplitude at zero momentum transfer. However, note that the coefficient of I_1 vanishes identically in the *renormalized* expression for the bound-state energies as shown in Eqs. (6.1) and (6.3) below.

In Appendix A, we calculate the leading terms as $k\to\infty$ of the modified Fredholm determinant (5.3) under the assumption that $\chi(r) - \bar{\chi}$ can be written as a superposition of Yukawa potentials

$$\chi(r) - \bar{\chi} = \int_{\mu_{\min}}^{\infty} d\mu\, C(\mu) \frac{e^{-\mu r}}{r} , \tag{5.13}$$

with $\mu_{\min} > 0$, for an arbitrary weight function $C(\mu)$. This is what one would expect of a potential arising from particle exchanges. The Fredholm series (5.3) allows us to present an expansion for large k, whose leading term is

$$[\arg D(k)] \underset{k\to\infty}{\sim} -\frac{1}{16\pi k} I_2 + O\left(\frac{1}{k^3}\right) , \tag{5.14}$$

with I_2 defined by (2.25). In Appendix B we derive the following dispersion relation for $\arg D(k)$:

$$\arg D(k) = -\left\{ 2 \sum_{\alpha,l}{}' \left[(2l+1) \arctan\left(\frac{\kappa_{\alpha l}}{k}\right) \right] + \frac{k}{\pi} \mathrm{P} \int_0^{\infty} dp \left(\frac{\ln |D(p)|^2}{p^2 - k^2} \right) \right\} . \tag{5.15}$$

The primed sum is again over only bound excitations of (2.19), and P indicates the principal value of the integral. Thus,

$$[\arg D(k)] \underset{k\to\infty}{\sim} \frac{1}{k}\left[-2 \sum_{\alpha,l}{}' (2l+1)\kappa_{\alpha l} + \frac{1}{\pi} \int_0^{\infty} dp \ln |D(p)|^2 \right] + O\left(\frac{1}{k^3}\right) . \tag{5.16}$$

Comparing this with (5.14) we obtain the trace identity

$$I_2 = 32\pi \left[\sum_{\alpha,l}{}' (2l+1)\kappa_{\alpha l} - \frac{1}{2\pi} \int_0^{\infty} dp \ln |D(p)|^2 \right] . \tag{5.17}$$

In addition, from (5.6), (5.8), and (5.15) we find

$$\sum_{l=0}^{\infty} (2l+1)\delta_l(k) = \frac{-k}{4\pi} I_1 + 2 \sum_{\alpha,l}{}' (2l+1) \arctan\left(\frac{\kappa_{\alpha l}}{k}\right) + \frac{k}{\pi} \mathrm{P} \int_0^{\infty} dp \frac{\ln |D(p)|^2}{p^2 - k^2} . \tag{5.18}$$

Note that the leading term as $k\to\infty$ gives I_1 in terms of the high-energy limit of $\ln \det S(k)$, which is simply related to the Born approximation for the forward scattering amplitude as discussed above. [See Eqs. (5.8)–(5.12).] Having obtained a trace identity for the first term in (2.27), we now consider the continuum part of the mode sum for the zero-point energy. Since the potential $\chi(r)-\bar{\chi}$ is, by assumption, spherically symmetric, the solutions of the Schrödinger equation (2.19) can be written

$$\Psi'_{\alpha l m}(\vec{r})=\frac{\psi'_l(k',r)}{r}Y_{lm}(\hat{r}). \qquad (5.19)$$

For convenience, we have dropped the subscripts on k'. The scattering states satisfy[10]

$$\psi'_l(k',r)\underset{r\to\infty}{\sim}\frac{\tau_l(k')}{k'}\sin\left(k'r+\delta_l-\frac{\pi}{2}l\right). \qquad (5.20)$$

