Three-body decays of the proton

Mark B. Wise
Department of Physics, Harvard University, Cambridge, Massachusetts 02138

R. Blankenbecler
Stanford Linear Accelerator Center, Stanford University, Stanford, California 94305

L. F. Abbott
Department of Physics, Brandeis University, Waltham, Massachusetts 02254

(Received 24 October 1980)

The rates for the three-body proton decays $p\to\pi\pi e^+$ are related to the rate for the decay $p\to\pi^0e^+$. This is done by making an ansatz for the form of the three-body amplitude which is consistent with current algebra and with the measured $\pi\pi$ final-state interactions. We find that the three-body decay rates are comparable with the rate for the two-body decay $p\to\pi^0e^+$.

I. INTRODUCTION

Grand unified models of the strong, weak, and electromagnetic interactions contain new interactions which can mediate baryon-number-violating nucleon decay.[1] If proton decay is characterized by a mass scale of order 10^{15} GeV,[2] as indicated by renormalization-group analysis, then only baryon-number-violating operators of the lowest possible dimension can contribute at an observable rate. Weinberg and Wilczek and Zee have enumerated the baryon-number-violating dimension-six operators consistent with Lorentz and $SU(3)\otimes SU(2)\otimes U(1)$ invariance.[3,4] For decays into nonstrange final states they are

$$Q_1=(\bar{d}^c_{\alpha R}u_{\beta R})(\bar{u}^c_{\gamma L}e_L-\bar{d}^c_{\gamma L}\nu_L)\epsilon_{\alpha\beta\gamma}\ ,$$

$$Q_2=(\bar{d}^c_{\alpha L}u_{\beta L})(\bar{u}^c_{\gamma R}e_R)\epsilon_{\alpha\beta\gamma}\ ,$$

$$Q_3=(\bar{d}^c_{\alpha L}u_{\beta L})(\bar{u}^c_{\gamma L}e_L-\bar{d}^c_{\gamma L}\nu_L)\epsilon_{\alpha\beta\gamma}\ ,$$

and

$$Q_4=(\bar{d}^c_{\alpha R}u_{\beta R})(\bar{u}^c_{\gamma R}e_R)\epsilon_{\alpha\beta\gamma}\ , \tag{1}$$

where the notation of Weinberg has been used. We have shown only those operators relevant to decays with a positron or electron antineutrino in the final state. Similar operators exist for decays with an antimuon or muon antineutrino in the final state. The operators Q_1 and Q_3 lead to right-handed antileptons in the final state while Q_2 and Q_4 lead to left-handed antileptons in the final state. Consequently it is convenient to decompose the effective Hamiltonian for proton decay so that

$$\mathcal{H}^{|\Delta B|=1}_{\text{eff}}=\mathcal{H}_+ +\mathcal{H}_-\ , \tag{2}$$

where

$$\mathcal{H}_+ = a_+Q_2+b_+Q_4+\text{H.c.} \tag{3a}$$

and

$$\mathcal{H}_- = a_-Q_1+b_-Q_3+\text{H.c.} \tag{3b}$$

The Wilson coefficients $a_\pm$ and $b_\pm$ depend on the specific grand unified model being considered. The contributions of $\mathcal{H}_+$ and $\mathcal{H}_-$ do not interfere if the mass of the final-state antilepton is neglected.[5] For example, in the decay $p\to\pi^0e^+$ the matrix elements for right-handed and left-handed positrons can be parametrized in the following manner:

$$\langle\pi^0e^+|\mathcal{H}_\pm(0)|p\rangle\equiv E_\pm\bar{v}^c_e(1+\gamma_5)u_p\ , \tag{4}$$

and the total rate (m is the nucleon mass)

$$\Gamma(p\to\pi^0e^+)=\frac{m}{8\pi}\,(|E_+|^2+|E_-|^2) \tag{5}$$

contains no interference between the contributions of $\mathcal{H}_+$ and $\mathcal{H}_-$.

