STABILITY AND INSTABILITY OF SCALAR FIELDS COUPLED TO GRAVITY

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Received 16 January 1986

We relate the stability criteria of Boucher for a scalar field coupled to gravity to the semi-classical tunnelling calculations of Coleman and DeLuccia. This enables us to prove that the Boucher conditions imply both stability when they are satisfied and instability when they are not. In addition, it provides physical insight into the nature of the Boucher conditions and indicates that the O(4)-invariant, thin-wall approximation used in semi-classical tunnelling computations is reliable.

Consider an extremum point $\phi = \bar{\phi}$ of a scalar potential $V(\phi)$ such that $V'(\bar{\phi}) = 0$. If $V(\bar{\phi}) \leqslant 0$, gravitational effects can stabilize the configuration $\phi = \bar{\phi}$ even if it is unstable in the absence of gravity [1–6]. The stability of a given configuration can best be established by using the criteria of Boucher [1]. The configuration $\phi = \bar{\phi}$ is stable if there exists a real function $f(\phi)$ satisfying the boundary condition

$$f(\bar{\phi}) = [-V(\bar{\phi})/3\kappa]^{1/2} \tag{1}$$

at the point $\phi = \bar{\phi}$ and the differential equation

$$2(f')^2 - 3\kappa f^2 = V \tag{2}$$

everywhere. In eqs. (1) and (2) κ is 8π times Newton's constant and a prime indicates differentiation with respect to ϕ. Boucher's elegant solution raises two questions. If eq. (2) is not satisfied everywhere, is the configuration unstable? What is the physical significance of the function $f(\phi)$? We will answer both of these questions in this note.

Eq. (2) can be rewritten in the form

$$f' = [(V + 3\kappa f^2)/2]^{1/2}, \tag{3}$$

which shows what happens when a potential fails to satisfy eq. (2). If V ever gets more negative than $3\kappa f^2$ then the square root in eq. (3) will become imaginary, violating the reality condition on f. The signal for this occurrence is a point $\phi = \phi_s$ where $f'(\phi_s) = 0$ and $V'(\phi_s) \neq 0$. It is easy to show from (3) that f will become imaginary in the immediate neighborhood of such a point.

Suppose that we integrate eq. (3) around a point $\phi = \bar{\phi}$ with the boundary condition (1) and find a point $\phi = \phi_s$ where $f' = 0$ and $V' \neq 0$ so that the conditions for stability are not satisfied. Does this imply that $\phi = \bar{\phi}$ is unstable? As we will now show the answer is yes.

If $V''(\bar{\phi}) - 3\kappa V(\bar{\phi})/4$ is positive then eq. (3) can always be integrated over a finite region around $\bar{\phi}$ before ϕ_s is reached. This is the case we will consider first. Semi-classical instability is signalled by the presence of a solution to the euclidean field equations satisfying "bounce" boundary conditions and having finite euclidean action [2]. If we look for O(4)-symmetric solutions we can write the metric as

$$ds^2 = d\xi^2 + \rho^2(\xi)\, d\Omega^2, \tag{4}$$

where $d\Omega$ is the solid angular measure on a three-sphere. We take ϕ to be a function of ξ only. The relevant field equations are

$$\ddot{\phi} + 3(\dot{\rho}/\rho)\dot{\phi} = V', \tag{5a}$$

$$(\dot{\rho}/\rho)^2 = 1/\rho^2 + \tfrac{1}{3}\kappa(\tfrac{1}{2}\dot{\phi}^2 - V). \tag{5b}$$

Here a dot denotes differentiation with respect to ξ

[1] Research supported by U.S. Department of Energy Contract DE-AC03-76-ER03230.
[2] Research supported by an Alfred P. Sloan Foundation Fellowship.

0370-2693/86/$ 03.50 © Elsevier Science Publishers B.V.
(North-Holland Physics Publishing Division)

and a prime is a ϕ derivative. The bounce boundary conditions are $\phi(\infty) = \bar{\phi}$, $\dot{\phi}(\infty) = 0$ and $\dot{\phi}(0) = 0$.

In order to solve eq. (5) we will assume that the term $1/\rho^2$ in eq. (5b) can be ignored in the region where $\dot{\phi}$ is appreciably different from zero. Of course, this approximation will not be valid for all potentials. Nevertheless, as we will show, our results are valid for all potentials. For now, however, let us assume that the potential being considered has the form needed to make this approximation a good one. Ignoring the $1/\rho^2$ term in eq. (5b), (5a) and (5b) can be combined to give

$$\ddot{\phi} \pm [3\kappa(\tfrac{1}{2}\dot{\phi}^2 - V)]^{1/2}\dot{\phi} = V'. \tag{6}$$

Next we define a function $f(\phi)$ in terms of the euclidean energy of the scalar field by

$$\tfrac{1}{2}\dot{\phi}^2 - V = 3\kappa f^2(\phi). \tag{7}$$

Note that the energy can be expressed as a function of ϕ only, as in eq. (7), if $\phi(\xi)$ is monotonic in ξ. This is true for the bounce solutions which are of interest to us here. Note also that the sign ambiguity in (6) will now appear as a sign ambiguity in the definition of f. If we multiply (6) by $\dot{\phi}$ and use the definition (7), then eq. (6) can be rewritten as

$$6\kappa f'f\dot{\phi} - 3\kappa f\dot{\phi}^2 = 0, \tag{8}$$

which has the non-trivial solution

$$\dot{\phi} = 2f'. \tag{9}$$

Finally, combining eqs. (7) and (9) gives

$$2(f')^2 - 3\kappa f^2 = V. \tag{10}$$

Eq. (10) is of course the Boucher condition (2). Thus, we find that Boucher's equation is equivalent to the euclidean O(4)-invariant field equations if we relate f to the euclidean energy as in eq. (7).