We now confine our system inside a large sphere of radius R by requiring (5.20) to vanish at the spherical boundary. Similarly, applying these boundary conditions to the solutions of (2.18) we must have, for sufficiently large R,

$$k'R+\delta_l-\frac{\pi}{2}l=n\pi=kR-\frac{\pi}{2}l. \qquad (5.21)$$

Thus, we find

$$\frac{\Delta n}{\Delta k}=\frac{R}{\pi} \qquad (5.22)$$

and

$$k'=k-\left(\frac{\delta_l}{R}\right). \qquad (5.23)$$

For a fixed k, we must require

$$l<l_{\max}, \qquad (5.24)$$

where

$$l_{\max}\propto kR \qquad (5.25)$$

in order to make the wave function vanish at $r=R$. [Since (5.20) and (5.21) break down for l near $l_{\max}$ we *cannot* simply use (5.21) to write $l_{\max}=(2/\pi)kR$. However, the exact value of $l_{\max}$ will not be needed, and errors from using (5.21) will vanish as $R\to\infty$.] Then, as $R\to\infty$, we can write our sum over continuum modes as an integral. Using (5.22),

$$\sum_{\alpha,l}^{c}(2l+1)\underset{R\to\infty}{\longrightarrow}\int_0^\infty\frac{dk}{\pi}R\sum_{l=0}^{l_{\max}}(2l+1). \qquad (5.26)$$

Note that (5.25) and (5.26) give the total number of modes proportional to R^3 and $\int_0^\infty k^2dk$ as they must. Now using (5.23),

$$(k'^2+m^2)^{1/2}-(k^2+m^2)^{1/2}=\left[\left(k-\frac{\delta_l}{R}\right)^2+m^2\right]^{1/2}-(k_2+m^2)^{1/2}$$
$$\underset{R\to\infty}{\sim}-\left(\frac{\delta_l}{R}\right)\frac{d}{dk}(k^2+m^2)^{1/2}=-\left(\frac{\delta_l}{R}\right)\frac{k}{(k^2+m^2)^{1/2}}. \qquad (5.27)$$

Then, combining (5.26) and (5.27),

$$\tfrac{1}{2}\sum_{\alpha,l}^{c}(2l+1)[(k'^2_{\alpha l}+m^2)^{1/2}-(k_{\alpha l}^2+m^2)^{1/2}]=-\tfrac{1}{2}\int_0^\infty\frac{kdk}{\pi}\left[\frac{1}{(k^2+m^2)^{1/2}}\sum_{l=0}^{l_{\max}}(2l+1)\delta_l(k)\right]. \qquad (5.28)$$

However,

$$\delta_l(k)\sim 0 \qquad (5.29)$$

for

$$l\gg kR_{\text{eff}}, \qquad (5.30)$$

where R_{eff} measures the effective size of the potential $\chi(r)-\bar{\chi}$. Then, as $R\to\infty$,

$$kR\gg kR_{\text{eff}} \qquad (5.31)$$

provided that[10]

$$\int_0^\infty r\,dr\,|\chi(r)-\bar{\chi}|<\infty. \qquad (5.32)$$

This is satisifed by any potential of the form (5.13) provided that $\mu_{\min}\neq 0$. As a result of (5.25), (5.29), (5.30), and (5.31) we may extend the sum over l in (5.28) to infinity, and then use (5.18) to write

$$\tfrac{1}{2}\sum_{\alpha,l}^{c}(2l+1)[(k'^2_{\alpha l}+m^2)^{1/2}-(k_{\alpha l}^2+m^2)^{1/2}]=I_1\int_0^\Lambda\frac{k^2dk}{8\pi^2}\left[\frac{1}{(k^2+m^2)^{1/2}}\right]$$
$$-\int_0^\Lambda\frac{kdk}{\pi}\left[\frac{1}{(k^2+m^2)^{1/2}}\sum_{\alpha,l}{}'(2l+1)\arctan\left(\frac{\kappa_{\alpha l}}{k}\right)\right]$$
$$-\int_0^\Lambda\frac{k^2dk}{2\pi^2}\left\{\frac{1}{(k^2+m^2)^{1/2}}\,\mathrm{P}\int_0^\infty dp\left[\frac{\ln|D(p)|^2}{p^2-k^2}\right]\right\}. \qquad (5.33)$$