Recently several estimates have been made for the two-body proton decay rates in the Georgi–Glashow SU(5) grand unified model.[6] In this paper we shall consider the three-body proton decay modes $p\to\pi\pi e^+$ in a model-independent manner. Since the operators $Q_1,\ldots,Q_4$ defined in Eq. (1) are purely isospin $\frac{1}{2}$, the final-state pions can either be in an $I=0$ or $I=1$ state; the $I=2$ final state is forbidden. To obtain crude estimates for $\Gamma(p\to\pi\pi(I=0)e^+)$ and $\Gamma(p\to\pi\pi(I=1)e^+)$ one can compute the rate for the decays $p\to\pi\pi e^+$ from the lowest-order diagrams of Fig. 1. Using Eqs. (4) and (5), neglecting the momentum dependence of the form factors $E_\pm$, and noting that the isospin properties of $Q_1,\ldots,Q_4$ imply $\langle\pi^-e^+|\mathcal{H}_\pm(0)|n\rangle=\sqrt{2}\,\langle\pi^0e^+|\mathcal{H}_\pm(0)|p\rangle$, these contributions give[7]

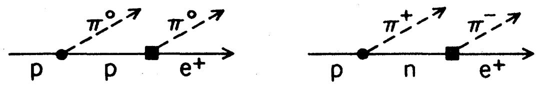

FIG. 1. Born or pole diagrams contributing to $p\to\pi\pi e^+$.

$$\frac{\Gamma(p\to\pi\pi(I=0)e^+)}{\Gamma(p\to\pi^0e^+)}=\frac{3g_r^{\,2}}{32\pi^2}\left(\frac{\pi^2}{3}-\frac{5}{2}\right)\approx 1.4 \qquad (6a)$$

and

$$\frac{\Gamma(p\to\pi\pi(I=1)e^+)}{\Gamma(p\to\pi^0e^+)}=\frac{g_r^{\,2}}{16\pi^2}\left(\frac{7}{2}-\frac{\pi^2}{3}\right)\approx 0.24\,, \qquad (6b)$$

when the pion mass is neglected, and where g_r is the pion-nucleon coupling constant

$$\frac{g_r^{\,2}}{4\pi}=14.6\,. \qquad (7)$$

Therefore, the naive expectation is that the three-body decay modes $p\to\pi\pi e^+$ should be significant in comparison with $p\to\pi^0e^+$.

Several improvements on the above estimate of the rates for the three-body decay modes $p\to\pi\pi(I=0)e^+$ and $p\to\pi\pi(I=1)e^+$ are possible. In Sec. II current algebra is used to gain information on the decay amplitudes when one of the pions is soft and in Sec. III dispersion-relation techniques are used to estimate the effects of final-state strong interactions. Concluding remarks are given in Sec. IV.

II. CURRENT-ALGEBRA CONSTRAINTS

When one of the pions is "soft," current algebra[8] can be used to gain information on the amplitudes for the decays $p\to\pi\pi e^+$. Consider firstly the decay $p\to\pi^0\pi^0e^+$. The invariant amplitude for this decay

$$a_{\pm}^{(0,0)}(p_2,p_3)\equiv\langle e^+(p_1)\pi^0(p_2)\pi^0(p_3)|\mathcal{H}_{\pm}(0)|p\rangle \qquad (8)$$

is a symmetric function of the pion momenta. Using the Lehmann-Symanzik-Zimmermann reduction formula, one finds

$$a_{\pm}^{(0,0)}(p_2,p_3)=\lim_{p_3^{\,2}\to\mu^2} i(\mu^2-p_3^{\,2})\int d^4x\, e^{ip_3\cdot x}\langle e^+(p_1)\pi^0(p_2)|T(\phi_0(x)\mathcal{H}_{\pm}(0))|p\rangle\,, \qquad (9)$$

where μ is the pion mass and ϕ_0 the neutral-pion field. Any field with the appropriate quantum numbers can be used for the pion in Eq. (9) provided it is appropriately normalized. The standard choice in current algebra is to relate the neutral-pion field to the third component of the axial-vector current by

$$\phi_0=\frac{\sqrt{2}}{\mu^2f_\pi}\,\partial^\mu A_\mu^{(3)}\,. \qquad (10)$$

Inserting this into Eq. (9), integrating by parts and taking the soft pion limit $p_3\to 0$, one finds

$$a_{\pm}^{(0,0)}(p_2,0)=\frac{-i\sqrt{2}}{f_\pi}\langle e^+(p_1)\pi^0(p_2)|[Q_5^{(3)},\mathcal{H}_{\pm}(0)]|p\rangle + \lim_{p_3\to 0}\frac{\sqrt{2}}{f_\pi}p_3^\mu\int d^4x\, e^{ip_3\cdot x}\langle e^+(p_1)\pi^0(p_3)|T(A_\mu^{(3)}(x)\mathcal{H}_{\pm}(0))|p\rangle\,. \qquad (11)$$

