Comments and Addenda

The section Comments and Addenda is for short communications which are not appropriate for regular articles. It includes only the following types of communications: (1) Comments on papers previously published in **The Physical Review** *or* **Physical Review Letters**. *(2) Addenda to papers previously published in* **The Physical Review** *or* **Physical Review Letters**, *in which the additional information can be presented without the need for writing a complete article. Manuscripts intended for this section must be accompanied by a brief abstract for information-retrieval purposes. Accepted manuscripts follow the same publication schedule as articles in this journal, and page proofs are sent to authors.*

Local supersymmetry transformations and fermion solutions in the presence of instantons

L. F. Abbott
Stanford Linear Accelerator Center, Stanford University, Stanford, California 94305

Marcus T. Grisaru*
CERN, Geneva, Switzerland
(Received 14 February 1978)

Local supersymmetry transformations are used to generate solutions of the Dirac equation in the presence of instantons. We show that *all* spin-1/2 zero-eigenvalue modes for an isovector fermion in an N-instanton field can be obtained by spacetime-dependent supersymmetry transformations, and that through additional supersymmetry operations these can be used to generate zero-eigenvalue solutions to the small-fluctuations problem for the Yang-Mills field. Similar problems for supergravity theories with a gravitational instanton are also discussed.

I. INTRODUCTION

Recently, considerable attention has been given to the problem of constructing zero-eigenvalue modes of the Dirac operator[1-4] in the presence of an instanton field.[5,6] In particular, Jackiw and Rebbi[1] have constructed the $4N$ zero-eigenvalue modes for an isovector fermion in an N-instanton field. It has been noted that four of these modes can be obtained by *global* supersymmetry transformations of the N-instanton solution itself[1,2] and this has been further discussed by Zumino.[7] As we shall show in Sec. II below, *all* $4N$ zero-eigenvalue solutions can be obtained by suitable *local* supersymmetry transformations. Although this technique does not lead to any simplification in obtaining the solutions, it does provide an interesting interpretation of them and suggests that in a supersymmetric model *all* zero-eigenvalue solutions can be obtained by local supersymmetry transformations.

Brown, Carlitz, and Lee[3] have linked the small-fluctuations problem for the Yang-Mills field to the Dirac problem discussed here. The $4N$ fermion zero-eigenvalue modes provide $8N$ zero-eigenvalue fluctuations of the Yang-Mills field and indicate that the complete N-instanton solution depends on $8N$-3 parameters.[3,8] In Sec. III, we derive this fermion-boson correspondence by supersymmetry arguments. Finally, in Sec. IV, we discuss aspects of the zero-eigenvalue problem for boson and fermion fields in supergravity theories.

Throughout this work, we start with solutions ϕ_i to the field equations of a supersymmetric theory and by *infinitesimal* supersymmetry transformations obtain solutions $\delta\phi_i$ to the linearized equations in the presence of the background fields ϕ_i.[9] In principle, supersymmetry requires that all spinors be Majorana and anticommuting. However, when we deal with infinitesimal transformations and linearized equations, these requirements may often be dropped.[7] In each case, one can explicitly verify that our solutions are valid when the spinors are complex c-number fields. In the following we shall use such spinors and work exclusively in Euclidean space.

II. THE DIRAC EQUATION IN AN N-INSTANTON FIELD

The theory of SU(2) gauge bosons coupled to isovector spin-$\frac{1}{2}$ (Majorana) fermions is globally supersymmetric.[10] Since we wish to obtain solutions to the Dirac equation by *local* supersymmetry transformations, we begin by coupling the theory to supergravity which gauges the original supersymmetry.[11] The system now contains the gravitational field $\hat{g}_{\mu\nu}$ (or vierbein field $\hat{e}^a_\mu$), a spin-$\frac{3}{2}$ field $\hat{\psi}_\mu$, the gauge field $\hat{A}^a_\mu$, and the isovector spin-$\frac{1}{2}$ field $\hat{\psi}^a$. We begin with the following solution to the classical field equations:

$$\begin{aligned} \hat{g}_{\mu\nu} &= \eta_{\mu\nu} , \\ \hat{\psi}_\mu &= 0 , \\ \hat{A}^a_\mu &= A^a_\mu , \\ \hat{\psi}^a &= 0 , \end{aligned} \tag{2.1}$$

where A^a_μ is an N-instanton solution.[5,6] Performing an infinitesimal *local* supersymmetry transformation on this solution gives[11]