Again we have cut off all divergent integrals. A straightforward integration gives

$$\int_0^\Lambda \frac{kdk}{\pi}\left[\frac{1}{(k^2+m^2)^{1/2}}\sum_{\alpha,l}{}' (2l+1)\arctan\left(\frac{\kappa_{\alpha l}}{k}\right)\right]$$
$$=\sum_{\alpha,l}{}' (2l+1)\left\{\frac{\kappa_{\alpha l}}{\pi}-\frac{m}{2}+\frac{(m^2-\kappa_{\alpha l}{}^2)^{1/2}}{\pi}\arctan\left[\frac{(m^2-\kappa_{\alpha l}{}^2)^{1/2}}{\kappa_{\alpha l}}\right]+\frac{\kappa_{\alpha l}}{\pi}\ln\left(\frac{2\Lambda}{m}\right)\right\}, \tag{5.34}$$

and in Appendix C we show that

$$\mathrm{P}\int_0^\Lambda \frac{k^2dk}{2\pi^2}\left[\frac{1}{(k^2+m^2)^{1/2}}\frac{1}{p^2-k^2}\right]=-\frac{1}{2\pi^2}\ln\left(\frac{2\Lambda}{m}\right)+\frac{1}{2\pi^2}\left(\frac{p^2}{p^2+m^2}\right)^{1/2}\ln\left[\frac{\sqrt{p^2}+(p^2+m^2)^{1/2}}{m}\right]. \tag{5.35}$$

Then, from (5.33), (5.34), and (5.35),

$$\tfrac{1}{2}\sum_{\alpha,l}{}^{c} (2l+1)[(k_{\alpha l}'^{\,2}+m^2)^{1/2}-(k_{\alpha l}{}^2+m^2)^{1/2}]$$
$$=I_1\int_0^\Lambda \frac{k^2dk}{8\pi^2}\left[\frac{1}{(k^2+m^2)^{1/2}}\right]$$
$$-\sum_{\alpha,l}{}' (2l+1)\left\{\frac{\kappa_{\alpha l}}{\pi}-\frac{m}{2}+\frac{(m^2-\kappa_{\alpha l}{}^2)^{1/2}}{\pi}\arctan\left[\frac{(m^2-\kappa_{\alpha l}{}^2)^{1/2}}{\kappa_{\alpha l}}\right]+\frac{\kappa_{\alpha l}}{\pi}\ln\left(\frac{2\Lambda}{m}\right)\right\}$$
$$-\int_0^\infty \frac{dp}{2\pi^2}\ln|D(p)|^2\left\{\left(\frac{p^2}{p^2+m^2}\right)^{1/2}\ln\left[\frac{\sqrt{p^2}+(p^2+m^2)^{1/2}}{m}\right]-\ln\left(\frac{2\Lambda}{m}\right)\right\}. \tag{5.36}$$

This completes our calculation of the mode sum.

VI. RENORMALIZED BOUND-STATE ENERGIES

Inserting the results (5.17) and (5.36) into the unrenormalized bound-state energy (2.27) and using the renormalization (3.7),

$$E[\{n\}]=N\Bigg(-\left(\frac{48\pi}{g}+\frac{1}{2\pi}\right)\left\{\sum_{\alpha,l}{}' [(2l+1)\kappa_{\alpha l}]-\frac{1}{2\pi}\int_0^\infty dp\ln|D(p)|^2\right\}$$
$$-\frac{1}{\pi}\ln\left(\frac{M}{2\Lambda}\right)\left\{\sum_{\alpha,l}{}' [(2l+1)\kappa_{\alpha l}]-\frac{1}{2\pi}\int_0^\infty dp\ln|D(p)|^2\right\}$$
$$+I_1\left\{\frac{3}{\lambda_0}(\mu_0^2-\overline{\chi})+\int_0^\Lambda \frac{k^2dk}{8\pi^2}\left[\frac{1}{(k^2+m^2)^{1/2}}\right]\right\}+\frac{1}{N}\sum_{\alpha,l,m}{}' n_{\alpha l m}(m^2-\kappa_{\alpha l}{}^2)^{1/2}$$
$$+\tfrac{1}{2}\sum_{\alpha,l}{}' (2l+1)(m^2-\kappa_{\alpha l}{}^2)^{1/2}-\sum_{\alpha,l}{}' (2l+1)\left\{\frac{\kappa_{\alpha l}}{\pi}+\frac{(m^2-\kappa_{\alpha l}{}^2)^{1/2}}{\pi}\arctan\left[\frac{(m^2-\kappa_{\alpha l}{}^2)^{1/2}}{\kappa_{\alpha l}}\right]+\frac{\kappa_{\alpha l}}{\pi}\ln\left(\frac{2\Lambda}{m}\right)\right\}$$
$$-\int_0^\infty \frac{dp}{2\pi^2}\ln|D(p)|^2\left\{\left(\frac{p^2}{p^2+m^2}\right)^{1/2}\ln\left[\frac{\sqrt{p^2}+(p^2+m^2)^{1/2}}{m}\right]-\ln\left(\frac{2\Lambda}{m}\right)\right\}\Bigg). \tag{6.1}$$

