Physica D 46 (1990) 449–457
North-Holland

CHAOS IN A NEURAL NETWORK CIRCUIT *

Thomas B. KEPLER, Sumeet DATT, Robert B. MEYER and L.F. ABBOTT
Physics Department, Brandeis University, Waltham, MA 02254, USA

Received 2 January 1990
Accepted 14 June 1990
Communicated by R.M. Westervelt

We have constructed a neural network circuit of four clipped, high-gain, integrating operational amplifiers coupled to each other through an array of digitally programmable resistor ladders (MDACs). In addition to fixed-point and cyclic behavior, the circuit exhibits chaotic behavior with complex strange attractors which are approached through period doubling, intermittent attractor expansion and/or quasiperiodic pathways. Couplings between the nonlinear circuit elements are controlled by a computer which can automatically search through the space of couplings for interesting phenomena. We report some initial statistical results relating the behavior of the network to properties of its coupling matrix. Through these results and further research the circuit should help resolve fundamental issues concerning chaos in neural networks.

1. Introduction

A neural network, whether biological or electronic, is a highly coupled system of nonlinear elements. Neural network research focuses on designing networks to perform tasks of practical or biological importance. Much attention has been given to the fixed-point behavior of networks[#1], that is, to networks which ultimately reach a static final state. Less attention has been paid to cyclic phenomena and less still to chaotic behavior. One reason for this is that chaotic behavior is more difficult to treat analytically and, as we will discuss, is very problematic for computer simulations. To remedy this situation we have constructed a small analog neural network circuit capable of exhibiting all three types of behavior but designed specifically for the analysis of chaos in networks. In this report we will describe the circuit and demonstrate its chaotic behavior and the approaches to chaos which it exhibits. We also present some initial statistical analysis relating the behavior of the network to properties of its coupling matrix. Further quantitative analysis of this type will be reported in a future publication.

Chaotic behavior is an interesting property of nonlinear dynamical systems and thus is worth studying in neural networks for its own sake. In addition, there are indications that chaotic behavior could be extremely useful in neural network parallel processors. One of the most difficult tasks which a neural network processor must face is distinguishing between two very similar inputs which are to be mapped to different outputs. It is well known that a defining characteristic of chaotic dynamics is its extreme sensitivity to nearly identical inputs and it may be that chaotic behavior can be incorporated in a controlled way to achieve the needed sensitivity for input discrimination. Before this potential can be realized, however, we must have a clear understanding of what sort of chaotic behavior is possible in networks, what produces this behavior and how it might be controlled and used. Chaotic behavior

*Research supported by Department of Energy Contract DE-AC0276-ER03230, National Science Foundation Grant DMR-8803582 and The Martin Fisher School of Physics.
[#1]For a review, see ref. [1].

has been studied theoretically in large neural networks [2] but because we will consider a very small network these results will not be applicable to our circuit. Chaotic behavior has been induced in a small network circuit by introducing time delays [3]. We will not consider any time delays in our work.

The key to constructing a truly general neural network circuit is to allow for all possible couplings and coupling strengths between the nonlinear elements. Previously, circuits have been built which allow all elements to be coupled to each other but only with a highly restricted set of strengths. Most existing circuits provide three choices, devices may be decoupled, coupled with positive coupling or negative coupling. Some circuits allow for the introduction of time delays [3] or of noise [4] but none to our knowledge have couplings which can easily be varied over a wide range of values. In the circuit described here, multiplying digital to analog converters (MDACs) provide the freedom to assign any one of 4096 strengths to each of the couplings which may in addition be either positive or negative. This gives us the necessary degrees of freedom to find sets of couplings which produce interesting chaotic behavior in the network.

From a research standpoint there is little point in constructing an analog circuit which can just as easily be simulated on a computer. At the same time, it is dangerous to put too much faith in results obtained from a circuit which cannot be understood well enough to be simulated. As we will show on the basis of idealized circuit equations, we are able to simulate the circuit we have constructed on a computer and match observed behavior extremely well. However, chaotic behavior cannot be systematically studied by computer alone without an enormous cost in computer time and resources. It is highly nontrivial in a computer simulation to find sets of couplings which exhibit chaotic behavior and to verify that chaos is really present. However, using the analog circuit this process is straightforward and can even be automated. Thus, we believe that the analog network circuit is an essential tool for studying chaotic behavior in neural networks.