$$\begin{aligned} \delta\hat{g}_{\mu\nu} &= 0 , \\ \delta\hat{\psi}_\mu &= \psi_\mu = 2\kappa^{-1}\partial_\mu\epsilon(x) , \\ \delta\hat{A}^a_\mu &= 0 , \\ \delta\hat{\psi}^a &= \psi^a = F^a_{\mu\nu}\Sigma_{\mu\nu}\epsilon(x) , \end{aligned} \tag{2.2}$$

where $F^a_{\mu\nu}$ is the N-instanton field tensor and $\Sigma_{\mu\nu} = \frac{1}{4}[\gamma_\mu, \gamma_\nu]$. Because of the local supersymmetry of the system, $\delta\hat{\psi}^a = \psi^a$ will satisfy the *linearized* field equation for $\hat{\psi}^a$ which, due to the supergravity coupling is now[11]

$$\gamma_\mu D^{ab}_\mu \psi^b = \tfrac{1}{2}\kappa F^a_{\alpha\beta}\gamma_\mu\Sigma_{\alpha\beta}\psi_\mu , \tag{2.3}$$

where

$$D^{ab}_\mu = \partial_\mu\delta^{ab} + A^c_\mu\epsilon^{acb} ; \tag{2.4}$$

the spin connection term is absent since $\hat{g}_{\mu\nu} = \eta_{\mu\nu}$. Therefore, $\psi^a = F^a_{\mu\nu}\Sigma_{\mu\nu}\epsilon(x)$ will be a solution to the Dirac equation in the presence of the N-instanton field $F^a_{\mu\nu}$ provided that we choose $\epsilon(x)$ so that the right-hand side of Eq. (2.3) vanishes,

$$F^a_{\alpha\beta}\gamma_\mu\Sigma_{\alpha\beta}\partial_\mu\epsilon(x) = 0 \tag{2.5}$$

where we have substituted $\psi_\mu = 2\kappa^{-1}\partial_\mu\epsilon(x)$ into Eq. (2.3). It can easily be verified directly that the ansatz

$$\psi^a = F^a_{\alpha\beta}\Sigma_{\alpha\beta}\epsilon(x) \tag{2.6}$$

satisfies the Dirac equation

$$\gamma_\mu D^{ab}_\mu\psi^b = 0 , \tag{2.7}$$

provided $\epsilon(x)$ satisfied (2.5). The introduction of supergravity fields was just a device to lead us to this result

Two obvious solutions to Eq. (2.5) are

$$\epsilon(x) = u \quad \text{and} \quad \epsilon(x) = \gamma\cdot xu , \tag{2.8}$$

where u is a constant spinor. When substituted into Eq. (2.6) they give the four solutions which have previously been generated by global supersymmetry transformations.[1,2]

Since the tensor $F^a_{\mu\nu}$ is self-dual for the N-instanton solution, $\Sigma_{\mu\nu}F^a_{\mu\nu}$ acts as a left-handed chiral projection operator. For this reason it is convenient to introduce a two-component notation. We define[1]

$$\psi^a = \begin{pmatrix} \psi^a_+ \\ \psi^a_- \end{pmatrix} , \quad \epsilon = \begin{pmatrix} \epsilon_+ \\ \epsilon_- \end{pmatrix} , \quad \Sigma_{\mu\nu} = \begin{pmatrix} \bar{\sigma}_{\mu\nu} & 0 \\ 0 & \sigma_{\mu\nu} \end{pmatrix} , \tag{2.9}$$

and we find that Eq. (2.6) gives left-handed solutions to the Dirac equation

$$\psi^a_- = F^a_{\mu\nu}\sigma_{\mu\nu}\epsilon_-(x) . \tag{2.10}$$

Furthermore, this relation can be inverted to give

$$\epsilon_-(x) = \frac{F^a_{\mu\nu}\sigma_{\mu\nu}\psi^a_-}{(F^a_{\mu\nu})^2} , \tag{2.11}$$

so that for every ψ^a_- which solves the Dirac equation an $\epsilon_-(x)$ can be found. [We have used the fact that for self-dual fields $(\sigma_{\mu\nu}F^a_{\mu\nu})^2 = (F^a_{\mu\nu})^2$.] We can write, for an N-instanton solution,

$$\epsilon_-(x) = 2\left[\frac{4f_{\mu\nu}\bar{\alpha}_\nu g^{(1,2)}_{\mu i} - f_{\mu\mu}\bar{\alpha}_\nu g^{(1,2)}_{\nu i}}{2f_{\mu\nu}{}^2 - f_{\mu\mu}{}^2}\right]u , \tag{2.12}$$