Two important features of this calculation now become apparent. First, by Eq. (3.5) we see that the term proportional to I_1 vanishes identically. Second, collecting all of the divergent logarithms we see that the result is renormalized and finite. Note that our trace identity (5.17) is the only result consistent with renormalizability, and that if we invoke renormalizability as a requirement this identity can be uniquely determined. This is an independent check of the results of Sec. V. Recalling (2.14), (4.1), and (4.2) we see that

$$-\frac{48\pi}{g}+\frac{1}{\pi}\ln\left(\frac{m}{M}\right)=\frac{1}{2\pi}\ln\rho_{\mathrm{II}}. \tag{6.2}$$

Then the renormalized bound-state energy is

$$E[\{n\}]=N\left(\sum_{\alpha,l}{}' (2l+1)\left\{\left[\frac{1}{2\pi}(\ln\rho_{\rm II}-1)-\frac{1}{\pi}\right]\kappa_{\alpha l}+\tfrac{1}{2}(m^2-\kappa_{\alpha l}{}^2)^{1/2}-\frac{(m^2-\kappa_{\alpha l}{}^2)^{1/2}}{\pi}\arctan\left[\frac{(m^2-\kappa_{\alpha l}{}^2)^{1/2}}{\kappa_{\alpha l}}\right]\right\}\right.$$
$$\left.+\frac{1}{N}\sum_{\alpha,l,m}{}' n_{\alpha l m}(m^2-\kappa_{\alpha l}{}^2)^{1/2}-\int_0^\infty\frac{dp}{2\pi^2}\ln|D(p)|^2\left\{\tfrac{1}{2}(\ln\rho_{\rm II}-1)+\left(\frac{p^2}{p^2+m^2}\right)^{1/2}\ln\left[\frac{\sqrt{p^2}+(p^2+m^2)^{1/2}}{m}\right]\right\}\right). \tag{6.3}$$

According to (2.16) we must choose $|D(k)|$ and the various $\kappa_{\alpha l}$ in (6.3) so as to minimize the total bound-state energy. The function $\chi(r)$ can then in principle be reconstructed by three-dimensional inverse-scattering methods.[9]

VII. BOUND-STATE CALCULATION

If the various $\kappa_{\alpha l}$ entering additively into the bound-state energy (6.3) are *independent*, then we can consider a single $\kappa=\kappa_{00}$ with occupation number $n=n_{000}$ without loss of generality. Note that with this assumption, all bound states are S states. A boundary condition on the variation (2.16) is that for $n=0$ we recover the vacuum state $\chi=\bar{\chi}$ which satisfies

$$|D(k)|=1. \tag{7.1}$$

However, the terms of (6.3) involving $|D(k)|$ are all independent of n. Thus, if $|D(k)|$ is independent of the $\kappa_{\alpha l}$, (7.1) must apply for *all* n. Under these assumptions, the bound-state energy is just

$$E[n]=N\left\{\left[\frac{1}{2\pi}(\ln\rho_{\rm II}-1)-\frac{1}{\pi}\right]\kappa+\left(\frac{n}{N}+\frac{1}{2}\right)(m^2-\kappa^2)^{1/2}-\frac{(m^2-\kappa^2)^{1/2}}{\pi}\arctan\left[\frac{(m^2-\kappa^2)^{1/2}}{\kappa}\right]\right\}. \tag{7.2}$$