Before describing the network circuit it may be useful to summarize what is known about neural network behavior especially with regard to fixed points, oscillation and chaos. It is well known that networks with symmetric couplings only exhibit fixed-point behavior with either discrete or continuous nodes and discrete [5] or continuous [6] updating. With parallel updating such networks may in addition exhibit simple cyclic behavior [7, 8]. When time delay is included conditions for fixed-point and oscillatory behavior have been given [9] and chaotic behavior has been observed [3]. When the network couplings are not symmetric much less is known [10]. However, it is clear from theory that asymmetric networks with high gain will exhibit chaos [2]. This is precisely the regime we intend to study, asymmetry and high gain, and it is surely the area where network behavior is least understood.

2. The circuit

The neural network circuit which we have built consists of four nonlinear amplifiers coupled to each other in all possible ways through programmable MDAC resistor ladders. We originally decided upon a four-node network because initial simulation had lead us to believe that this was the smallest number of nodes which could exhibit chaos without time delay. After building the circuit we found that in fact three nodes can produce chaos. This raises the interesting possibility of performing an analytic analysis on our circuit equations since, when certain approximations are made, they are tractable in the three-node case. We are also now in the process of determining the difference between chaotic behavior in three- and four-node networks, but in this paper we will concentrate on the behavior of the four-node circuit. The four nonlinear circuit elements we use are clipped, high-gain, integrating operational amplifiers arranged as shown in fig. 1. Feedback

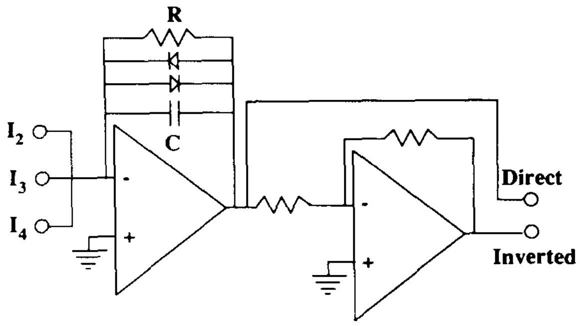

Fig. 1. Circuit diagram for one node of the network. I_2, I_3 and I_4 refer to inputs coming from other network nodes through the MDAC array. The direct or the inverted output is coupled through analog switches and the MDAC array to the other nodes.

diodes provide the nonlinearity by clipping the gain. The output of each of these nonlinear integrators is inverted by an additional unity gain op-amp so that both an inverted and noninverted output is available at each node. The input to any node is the sum of currents obtained from the outputs of all other nodes. Thus, in fig. 1, I_2, I_3 and I_4 are currents coming from the outputs or the inverted outputs of the other three nodes through analog switches and MDACs. The coupling of node j to node i is determined by a coupling matrix J_{ij} which is stored in a computer controlling the operation of the circuit and is down-loaded into latches in the programmable MDACs and into latches controlling the analog switches. The sign of J_{ij} determines whether node i receives its input from the noninverted ($J_{ij} > 0$) or inverted ($J_{ij} < 0$) output of node j. The inverted or noninverted output goes through an MDAC before connecting with one of the node inputs. The idealized MDAC represents a fixed resistive load but has a feed-through conductance which is variable between 0 and 10^{-4} mho in 4096 equal steps. The specific value of the feed-through conductance for the MDAC connecting node j to node i is determined by the magnitude of the matrix element J_{ij}. The only other circuit elements are analog switches which allow us to feed large positive or negative currents directly into the node inputs for fixing initial conditions on the network, and an analog to digital converter used in computer evaluation of circuit output.