where u is an arbitrary two-component spinor,

$$\begin{aligned} f_{\mu\nu} &= \partial_\mu\partial_\nu\left(\frac{1}{\rho}\right) , \\ g^{(1,2)}_{\mu i} &= \partial_\mu\left(\frac{1}{\rho^2}M^{(1,2)}_i\right) , \end{aligned} \tag{2.13}$$

and $\bar{\alpha}_\mu$, ρ, and $M^{(1,2)}_i$ are as given in Ref. 1. Substituting Eq. (2.12) into (2.10) then gives the $4N$ solutions of Jackiw and Rebbi.[1]

III. SMALL FLUCTUATIONS OF THE YANG-MILLS FIELD

The instanton field A^a_μ and the zero-eigenvalue mode ψ^a that we have found in the previous section form a solution to the full, coupled SU(2) field equations. The spinor ψ^a is a chiral eigenstate $\gamma_5\psi^a = -\psi^a$ and in Euclidean space $\bar{\psi} = \psi^\dagger$. As a result, the isovector current for ψ^a vanishes,

$$\begin{aligned} \epsilon^{abc}\bar{\psi}^b\gamma_\mu\psi^c &= \epsilon^{abc}\bar{\psi}^b\gamma_5\gamma_\mu\gamma_5\psi^c \\ &= -\epsilon^{abc}\bar{\psi}^b\gamma_\mu\psi^c = 0 , \end{aligned} \tag{3.1}$$

and the coupled Yang-Mills field equation

$$D^{ab}_\mu F^b_{\mu\nu} = g\epsilon^{abc}\bar{\psi}^b\gamma_\nu\psi^c = 0 \tag{3.2}$$

is satisfied. In addition, ψ^a satisfies the Dirac equation in the presence of the field A^a_μ. Thus we may take the solution $\hat{A}^a_\mu = A^a_\mu$, $\hat{\psi}^a = \psi^a$ and perform a global supersymmetry transformation (it is not possible now to find a local transformation for which the supergravity fields decouple) to obtain

$$\delta \hat{A}^a_\mu = i\bar{\eta}\gamma_\mu \psi^a ,$$
$$\delta \hat{\psi}^a = F^a_{\mu\nu} \Sigma_{\mu\nu}\eta . \tag{3.3}$$

By our usual supersymmetry arguments, the expression for $\delta\hat{A}^a_\mu$ in Eq. (3.3) generates solutions to the linearized Yang-Mills field equations. In particular, $A^a_\mu + \delta\hat{A}^a_\mu$ (to first order in $\delta\hat{A}^a_\mu$) gives a self-dual solution to the sourceless Yang-Mills equations. The argument is due to Zumino[7] and is based on the identity

$$\gamma_\nu D_\mu - \gamma_\mu D_\nu = \tfrac{1}{2}\epsilon_{\nu\mu\tau\rho}\gamma_5(\gamma_\tau D_\rho - \gamma_\rho D_\tau) + \tfrac{1}{2}(\gamma_\nu\gamma_\mu - \gamma_\mu\gamma_\nu)\gamma_\tau D_\tau \tag{3.4}$$

applied to $A^a_\mu + \delta\hat{A}^a_\mu$. Therefore, if $\epsilon(x)$ is chosen to satisfy Eq. (2.5),

$$\delta\hat{A}^a_\mu = i\bar{\eta}\gamma_\mu\Sigma_{\alpha\beta}\epsilon(x)F^a_{\alpha\beta} \tag{3.5}$$

gives the zero-eigenvalue solutions to the small-fluctuations problem for the Yang-Mills field about an N-instanton solution. Note that since $\Sigma_{\alpha\beta}F^a_{\alpha\beta}\epsilon(x)$ is pure left-handed only the right-handed components of η will enter into Eq. (3.5). Then there are two independent choices for η and the $4N$ solutions to the Dirac equation generate $8N$ small fluctuations for the Yang-Mills field. Furthermore, again since ψ^a satisfies the Dirac equation, $\delta\hat{A}^a_\mu$ automatically satisfied the background gauge condition

$$D^{ab}_\mu(A)\,\delta\hat{A}^b_\mu = 0 . \tag{3.6}$$

IV. SUPERGRAVITY

We consider now a theory of supergravity[12] (or extended supergravity[13]—but for simplicity we discuss here the pure supergravity case). We begin with a solution to the classical field equations