From Eq. (1) it is easy to relate the equal time commutators of the axial-vector charges $\vec{Q}_5$ to those of the isospin charge $\vec{I}$,

$$[\vec{Q}_5,\mathcal{H}_{\pm}]=\pm[\vec{I},\mathcal{H}_{\pm}]\,. \qquad (12)$$

This can be used to evaluate the commutator term in Eq. (11). The second term in Eq. (11) can be evaluated by diagrammatic technique using the axial-vector current–proton vertex $(g_A/2)\gamma_\mu\gamma_5$. From Eq. (4) we find

$$a_{\pm}^{(0,0)}(p_2,p_3)\underset{p_3\to 0}{\longrightarrow}\frac{-i\sqrt{2}}{2f_\pi}E_{\pm}\bar{v}_e^{\,c}(1\pm\gamma_5)\left(\frac{mg_A\not{p}_3}{p\cdot p_3}-1+g_A\right)\gamma_5u_p \qquad (13)$$

when pion mass is neglected.

In the Introduction rates for $p\to\pi\pi e^+$ were given that were computed using the Born diagrams in Fig. 1. However, the amplitude arising from Fig. 1 is not consistent with the current-algebra relation in Eq. (13). The diagrams in Fig. 1 are expected to vary more strongly over the kinematically allowed region than other contributions, for example, from diagrams with higher-mass intermediate states (e.g., N^*). Consequently we assume that these contributions can be approximated by a constant over this kinematical region. We further assume that the amplitude for $p\to\pi^0e^+$, when the proton is virtual, is well approximated by that for a physical proton—i.e., we neglect the off-shell dependence of $E_{\pm}$. These assumptions then lead to the following ansatz for the amplitude for $p \to \pi^0\pi^0 e^+$:

$$a_\pm^{(0,0)}(p_2,p_3) = -iE_\pm \bar{v}_e^c(1 \pm \gamma_5)\left[g_r\left(\frac{\not{p}_2}{2p\cdot p_2} + \frac{\not{p}_3}{2p\cdot p_3}\right) + \frac{g_r}{m}C_0\right]\gamma_5 u_p \ . \quad (14)$$

Our guiding principle (assumption) here is that the (PCAC) (partial conservation of axial-vector current) constraints are to be satisfied by adding terms to the *lowest* angular momentum states for *each* isospin value.

The decay rate following from this amplitude is easily calculated when the mass of the pion is neglected:

$$\Gamma(p \to \pi^0\pi^0 e^+) = \frac{(|E_+|^2 + |E_-|^2)}{256\pi^3} m g_r^{\,2} J_0 \ , \quad (15)$$

where

$$J_0 = \left(\frac{\pi^2}{3} - \frac{5}{2}\right) - C_0 + \tfrac{1}{3}C_0^{\,2} \ ,$$

or equivalently

$$\frac{\Gamma(p \to \pi^0\pi^0 e^+)}{\Gamma(p \to \pi^0 e^+)} = \frac{g_r^{\,2}}{32\pi^2} J_0 \ . \quad (16)$$

The constant C_0 can be determined by requiring that the limit of Eq. (14) when $p_3 \to 0$ agrees with Eq. (13). The Goldberger-Treiman[9] relation

$$g_r = \frac{\sqrt{2}\, m g_A}{f_\pi} \quad (17)$$

then gives

$$C_0 = \frac{1}{2}\left(3 - \frac{1}{g_A}\right) \ . \quad (18)$$

Inserting this into Eq. (16) yields $\Gamma(p \to \pi^0\pi^0 e^+) \approx 0.06\Gamma(p \to \pi^0 e^+)$. Since the amplitude for $p \to \pi\pi(I=2)e^+$ vanishes, the rate for $p \to \pi^0\pi^0 e^+$ is one third that for $p \to \pi\pi(I=0)e^+$. Thus

$$\Gamma(p \to \pi\pi(I=0)e^+) \approx 0.17(p \to \pi^0 e^+) \ . \quad (19)$$

Next we consider the charged-pion final state and the constraints imposed on the invariant amplitudes