The components chosen for this circuit design were intended to keep the behavior of the circuit close to an idealized model. The op-amps are FET input devices with negligible input current and maximum input offset voltage in the millivolt range. The MDACs are analog devices AD7548 which have a fixed input impedance of about 11 kΩ, and function by dividing input current between two resistive paths, one to ground and the other to the virtual ground input of an op-amp. The current division is approximately linear and monotonic to 12 bits accuracy if the virtual ground offset voltage is small enough. We calibrated the MDACs and found at most about 10% variation in their input impedance and 11-bit accuracy in our circuit. We could have trimmed the impedance with a series resistor, and selected better op-amps, but as discussed below this did not seem necessary for understanding circuit behavior. By selecting $R = 10^6$ Ω and $C = 0.1$ μF we assured that the oscillatory frequency of the circuit was in the range of a few Hz to a few kHz, well below the range in which nonideal behavior of the op-amps or diodes could be important. For each diode element we actually used four IN4148 diodes in series to raise output voltages and make input offset voltages less significant. For comparison, we also used a single IN4148 for the diode, thus lowering voltages by a factor of 4, reduced the value of the feedback capacitor by a factor of 10, thereby raising frequencies by a similar factor, and removed the feedback resistors, setting $J_{ii} = 0$, with no important changes in the nature of the circuit behavior for a given coupling matrix. This made us confident that our circuit element choices were not crucial for circuit behavior. As another check on the circuit, we tried exchanging connection elements between nodes, and permuting the coupling matrix elements to effectively interchange nodes. Again no significant change in circuit behavior was observed, indicating that small differences among the op-amps and MDACs are not important.

3. Circuit equations

We can model the circuit described above by considering the node op-amp to be a perfect device having exactly zero input current and zero off-set voltage between its inputs. We use the simple current voltage relation

$$I = \tfrac{1}{2}a(\mathrm{e}^{-bV} - 1) \tag{3.1}$$

for the diode element where $a = 5.5 \times 10^{-9}$ A and $b = 21/n$ V^{-1} with n the number of diodes linked in series. Then, the output voltages V_i for the four nodes $i = 1, 2, 3, 4$ are determined by the following four, coupled, first-order differential equations:

$$-C\frac{\mathrm{d}V_i}{\mathrm{d}t} = \sum_{i=1}^{4} J_{ij}V_j + a\sinh(bV), \tag{3.2}$$

where C is the value of the feedback capacitor (see fig. 1),

$$J_{ii} = 1/R \tag{3.3}$$

and for $i \neq j$

$$J_{ij} = \pm\sigma_{ij} \tag{3.4}$$

with R the value of the feedback resistor (see fig. 1), σ_{ij} the feed-through conductance of the MDAC through which the signal from node j to node i passes and the plus or minus depends on whether the inverted or direct output of node j is used. Simulation work has shown that the sinh function in the circuit equations may be replaced by any function having the same general form without appreciably modify circuit behavior.

The above equations are somewhat different than those of standard, analog, continuous-time networks. Denker [11] has discussed the use of this neural model which he calls a virtual ground neuron. From a biological standpoint, both models assume that the fundamental nonlinearity in a biological neuron is action potential firing. However, in the usual model this nonlinearity enters because synapses are only modelled to transmit a signal in the presence of action potential spikes. In the model we have used, the synapses are linear, transmitting a signal which depends on the average membrane potential of the presynaptic cell even in the absence of spiking. The fundamental nonlinear role of action potential firing is instead to limit the average cell potential from rising far beyond the action potential threshold.

We have simulated these circuit equations on a computer. With random choices of the matrix J_{ij} we typically find simple fixed-point (all V_i static) behavior or simple periodic behavior (V_i periodic functions of time). However, after extensive trials we were able to find chaotic behavior as is shown in figs. 2b and 2d. Here we have plotted the output of node 1, V_1, against that of node 2, V_2, (a and b) and also against V_3 (c and d). When the

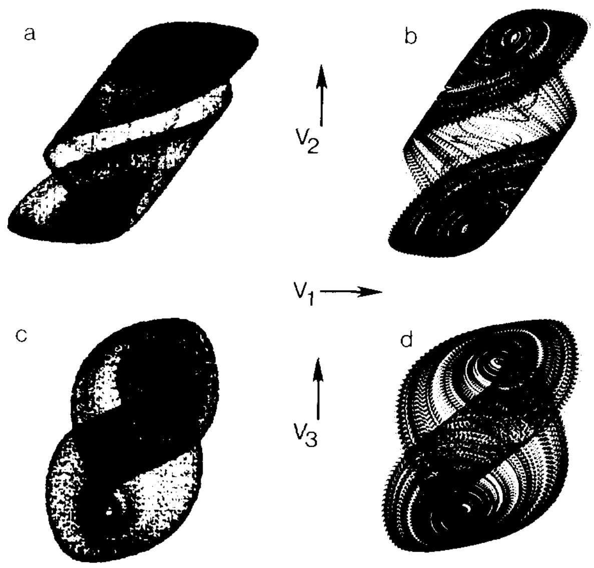

Fig. 2. A comparison of chaotic results obtained from a computer simulation (b and d) with those coming from the circuit (a and c) using the same coupling matrix.

same matrix which produced figs. 2b and 2d on the computer is programmed into the actual circuit and the same voltages are plotted against each other the result is figs. 2a and 2c. The similarity between these two sets of figures convinces us that the above idealized equations do an excellent job of approximating the real behavior of the circuit and that the interesting chaotic behavior in the real circuit is not the result of other effects which we have taken into consideration.