$$\hat{e}^a_\mu = e^a_\mu ,$$
$$\hat{\psi}_\mu = 0 , \tag{4.1}$$

where e^a_μ could represent an instanton-type solution to the gravitational field equations.[14] Performing an infinitesimal local supersymmetry transformation on these fields gives[12]

$$\delta\hat{e}^a_\mu = 0 ,$$
$$\delta\hat{\psi}_\mu = \psi_\mu = 2\kappa^{-1}D_\mu\epsilon(x) , \tag{4.2}$$

where D_μ is the covariant derivative for the vierbein field e^a_μ. Because of the supersymmetry of the model, ψ_μ will satisfy the linearized spin-$\frac{3}{2}$ field equation which is just the covariant Rarita-Schwinger equation

$$\epsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\nu D_\rho\psi_\sigma = 0 . \tag{4.3}$$

We have thus generated zero-eigenvalue modes of the Rarita-Schwinger equation in the background gravitational field given by e^a_μ by local supersymmetry transformation in analogy with our treatment of the Dirac equation in Sec. II. However, an important difference between the two cases is that supersymmetry is a *gauge* symmetry of the Rarita-Schwinger equation. As a result, even if we fix a gauge for the Rarita-Schwinger field (such as $\gamma^\mu\psi_\mu = 0$), we find that the solutions of Eq. (4.2) are pure gauges and are not physically relevant.

A similar problem arises when we treat small fluctuations of the gravitational field around the background field e^a_μ. Suppose we have a physical solution to Eq. (4.3) (not a pure gauge), ψ_μ. Recall that in Sec. III we noted that our Dirac solutions had zero isocurrent and so they formed along with the instantion field a solution to the coupled Yang-Mills-Dirac system. We then generated zero-eigenvalue modes of the Yang-Mills field by supersymmetry transformation. In the present case, we note that the fields

$$\hat{e}^a_\mu = e^a_\mu ,$$
$$\hat{\psi}_\mu = \psi_\mu \tag{4.4}$$

form a solution to the coupled Rarita-Schwinger-Einstein equations [the supergravity equations without the quartic $(\bar{\psi}\psi)^2$ term in the Lagrangian]. This is because we can always choose ψ_μ to be a γ_5 eigenstate. Then, if we choose such eigenstates the energy-momentum tensor for the Rarita-Schwinger field vanishes:

$$T_{\alpha\beta} = \tfrac{1}{2}\epsilon^{\mu\beta\rho\sigma}\bar{\psi}_\mu\gamma_5\gamma_\alpha D_\rho\psi_\sigma = 0 , \tag{4.5}$$

since $\bar{\psi}_\mu = \psi^\dagger_\mu$ in Euclidean space. This is in complete analogy with the vanishing of the Dirac isocurrent in Sec. III. Consider now an infinitesimal supersymmetry transformation

$$\delta\hat{e}^a_\mu = \kappa\bar{\eta}\gamma^a\psi_\mu ,$$
$$\delta\hat{\psi}_\mu = 2\kappa^{-1}D_\mu(e^a_\nu)\eta \tag{4.6}$$

around the previous solution. It is known (see the first paper in Ref. 12) that in general the Rarita-Schwinger-Einstein Lagrangian is not invariant under supersymmetry transformations unless one adds a quartic $(\bar{\psi}\psi)^2$ term to it and a quadratic $(\bar{\psi}\psi)$ term to the transformation law for ψ_μ. However, we observe that this additional term [$\Delta\mathcal{L}_{3/2}$ of Eq. (10) in the first paper of Ref. 12] contains an overall factor $\bar{\psi}_\lambda\gamma_d\psi_\rho$ which will vanish in Euclidean space since we choose ψ_μ to be a chiral eigenstate $\gamma_5\psi_\mu = \pm\psi_\mu$. Therefore for variations around such solutions we have invariance of the Rarita-Schwinger-Einstein system itself. Note that just as in the Yang-Mills case, for a ψ_μ of one chirality only those components of η having opposite chirality will enter into Eq. (4.6) for δe^a_μ.