$$a_\pm^{(+,-)}(p_2,p_3) \equiv \langle e^+(p_1)\pi^+(p_2)\pi^-(p_3)|\mathcal{H}_\pm(0)|p\rangle \quad (20)$$

by current algebra. Proceeding as before, we find

$$a_\pm^{(+,-)}(p_2,0) = \mp\frac{i}{f_\pi}\langle e^+(p_1)\pi^+(p_2)|[I_+,\mathcal{H}_\pm(0)]|p\rangle \ , \quad (21)$$

where Eq. (12) has been used. There is no pole term in this amplitude and the isospin operator acting on the states yields

$$a_\pm^{(+,-)}(p_2,p_3) \underset{p_3\to 0}{\longrightarrow} \frac{-i}{f_\pi}\sqrt{2}\, E_\pm \bar{v}_e^c(1\pm\gamma_5)\gamma_5 u_p \ . \quad (22)$$

Alternatively, when the π^+ is removed from the final state of the matrix element in Eq. (21) and its momentum is taken to zero, we find that[10]

$$a_\pm^{(+,-)}(0,p_3) = \lim_{p_2\to 0} -\frac{p_2^\mu}{f_\pi}\int d^4x\, e^{ip_2\cdot x}\langle e^+(p_1)\pi^-(p_3)|T(A_\mu^{(-)}(x)\mathcal{H}_\pm(0))|p\rangle \ . \quad (23)$$

The commutator terms vanish in this case and the right-hand side of Eq. (23) can be evaluated with diagrammatic techniques using the proton–neutron–axial-vector-current coupling $+g_A\gamma_\mu\gamma_5$. The result is

$$a_\pm^{(+,-)}(p_2,p_3) \underset{p_2\to 0}{\longrightarrow} \frac{i\sqrt{2}\,g_A}{f_\pi}E_\pm \times \bar{v}_e^c(1\pm\gamma_5)\left(1 + \frac{\not{p}_2 m}{p\cdot p_2}\right)\gamma_5 u_p \quad (24)$$

when the pion mass is neglected.

The amplitudes for $p \to \pi\pi(I=0)e^+$ and $p \to \pi\pi(I=1)e^+$ are related to those for $p \to \pi^+\pi^- e^+$ by the relations

$$a_\pm^{(I=0)}(p_2,p_3) = \tfrac{1}{2}\sqrt{3}\,[a_\pm^{(+,-)}(p_2,p_3) + a_\pm^{(+,-)}(p_3,p_2)] \quad (25)$$

and

$$a_\pm^{(I=0)}(p_2,p_3) = \frac{1}{\sqrt{2}}[a_\pm^{(+,-)}(p_2,p_3) - a_\pm^{(+,-)}(p_3,p_2)] \ . \quad (26)$$

Using Eqs. (24), (23), and (4)

$$a_\pm^{(I=0)}(p_2,p_3) \underset{p_2\to 0}{\longrightarrow} \frac{i\sqrt{6}}{2f_\pi}E_\pm \bar{v}_e^c(1\pm\gamma_5) \times\left(\frac{mg_A}{p\cdot p_2}\not{p}_2 - 1 + g_A\right)\gamma_5 u_p \quad (27)$$

and

$$a_\pm^{(I=1)}(p_2,p_3) \underset{p_2\to 0}{\longrightarrow} \frac{i}{f_\pi}E_\pm \bar{v}_e^c(1\pm\gamma_5) \times\left(\frac{g_A m}{p\cdot p_2}\not{p}_2 + 1 + g_A\right)\gamma_5 u_p \ . \quad (28)$$

The isospin-zero amplitude in Eq. (27) leads to the same result as was derived from our discussion of $p \to \pi^0\pi^0 e^+$. The isospin-one amplitude resulting from the Born diagrams in Fig. 1 does not satisfy the constraint given in Eq. (28). A simple ansatz for the $I=1$ amplitude that is consistent with current algebra, Bose statistics, and with our intuition that most of the kinematical variation of the amplitude (apart from the effects of $\pi\pi$ final-state interactions) comes from the Born diagrams of Fig. 1 is