The success of our computer simulation might suggest that chaotic behavior could be analyzed using only computer simulations. However, this would be prohibitively time consuming. Each computer simulation must run for a long time to insure that transient behavior has been eliminated. Minor adjustments in the values of the couplings J_{ij} require re-running the simulation. These two facts make performing repeated simulations quite tedious and since, as we have found, chaotic coupling matrices are quite rare it is extremely time consuming to find them by computer simulation. In fact, during extensive computer simulation we found only two examples of chaos one of which is shown in fig. 2. With the circuit we have found numerous examples in a fraction of the time.

4. Circuit behavior

As in the computer simulation, most random coupling matrices programmed into our circuit produce either fixed-point or cyclic behavior. However, approximately one in a few hundred matrices results in chaotic behavior on highly structured strange attractors. We have verified that chaotic behavior is present by performing a spectral analysis of a given node's output voltage using a high-resolution audio spectrum analyzer. We can approach these attractors by varying one of the matrix elements slowly. Here the fact that we have 4096 possible values for each matrix element is very important since the onset of chaos is highly sensitive to changes in the matrix.

As the matrix elements are adjusted in various ways near or in a chaotic region, the circuit exhibits period-doubling and quasiperiodic routes to chaos, as well as sudden, perhaps crisis induced, intermittent expansions of strange attractors [12]. We have even found an attractor which shows all three transitions depending on which matrix element is varied on the approach. Fig. 3 shows a simple cycle (a) becoming a quasiperiodic torus (b, c) and then becoming chaotic (d). In fig. 4, a period-doubling route (a–d) leads to chaotic behavior and then an intermittent expansion of the strange attractor is seen (e, f). These sudden expansions are associated with a restoration of the $V_i \to -V_i$ (for all i) symmetry of the circuit equations. Although the circuit equations (3.2) are invariant under such a symmetry, many solutions to them (for example, fixed points) are not. When a strange attractor lacks this symmetry, we observe that at first it grows steadily as some matrix element is varied. At some point in this growth, it jumps across to its mirror image (which is, of course, also a stable attractor) for several cycles and then back again (fig. 4e). We have seen this intermittency occurring as slowly as 1 Hz in a circuit operating at a fundamental frequency of about 1000 Hz. As the matrix element is varied still further, the mirror image attractors merge and the symmetry is restored (fig. 4f). All of these behaviors were found and explored using the analog circuit, but for clarity figs. 3 and 4 are computer generated through numerical integration of eqs. (3.2). The existence of period-doubling and quasiperiodic and chaotic behaviors has been verified by examining Fourier transforms of the circuit output using an audio spectrum analyzer.

5. Analysis of circuit couplings

A natural question arises in studying the different behaviors which this network circuit can exhibit: Is there some property or set of properties of the coupling matrix which can be used to