A convenient substitution is

$$\kappa=m\sin\theta. \tag{7.3}$$

Since $\kappa\geq 0$ and $\omega\geq 0$ we require

$$0\leq\theta\leq\pi/2. \tag{7.4}$$

Then, in terms of θ,

$$E[n]=Nm\left[\frac{1}{2\pi}(\ln\rho_{\rm II}-1)\sin\theta+\frac{n}{N}\cos\theta+\frac{1}{\pi}(\theta\cos\theta-\sin\theta)\right]. \tag{7.5}$$

To minimize this we set

$$\frac{dE[n]}{d\theta}=0, \tag{7.6}$$

obtaining

$$\tfrac{1}{2}(\ln\rho_{\rm II}-1)\cos\theta=\left(\frac{n}{N}\pi+\theta\right)\sin\theta. \tag{7.7}$$

This would appear to have solutions in the allowed range of (7.4) for

$$\ln\rho_{\rm II}>1. \tag{7.8}$$

This is paradoxical, since the binding energy of the two-body bound state, as determined from the Green's functions,[6] *increases* as $\ln\rho_{\rm II}$ *decreases* from 1 to -1. Further, taking a second derivative we find

$$\frac{d^2E[n]}{d\theta^2}=-\left[\frac{1}{2\pi}(\ln\rho_{\rm II}-1)\sin\theta+\frac{n}{N}\cos\theta+\frac{1}{\pi}(\sin\theta+\theta\cos\theta)\right]<0, \tag{7.9}$$

so these solutions are unstable *maxima* not minima of the bound-state energy. Furthermore, if we set $n=0$ in Eq. (7.7) we *do not* recover the vacuum state $\theta=\kappa=0$.

Equation (7.7) seems to suggest that a tower of unstable bound states exists in the model. However, the fact that these solutions do not go over into the stable vacuum $\chi=\bar{\chi}=\bar{\chi}_{\rm II}$ for $n=0$ suggests that the bound states are being built on the *unstable* vacuum, $\bar{\chi}=\bar{\chi}_{\rm I}$, discussed in Sec. IV. As a result, for nonzero n we conjecture that the assumption of a single bound excitation $\kappa=\kappa_{00}$ together with Eq. (7.1) imply that

$$\chi(r)\xrightarrow[r\to\infty]{}\bar{\chi}_{\rm I} \tag{7.10}$$

for these solutions, which is the wrong limit. This clearly would give our apparent bound states an infinite energy proportional to the volume of space, and removes them completely from the spectrum measured relative to the true, stable vacuum, $\bar{\chi}_{\rm II}$. Why then does Eq. (7.5) predict a finite energy for these states? We can find the answer by going back to our basic Schrödinger equation (2.19). In performing the mode sums for this equation we have tacitly assumed that the continuum excitations begin at $k'^2=0$. [See Eq. (5.26) for example.] This is true only if

$$\chi(r)\xrightarrow[r\to\infty]{}\bar{\chi}=\bar{\chi}_{\rm II}. \tag{7.11}$$

If instead (7.10) holds, the continuum actually begins at $k'^2<0$ since from (4.7)

$$\bar{\chi}_{\rm I}<\bar{\chi}_{\rm II}. \tag{7.12}$$

This means that for such states we have left out a piece of the continuum mode sum, which will contribute an infinite term to the bound-state energies which is proportional to the volume of space. Our conclusion then is that the solutions to Eq. (7.7) are not in the spectrum at all either as stable *or* unstable bound states.

VIII. DISCUSSION

At the end of Sec. VI, we noted that once the $|D(k)|$ and $\kappa_{\alpha l}$ have been determined by minimizing the bound-state energy, the function $\chi(r)$ can in principle be reconstructed by three-dimensional inverse-scattering methods.[9] However, there is an existence problem associated with this procedure.[9] All sets of scattering data do not necessarily lead to local potentials $\chi(r)-\bar{\chi}$. Thus, to be self-consistent we cannot really vary $|D(k)|$ and $\kappa_{\alpha l}$ freely, but rather we must restrict our variations to sets of scattering data which give local potentials upon inversion. The existence problem for the three-dimensional inverse-scattering method is not well understood, and it is not clear how the locality condition should restrict our variations.