$$a_\pm^{(I=1)}(p_2,p_3) = i\sqrt{2}\, E_\pm g_r \bar{v}_e^c (1\pm\gamma_5) \times\left[\left(\frac{\not p_2}{2p\cdot p_2} - \frac{\not p_3}{2p\cdot p_3}\right) + \frac{C_1}{m^2}(\not p_2 - \not p_3) + \frac{2}{m^3} D_1 p\cdot(p_2-p_3)\right]\gamma_5 u_p\ , \quad (29)$$

where C_1 and D_1 are constants. The rate following from this amplitude is

$$\Gamma(p\to\pi\pi(I=1)e^+) = \frac{(|E_+|^2+|E_-|^2)}{128\pi^3} m g_r^{\,2} J_1 \quad (30)$$

when the pion mass is neglected, and where

$$J_1 = [(\tfrac{7}{2} - \tfrac{1}{3}\pi^2) + \tfrac{1}{3}C_1 + \tfrac{1}{6}C_1^2 - \tfrac{1}{6}C_1 D_1 - \tfrac{1}{9}D_1 + \tfrac{1}{15}D_1^2]\ ,$$

or equivalently

$$\frac{\Gamma(p\to\pi\pi(I=1)e^+)}{\Gamma(p\to\pi^0 e^+)} = \frac{g_r^{\,2}}{16\pi^2} J_1\ . \quad (31)$$

The constants C_1 and D_1 are constrained by Eq. (28) to satisfy

$$C_1 - D_1 = -\frac{1}{2}\left(1-\frac{1}{g_A}\right)\ .$$

To parametrize the freedom in the choice of C_1 and D_1 we write

$$C_1 = -\frac{1}{2}\left(1-\frac{1}{g_A}\right)(1-b)\ ,$$
$$D_1 = +\frac{1}{2}\left(1-\frac{1}{g_A}\right)b\ . \quad (32)$$

Thus the choice $b=0$ means that $D_1=0$, and $b=1$ means that $C_1=0$.

In this section we have attempted to improve the naive estimates made in the Introduction using current algebra to gain information about the amplitudes for $p\to\pi\pi e^+$ when one of the pions is soft. We then extrapolated over the whole kinematical region assuming that most of the variation of the amplitude arises from the Born diagrams in Fig. 1. Recall that this procedure resulted in a reduction of the rate for $p\to\pi\pi(I=0)e^+$ by roughly a factor of 10 but a negligible reduction in the rate for $p\to\pi\pi(I=1)e^+$ for $b=0$ or 1 (see Table I).

TABLE I. $\Gamma(p\to\pi\pi e^+)/\Gamma(p\to\pi^0 e^+)$.

	Born	Born + PCAC	Born + PCAC + rescattering
$I=0$	1.38	0.17	0.24
$I=1$	0.24	0.24 ($b=0$) 0.23 ($b=1$)	1.6 ($b=0$) 1.5 ($b=1$)

However, these computations have neglected the effects of strong-interaction final-state $\pi\pi$ interactions. Since there is considerable phase space available for the pions, their final-state interactions can be dynamically significant. In the case where the pions are in an $I=1$ state, a large enhancement of the rate from the final-state interactions is expected since they can form a ρ resonance. In Sec. III we estimate the effects of final-state interactions for both the $J=0$ and 1 final states.

III. FINAL-STATE INTERACTIONS

Up to this point our discussion has neglected the strong interactions of the pions in the final state. To include these effects we must first decompose the amplitudes for $p\to\pi\pi e^+$ into partial waves. The $p\to\pi\pi(I=0,1)e^+$ amplitudes $a_\pm^{(I=0,1)}$ satisfy a unitarity constraint which follows from a consideration of the crossed diagram shown in Fig. 2. Let s be the square of the $\pi\pi$ center-of-mass momentum. In the "physical" region, $s>4\mu^2$, the absorptive part of the $ep\to\pi\pi$ amplitudes $a_\pm^{(I=0,1)}$ satisfy[11]

$$\mathrm{Abs}\,a_\pm^{(I=0,1)}(s;\hat{p}\cdot\hat{q}) = \frac{1}{32\pi}\left(1-\frac{4\mu^2}{s}\right)^{1/2}\int\frac{d\Omega_l}{4\pi}\mathfrak{M}^{(I=0,1)*}(s;\hat{l}\cdot\hat{q}) \times a_\pm^{(I=0,1)}(s;\hat{l}\cdot\hat{p})\ . \quad (33)$$