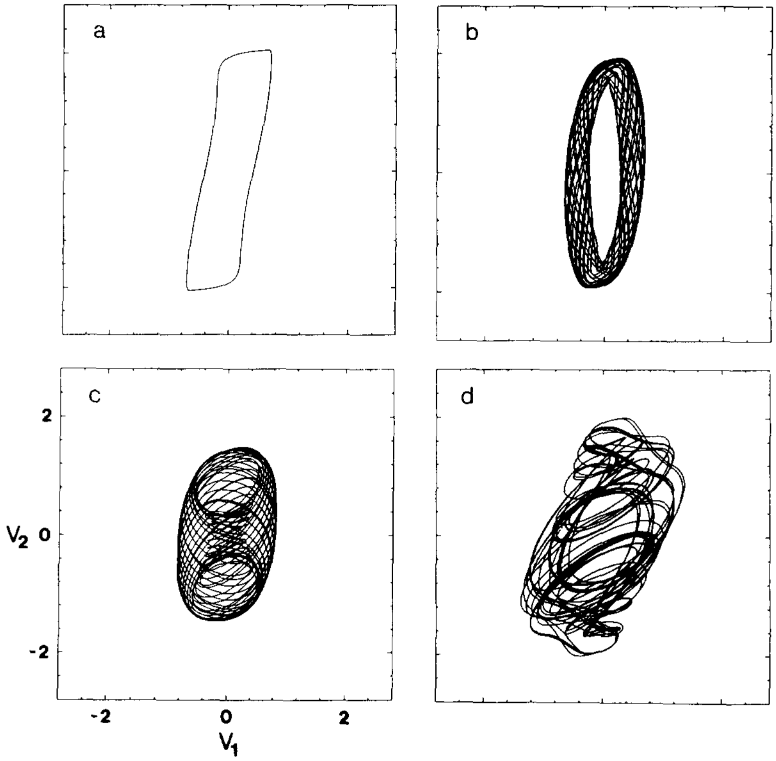

Fig. 3. A quasiperiodic approach to chaos.

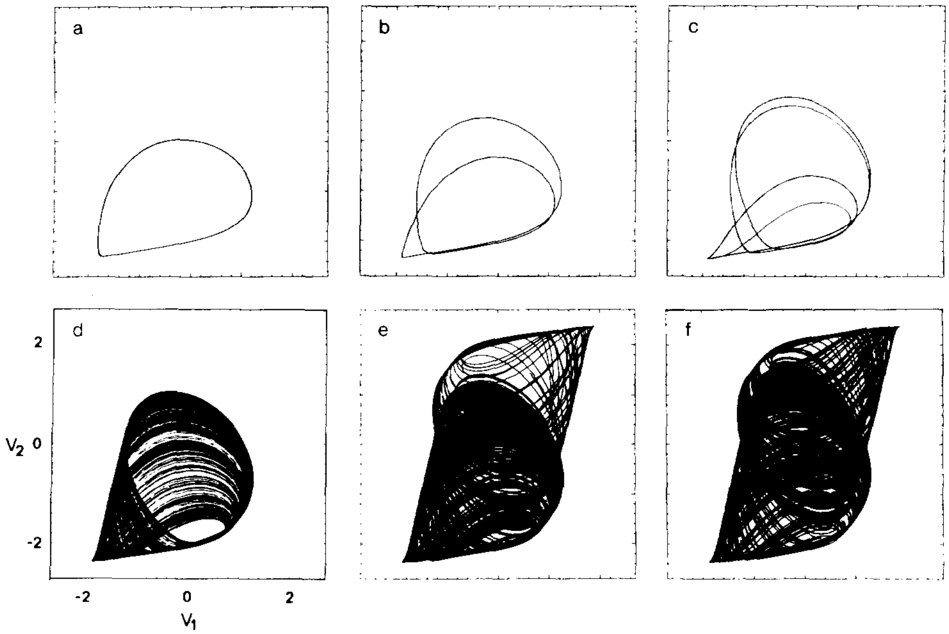

Fig. 4. A period-doubling approach to chaos with intermittent expansion of the attractor.

characterize and predict the behavior of a network interacting through that matrix? As a first step in answering this question we have performed a statistical analysis of the matrices we have found giving fixed-point, cyclic or chaotic behavior. Our analysis is based on the study of 33 matrices producing fixed-point behavior, 30 matrices giving cycles and 25 chaotic matrices. The quantities we have chosen to characterize these matrices are products of matrix values around closed loops. Such quantities have proven useful in many analyses of complex systems [13] because they show, among other things, the presence or absence of frustration, the negative products of couplings around loops. We computed expectation values of these quantities by averaging over all possible closed loops for a given matrix and averaging over all of the matrices we have which produce a given type of behavior. Thus we define