Another problem is whether, given the locality condition, we can still vary the $|D(k)|$ and the various $\kappa_{\alpha l}$ *independently*. Note that the assumption of the independence of these parameters was used extensively in Sec. VII. If the various bound excitations $\kappa_{\alpha l}$ are not independent, the bound-state spectrum would have to be reexamined with a knowledge of their interdependence. However, it is not clear what the relationships of the different $\kappa_{\alpha l}$ would be. Another possibility, suggested by our previous study of the Green's functions of this model, as discussed in Sec. IV, is that for an S-wave bound excitation κ there is an *associated* resonance (and possibly associated bound excitations in other angular momentum states) making $|D(k)|$ different from unity and dependent on κ.

Although the resolution of the above problems rests on a more complete understanding of the uniqueness and existence problem for the three-dimensional inverse-scattering method, another issue should be considered. It is well known that time-independent soliton-like solutions to the *classical* equations of motion for a scalar field theory cannot be stable in four dimensions. In our problem this result does not directly apply since we have included time-dependent ϕ_a fields and quantum corrections in our calculations. However, it may be that our difficulties arise from a similar sort of instability. If so, the ultimate success of semiclassical methods in four dimensions must await their application to field theories with spin.

APPENDIX A

The series expansion of the modified Fredholm determinant allows us to obtain an expression valid at large k. From (5.1)–(5.3) one can show that

$$D(k) \underset{k\to\infty}{\sim} 1-\tfrac{1}{2}\int d^3r_1 d^3r_2 K_{12}K_{21}+\cdots\,, \qquad \text{(A1)}$$

with the omitted terms vanishing more rapidly with k. Then, from (5.1) and (5.2), using the form (5.13), and performing the necessary Fourier transforms,

$$D(k) \underset{k\to\infty}{\sim} 1-\frac{1}{8\pi^4}\int d\mu_1 d\mu_2 [C(\mu_1)C(\mu_2)I(\mu_1,\mu_2)]\,, \qquad \text{(A2a)}$$

where

$$I(\mu_1,\mu_2)=\int d^3p_1 d^3p_2\left(\frac{1}{k^2-p_1^2+i\epsilon}\right)\left(\frac{1}{k^2-p_2^2+i\epsilon}\right)\times\left[\frac{1}{(\vec{p}_1-\vec{p}_2)^2+\mu_1^2}\right]\left[\frac{1}{(\vec{p}_1-\vec{p}_2)^2+\mu_2^2}\right]. \qquad \text{(A2b)}$$

We can perform the angular integrations by combining the last two denominators. Then, defining

$$A^2=\mu_1^2\alpha+\mu_2^2(1-\alpha)\,, \qquad \text{(A3)}$$

we have

$$I(\mu_1,\mu_2)=\int_0^1 d\alpha\, 16\pi^2\int_0^\infty p_1^2dp_1\int_0^\infty p_2^2dp_2\left(\frac{1}{k^2-p_1^2+i\epsilon}\right)\left(\frac{1}{k^2-p_2^2+i\epsilon}\right)\left[\frac{1}{(p_1-p_2)^2+A^2}\right]\left[\frac{1}{(p_1+p_2)^2+A^2}\right]. \qquad \text{(A4)}$$

The remaining momentum integrals can be evaluated by contour integration. The result is

$$I(\mu_1,\mu_2)=2\pi^4\int_0^1 d\alpha\left[\frac{1+2i(k/A)}{4k^2+A^2}\right]. \qquad \text{(A5)}$$

Thus, recalling (A3), we get

$$I(\mu_1,\mu_2)\underset{k\to\infty}{\sim}\frac{i\pi^4}{k}\int_0^1 d\alpha\left(\frac{1}{A}\right)=\frac{i2\pi^4}{k}\left(\frac{1}{\mu_1+\mu_2}\right). \qquad \text{(A6)}$$