We have shown that eq. (2) is equivalent to the euclidean field equations – what about eq. (1)? Eqs. (1) and (2) together imply that $f'(\bar{\phi}) = 0$ which from eq. (9) implies that when $\phi = \bar{\phi}$, $\dot{\phi} = 0$. This is one of the boundary conditions for a bounce – that it ends up at rest. Thus, in order to construct a bounce solution only one condition remains to be satisfied, it must start at rest. If we let $\phi = \phi_s$ be the starting point for the bounce then $\dot{\phi}$ must equal zero when $\phi = \phi_s$. From eq. (9) this implies that there must exist a point where $f'(\phi_s) = 0$. In addition, V' must be non-vanishing at $\phi = \phi_s$ or the solution we have obtained will be trivial. These are precisely the conditions for which Boucher's stability criterion (2) fails at the point $\phi = \phi_s$.

We thus have a complete relation between the Boucher conditions and the conditions necessary for a bounce solution to exist. When we integrate eq. (3) we are just integrating the bounce equations backwards from $\phi = \bar{\phi}$. The point $\phi = \phi_s$ where Boucher's condition fails is the starting point for the bounce. Thus, by solving condition (2) we are just checking for the existence of an O(4)-invariant bounce solution. If eq. (2) is satisfied everywhere then no bounce exists and the configuration $\phi = \bar{\phi}$ is stable. If eq. (2) fails then the point $\phi = \phi_s$ is the starting point for a bounce solution and in the region between $\phi = \bar{\phi}$ and $\phi = \phi_s$ the function f which has been constructed by integrating (2) gives the energy and velocity of the bounce through eqs. (7) and (9). The euclidean action of the bounce can be determined from the function f through eqs. (7) and (9) and if it is finite $\phi = \bar{\phi}$ is unstable.

We have assumed up to now that the term $1/\rho^2$ in eq. (5b) could be ignored. If $V(\bar{\phi}) \leqslant 0$, ρ increases without bound as a function of ξ. Thus, $1/\rho^2$ can be arbitrarily small in the region where $\dot{\phi}$ is non-zero

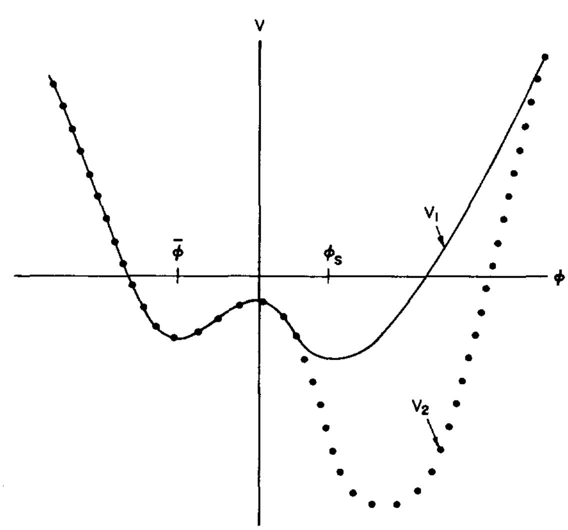

Fig. 1. V_1 satisfies the conditions discussed in the text. Since $V_2 \leqslant V_1$ and $V_1(\bar{\phi}) = V_2(\bar{\phi})$, if $\bar{\phi} = \phi$ is unstable for V_1 it is unstable for V_2 as well.

if ϕ lingers in the neighborhood of ϕ_s for a long time before proceeding to $\bar{\phi}$. The curve V_1 in fig. 1 indicates a potential for which this approximation is valid. What about a potential like V_2 in fig. 1 for which the approximation is not good? In this case we can use the fact that any potential which is everywhere equal to or greater than a stable potential is also stable, provided that the two potentials are equal at the point $\bar{\phi}$. Conversely, any potential which is everywhere less than or equal to an unstable potential is also unstable, again assuming that they are equal at $\bar{\phi}$. We can prove that the point $\phi = \bar{\phi}$ is unstable for the potential V_2 by using the arguments given above to prove that it is unstable for V_1. Thus, our proof is general despite the use of this approximation.

If $V''(\bar{\phi}) - 3\kappa V(\bar{\phi})/4 < 0$ eq. (2) cannot be solved anywhere in the neighborhood of $\bar{\phi}$. In this case the configuration $\phi = \bar{\phi}$ is classically unstable to small fluctuations as has been shown by Breitenlohner and Freedman [3]. The analysis is quite involved because of the absence of a space-like Cauchy surface in anti-de Sitter space. We refer the interested reader to ref. [3].

We have seen that a scalar field configuration $\phi = \bar{\phi}$ which fails to satisfy the Boucher conditions (1) and (2) is either classically or semi-classically unstable and that in the latter case, the function $f(\phi)$ constructed in (1) and (2) determines the tunnelling probability and bubble formation process.

We are grateful to Marc Grisaru for helpful discussions.

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