$$x(2) = \frac{\langle J_{ij} J_{ji} \rangle}{|J|^2}, \tag{5.1}$$

$$x(3) = \frac{\langle J_{ij} J_{jk} J_{ki} \rangle}{|J|^3}, \tag{5.2}$$

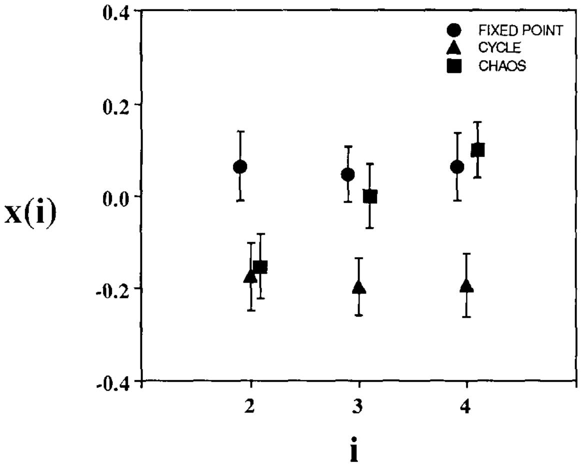

Fig. 5. $x(i)$ as defined in the text for matrices producing fixed-point, cyclic and chaotic behavior. Note the distinctly different patterns for these three cases.

and

$$x(4) = \frac{\langle J_{ij} J_{jk} J_{km} J_{mi} \rangle}{|J|^4}, \tag{5.3}$$

where

$$|J| = \sqrt{\langle J_{ij}^2 \rangle} \tag{5.4}$$

and the angular brackets indicate an average over all index values and over all matrices which we have.

Another useful set of quantities which give the magnitude rather than the signs of products around loops are defined similarly as

$$y(2) = \frac{\langle (J_{ij} J_{ji})^2 \rangle}{|J|^4}, \tag{5.5}$$

$$y(3) = \frac{\langle (J_{ij} J_{jk} J_{ki})^2 \rangle}{|J|^6}, \tag{5.6}$$

and

$$y(4) = \frac{\langle (J_{ij} J_{jk} J_{km} J_{mi})^2 \rangle}{|J|^8}. \tag{5.7}$$

We leave out $y(1)$ because it is identically equal to one. In addition, for all types of matrices $x(1)$ is consistent with zero so we will not include it in our discussion. These quantities do not of course provide a complete description of a matrix or even of its potential for producing chaotic behavior. However, they do seem to provide an interesting indicator of what behavior a matrix might produce.

The results for these quantities averaged over sets of matrices giving either fixed-point, cyclic or chaotic behavior are shown in figs. 5 and 6. Matrices producing the three types of behavior show markedly different results. In fig. 5, matrices giving fixed point behavior have positive x values indicating a lack of frustration. Matrices leading to cyclic behavior have consistently negative x values indicating large amounts of frustration in all loops. Matrices producing chaos have a completely different pattern: negative $x(2)$, an essen-

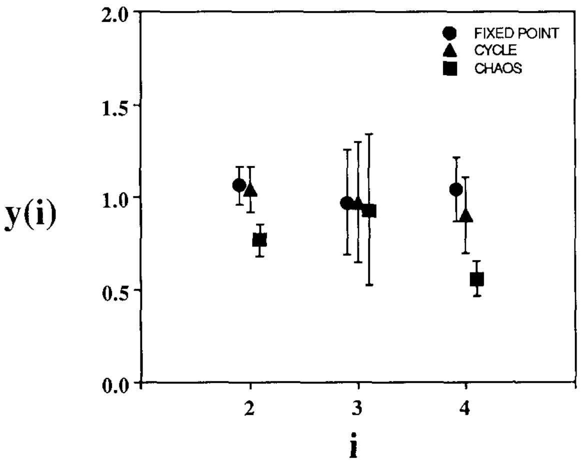

Fig. 6. $y(i)$ as defined in the text for matrices producing fixed-point, cyclic and chaotic behavior.

tially zero value for $x(3)$ and positive $x(4)$. This indicates frustration only in two-site loops. More is learned about chaotic matrices in fig. 6, although the results are somewhat less dramatic. Matrices exhibiting fixed-point and cyclic behavior have all y essentially equal to one. However for matrices exhibiting chaotic behavior, $y(4)$ and especially $y(2)$ are significantly smaller than one. We believe that there is a good reason for this. The chaos-producing matrices have frustration in two-member loops which if too strong would produce cyclic behavior. Therefore it is quite weak. Likewise the lack of frustration in four-site loops might lead to fixed-point behavior if it was too strong. Only $y(3)$ is equal to one since it corresponds to an $x(3)$ of zero, which would not strongly lead to either fixed-point or cyclic behavior. More matrices need to be accumulated to strengthen the statistics of what we have seen, but it is remarkable that the signature of the matrices producing chaos is so clearly seen.