Then, from (A2),

$$D(k) \underset{k\to\infty}{\simeq} 1 - \frac{i}{4k}\int d\mu_1 d\mu_2 \left[\frac{C(\mu_1)C(\mu_2)}{\mu_1+\mu_2}\right] + O\left(\frac{1}{k^2}\right). \tag{A7}$$

However, a potential of the form (5.13) used in conjunction with (2.25) gives

$$I_2 = 4\pi \int d\mu_1 d\mu_2 \left[\frac{C(\mu_1)C(\mu_2)}{\mu_1+\mu_2}\right]. \tag{A8}$$

Thus

$$D(k) \underset{k\to\infty}{\simeq} 1 - \frac{i}{16\pi k} I_2 + O\left(\frac{1}{k^2}\right) \tag{A9}$$

and we recover Eq. (5.14). Parenthetically we remark that because of the reality condition $D(k) = D^*(-k)$, $|D(k)|$ and arg $D(k)$ are even and odd functions of k, respectively.

APPENDIX B

The modified Fredholm determinant $D(k)$ is analytic in the upper half of the complex k plane and satisfies[8,9]

$$D(k) \underset{|k|\to\infty}{\sim} 1. \tag{B1}$$

It has simple zeros at the bound-state values $k = i\kappa_{\alpha l}$, so we can write

$$D(k) = \prod_{\alpha,l,m}{}' \left(\frac{k - i\kappa_{\alpha l}}{k + i\kappa_{\alpha l}}\right) D'(k) \tag{B2a}$$

$$= \prod_{\alpha,l}{}' \left(\frac{k - i\kappa_{\alpha l}}{k + i\kappa_{\alpha l}}\right)^{2l+1} D'(k) \tag{B2b}$$

or

$$\ln D(k) = -2i \sum_{\alpha,l}{}' (2l+1) \arctan\left(\frac{\kappa_{\alpha l}}{k}\right) + \ln D'(k). \tag{B3}$$

Now because $D'(k)$ is analytic in the upper half-plane, satisfies (B1), and is free from zeros, we can write

$$\ln D'(k) = \frac{k}{\pi i}\int_{-\infty}^{\infty} dp \left[\frac{\ln D'(p)}{p^2 - k^2 - i\epsilon}\right]. \tag{B4}$$

But we further have the reality condition

$$D'(-k) = D'^*(k), \tag{B5}$$

and along the real k axis (B2) implies

$$|D(k)|^2 = |D'(k)|^2. \tag{B6}$$

Then combining (B3)–(B6), we find

$$\ln D(k) = -2i \sum_{\alpha,l}{}' (2l+1) \arctan\left(\frac{\kappa_{\alpha l}}{k}\right) + \frac{k}{\pi i}\int_0^{\infty} dp \left[\frac{\ln|D(p)|^2}{p^2 - k^2 - i\epsilon}\right]. \tag{B7}$$

Finally, noting that

$$\arg D(k) = \frac{1}{2i}\left[\ln D(k) - \ln D^*(k)\right], \tag{B8}$$

we obtain (5.15) of the text.

APPENDIX C

We wish to compute the principal-value integral

$$P\int_0^{\Lambda} \frac{k^2 dk}{2\pi^2}\left[\frac{1}{(k^2+m^2)^{1/2}}\right]\left(\frac{1}{p^2-k^2}\right) = -\frac{1}{2\pi^2}\ln\left(\frac{2\Lambda}{m}\right) + \frac{p^2}{2\pi^2} I \tag{C1}$$

with

$$I = P\int_0^{\infty} \frac{dk}{(k^2+m^2)^{1/2}}\left(\frac{1}{p^2-k^2}\right). \tag{C2}$$

Now define

$$\sqrt{s} = (k^2+m^2)^{1/2}, \tag{C3}$$

$$\sqrt{t} = (p^2+m^2)^{1/2} \tag{C4}$$

and consider the integral

$$J(t) = \tfrac{1}{2}\int_{m^2}^{\infty} \frac{ds}{[s(s-m^2)]^{1/2}}\left(\frac{1}{s-t-i\epsilon}\right). \tag{C5}$$