The variation

$$\delta\hat{e}^a_\mu = \kappa\bar{\eta}\gamma^a\psi_\mu , \tag{4.7}$$

where ψ_μ is a solution of the Rarita-Schwinger equation, produces a variation in the metric

$$\delta\hat{g}_{\mu\nu} = \kappa\bar{\eta}(\gamma_\mu\psi_\nu + \gamma_\nu\psi_\mu) , \tag{4.8}$$

which satisfies the linearized field equations. The corresponding variation in the spin connection is[12]

$$\delta\omega_{\mu ab} = -e^{-1}\bar{\eta}\gamma_5\gamma_\mu\epsilon_{abcd}D_c\psi_d . \tag{4.9}$$

Note that additional terms usually found in $\delta\omega_{\mu ab}$ (see Ref. 12) are absent here because ψ_μ satisfies the Rarita-Schwinger equation. Now for any ψ_μ which satisfies the Rarita-Schwinger equation we have the identity

$$\epsilon_{abcd}D_c\psi_d = \gamma_5(D_a\psi_b - D_b\psi_a) . \tag{4.10}$$

Then since ψ_μ is a chiral eigenstate we can easily show that $\delta\omega_{\mu ab}$ of Eq. (4.9) is self-dual (or anti-self-dual). A self-dual spin connection will in turn generate a self-dual curvature $R_{\mu\nu ab}$.

However, we now run into the problem of isolating from the zero-eigenvalue modes of Eq. (4.8) those which are physical and not just pure gauges. We have no general procedure for doing this and so have been unable to establish a gauge invariant method for counting these modes.

ACKNOWLEDGMENTS

Much of this work was done at the Aspen Center for Physics and we thank the Center for its hospitality. We have benefitted from conversations with Tohru Eguchi, Peter van Nieuwenhuizen, Hugh Pendleton, and Howard Schnitzer. One of us (M.T.G.) also wishes to thank B. Zumino for helpful discussions. The work of L.F.A. was supported by the Department of Energy. The work of M.T.G. was supported in part by the National Science Foundation under Grant No. PHY-76-02054.

---

*Permanent address: Department of Physics, Brandeis University, Waltham, Ma 02154.

[1]R. Jackiw and C. Rebbi, Phys. Rev. D 16, 1052 (1977).

[2]S. Chada, A. D'Adda, P. DiVecchia, and F. Nicodemi, Phys. Lett. 67B, 103 (1977).

[3]L. S. Brown, R. D. Carlitz, and C. Lee, Phys. Rev. D 16, 417 (1977).

[4]B. Grossman, Phys. Lett. 61A, 86 (1977); J. Kiskis, Phys. Rev. D 15, 2329 (1977); A. A. Belavin and A. M. Polyakov, Nucl. Phys. B123, 429 (1977).

[5]A. A. Belavin, A. M. Polyakov, A. S. Schwartz, and Yu. S. Tyupkin, Phys. Lett. 59B, 85 (1976).

[6]E. Witten, Phys. Rev. Lett. 38, 121 (1977); E. Corrigan and D. B. Fairlie, Phys. Lett. 67B, 69 (1977); G. 't Hooft (unpublished); F. Wilczek, Princeton University report (unpublished); R. Jackiw, C. Nohl, and C. Rebbi, Phys. Rev. D 15, 1642 (1977).

[7]B. Zumino, Phys. Lett. 69B, 369 (1977).

[8]R. Jackiw and C. Rebbi, Phys. Lett. 67B, 189 (1977).

[9]A similar technique for generating solutions, and its finite analog, are discussed in N. S. Baaklini, S. Ferrara, and P. van Nieuwenhuizen, Lett. Nuovo Cimento 20, 113 (1977).

[10]A. Salam and J. Strathdee, Nucl. Phys. B76, 477 (1974); S. Ferrara, J. Wess, and B. Zumino, Phys. Lett. 51B, 239 (1974); S. Ferrara and B. Zumino, Nucl. Phys. B76, 413 (1974).

[11]S. Ferrara, F. Gliozzi, J. Scherk, and P. van Nieuwenhuizen, Nucl. Phys. B117, 333 (1976).

[12]D. Z. Freedman, P. van Nieuwenhuizen, and S. Ferrara, Phys. Rev. D 13, 3214 (1976); S. Deser and B. Zumino, Phys. Lett. 62B, 335 (1976).

[13]S. Ferrara and P. van Nieuwenhuizen, Phys. Rev. Lett. 37, 1669 (1976); D. Freedman, *ibid*. 38, 105 (1977); J. Scherk and B. Zumino, Phys. Lett. 66B, 35 (1977).

[14]T. Eguchi and P. G. O. Freund, Phys. Rev. Lett. 37, 1251 (1976); S. Hawking, Phys. Lett. 60A, 81 (1977).