6. Future research

In this report, we have introduced a network circuit with completely programmable couplings, qualitatively described its behavior and given some initial statistical analysis of that behavior. Although our results are a beginning, much more will have to be done before we can begin to completely characterize matrices which produce chaos. For example, at present we have no precise idea how numerous such matrices are. To attack and hopefully to answer some of these questions we have automated the search for interesting couplings, getting the controlling computer to generate matrices and analyze the resulting circuit output. This allows for the trial of hundreds of thousands of random matrices and the analysis of hundreds of matrices which produce chaos. From these data we hope to arrive at a greater understanding of the couplings which produce chaos and how chaotic behavior depends on initial conditions with the intention of using this information in the construction of neural network parallel processors.

Going beyond the simplest four-node circuit, we note that larger analog networks can be constructed using mixed analog and digital techniques with a multiplexed coupling scheme using one MDAC (or one MDAC per node) and a switched array of integrators and holding amplifiers feeding the inputs. With this technique circuit complexity scales linearly rather than quadratically with node number. Such a network circuit will be able to simulate feed-forward or recursive, single- or multi-layered networks by appropriate choice of the form of the coupling matrix.

Going further still, it is possible to investigate fully digital networks incorporating very fast digital CMOS multiply/add signal processing chips rather than MDACs and integrating op-amps. A 16-node circuit of this design will operate at a minimum of 160 million instructions per second, and will be as fully programmable as our analog circuit. This will provide an inexpensive research tool for studying the dynamics of fully general networks that will rival supercomputer simulations.

With a better understanding of chaos in networks we may ultimately be able to control chaotic behavior as well as we now can control fixed-point behavior. If so, there is no doubt that chaos will become an important part of network design and construction.

Acknowledgements

We wish to thank Geoffrey Haigh for technical assistance and advice, C. Marcus for helpful discussions and Analog Devices Corporation for providing many of the circuit elements.

References

[1] D.J. Amit, Modelling Brain Function: The World of Attractor Neural Networks (Cambridge Univ. Press, Cambridge, 1989).

[2] K.E. Kürten and J.W. Clark, Phys. Lett. A 114 (1986) 413;
S. Shinomoto, Prog. Theor. Phys. 75 (1986) 1313:
U. Riedel, R. Kühn and J.L. van Hemmen, Phys. Rev. A 38 (1988) 1105;
K.E. Kürten, Phys. Lett. A 129 (1988) 157;
H. Sompolinsky, A. Crisanti and H.J. Sommers, Phys. Rev. Lett. 61 (1988) 259;
M.Y. Choi and B.A. Hubermann, Phys. Rev. A 28 (1983) 1204.

[3] C.M. Marcus and R.M. Westervelt, in: Advances in Neural Information Processing Systems I, ed. D.S. Touretzky (Morgan Kaufmann, San Mateo, CA, 1989).

[4] M. Meidan, D.J. Amit, H. Gutfreund and H. Sompolinsky in: Neural Networks from Models to Applications, eds. L. Personnaz and G. Dreyfus (I.D.S.E.T., Paris, 1989) p. 766.

[5] J.J. Hopfield, Proc. Natl. Acad. Sci. USA 79 (1982) 2554.

[6] J.J. Hopfield, Proc. Natl. Acad. Sci. USA 81 (1984) 3008;
M.A. Cohen and S. Grossberg, IEEE Trans. SMC-13 (1983) 815.

[7] E. Goles-Schaac, F. Fogelmann-Soulie and D. Pellegrin, Discr. Appl. Math. 12 (1985) 261;
E. Goles and G.Y. Vishniac, in: Neural Networks for Computing, ed. J.S. Denker (Am. Inst. Phys., New York, 1986).

[8] C.M. Marcus and R.M. Westervelt, Phys. Rev. A 40 (1989) 501.

[9] C.M. Marcus and R.M. Westervelt, Phys. Rev. A 39 (1989) 347.

[10] H. Gutfreund, J.D. Reger and A.P. Young, J. Phys. A 21 (1988) 2775.

[11] J.S. Denker, in: Neural Network for Computing, ed. J.S. Denker (Am. Inst. Phys., New York, 1986).

[12] G. Grebogi, E. Ott and J.A. Yorke, Physica D 7 (1982) 181; Phys. Rev. Lett. 48 (1982) 1507.

[13] G. Toulouse, Commun. Phys. 2 (1977) 115.