Note that

$$\mathrm{Re}[J(t)] = -I \tag{C6}$$

and

$$\mathrm{Im}[J(t)] = \frac{\pi}{2}\left\{\frac{1}{[t(t-m^2)]^{1/2}}\right\}\theta(t-m^2). \tag{C7}$$

$J(t)$ then satisfies the dispersion relation

$$J(t) = \frac{1}{\pi}\int_{m^2}^{\infty} \frac{ds}{s-t-i\epsilon}\,\mathrm{Im}\,J(s). \tag{C8}$$

It is related to the two-body phase-space integral in two dimensions, and is given by

$$J(t) = -\frac{1}{[t(t-m^2)]^{1/2}}\ln\left[\frac{\sqrt{t}+(t-m^2)^{1/2}}{m}\right] + i\frac{\pi}{2}\left\{\frac{1}{[t(t-m^2)]^{1/2}}\right\} \text{ for } t>m^2. \tag{C9}$$

This gives upon changing back to the variables (C3) and (C4) and using (C6)

$$I = \frac{1}{[p^2(p^2+m^2)]^{1/2}}\ln\left[\frac{\sqrt{p^2}+(p^2+m^2)^{1/2}}{m}\right]. \tag{C10}$$

Then from (C1) and (C10) we arrive at (5.35).

*Research supported in part by ERDA under Contract No. E(11-1)3230.

[1]R. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. D 10, 4114 (1974); 10, 4130 (1974); 10, 4138 (1974); 11, 3424 (1975); 12, 2443 (1975). An equivalent method based on taking matrix elements of the field equations has been given in J. Goldstone and R. Jackiw, *ibid.* 11, 1486 (1975); A. Klein and F. Krejs, *ibid.* 12, 3112 (1975); 13, 3282 (1976); 13, 3295 (1976); S.-J. Chang, *ibid.* 12, 1071 (1975).

[2]L. F. Abbott, Phys. Rev. D 14, 552 (1976).

[3]C. R. Nohl, Princeton report (unpublished); D. Campbell, Los Alamos report (unpublished); S. Y. Lee and A. Gavrielides, Phys. Rev. D 12, 3880 (1975); S.-S. Shei, *ibid.* 14, 535 (1976).

[4]V. E. Zakharov and L. D. Faddeev, Func. Anal. Appl. 5, 280 (1972); L. D. Faddeev, Trudy Mat. in-ta im. V. A. Steklova 73, 314 (1964) (Am. Math. Soc. Trans. Ser. 2, 65, 139).

[5]Different methods have been applied to a class of scalar field theories in four space-time dimensions by R. Friedberg, T. D. Lee, and A. Sirlin, Phys. Rev. D 13, 2739 (1976). In particular, these authors emphasize the application of variational trial solutions for the study of solitons in four dimensions.

[6]L. F. Abbott, J. S. Kang, and H. J. Schnitzer, Phys. Rev. D 13, 2212 (1976). See also M. Kobayashi and T. Kugo, Prog. Theor. Phys. 54, 1537 (1975); R. Haymaker, Phys. Rev. D 13, 968 (1975). The Hartree approximation for the $O(N)$ model in the context of relativistic field theory was first developed by L. Dolan and R. Jackiw, *ibid.* 9, 3320 (1974); H. J. Schnitzer, *ibid.* 10, 1800 (1974); 10, 2042 (1974); S. Coleman, R. Jackiw, and H. D. Politzer, *ibid.* 10, 2491 (1974).

[7]H. J. Schnitzer, Nucl. Phys. B109, 297 (1976).

[8]See for example R. Newton, *Scattering Theory of Waves and Particles* (McGraw-Hill, New York, 1966).

[9]R. Newton, in *Scattering Theory in Mathematical Physics*, edited by J. A. Lavita and J.-P. Marchand (Reidel, Dordrecht, Holland, 1974).

[10]See for example, V. De Alfaro and T. Regge, *Potential Scattering* (North-Holland, Amsterdam, 1965).