Here we are working in the $\pi\pi$ center-of-mass coordinate system where

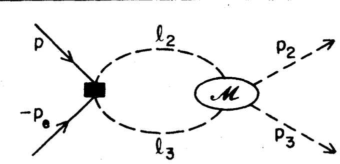

FIG. 2. $ep\to\pi\pi$ scattering diagram used in derivation of Eq. (33).

$$p_2+p_3=k=(\sqrt{s},\vec{0}),\quad \tfrac{1}{2}(p_2-p_3)=q=(0,\vec{q}),$$
$$l_2+l_3=k=(\sqrt{s},\vec{0}),\quad \tfrac{1}{2}(l_2-l_3)=l=(0,\vec{l}),$$

and

$$p=(E,\vec{p}),\quad -p_e=(e,-\vec{p}).$$

From $p-p_e=k=(\sqrt{s},\vec{0})$ it is easy to show that for $\mu=0$,

$$e=\frac{s-m^2}{2\sqrt{s}},\quad E=\frac{s+m^2}{2\sqrt{s}}\ \text{and}\ |\vec{p}|=\frac{m^2-s}{2\sqrt{s}}\ . \tag{34}$$

In Eq. (33) $\mathfrak{M}^{(I=0,1)}$ is the isospin-zero or -one pion-pion scattering amplitude. The $p\to\pi\pi e^+$ amplitude $a_\pm^{(I=0,1)}$ can be expressed in terms of two types of form factors. Suppressing the isospin superscripts

$$a_\pm=i\bar{v}_e^c(1\pm\gamma_5)(A_\pm+2B_\pm\not{q})\gamma_5 u_p\,. \tag{35}$$

The unitarity constraint for the crossed process $ep\to\pi\pi$, given in Eq. (33), implies that

$$\bar{u}_e^c(1\pm\gamma_5)(\text{Im}A_\pm-2\,\text{Im}B_\pm\not{q})u_p\equiv\bar{u}_e^c(1\pm\gamma_5)(\tilde{a}_\pm-2\not{b}_\pm)u_p\,, \tag{36}$$

where

$$\tilde{a}_\pm=\frac{1}{32\pi}\left(1-\frac{4\mu^2}{s}\right)^{1/2}\int\frac{d\Omega_l}{4\pi}\mathfrak{M}^*(s;\hat{l}\cdot\hat{q})A_\pm(s;\hat{p}\cdot\hat{l})\,, \tag{37}$$

$$\vec{b}_\pm=\frac{1}{32\pi}\left(1-\frac{4\mu^2}{s}\right)^{1/2}\int\frac{d\Omega_l}{4\pi}\mathfrak{M}^*(s;\hat{l}\cdot\hat{q})B_\pm(s;\hat{p}\cdot\hat{l})\,\vec{l}\,, \tag{38}$$

and $b_\pm^0=0$.

Multiplying Eq. (36) by $\bar{u}_p u_e^c$ and summing over the electron and proton spins gives

$$\text{Im}A_\pm-\frac{2mp_e\cdot q}{p_e\cdot p}\,\text{Im}B_\pm=\tilde{a}_\pm-\frac{2mb_\pm\cdot p_e}{p_e\cdot p}\,. \tag{39}$$

The $\pi\pi$ scattering amplitude $\mathfrak{M}$ has the partial-wave expansion

$$\mathfrak{M}(s;\hat{l}\cdot\hat{q})=\frac{32\pi}{(1-4\mu^2/s)^{1/2}}\times\sum_{J=0}^{\infty}(2J+1)P_J(\hat{l}\cdot\hat{q})e^{i\delta_J}\sin\delta_J\,. \tag{40}$$

The phase shift for the Jth partial wave δ_J depends only on s, and thus it is evident from Eq. (39) that when one makes the expansion

$$A_\pm(s;z)+\frac{2m|\vec{q}|z}{\sqrt{s}}B_\pm(s;z)=\sum_J(2J+1)P_J(z)f_\pm^J(s)\,, \tag{41}$$

where $z=\hat{p}\times\hat{q}$, the unitarity constraint becomes

$$\text{Im}f_\pm^J(s)=e^{-i\delta_J}\sin\delta_J f_\pm^J(s)\,. \tag{42}$$

Next we multiply Eq. (36) by $\bar{u}_p\not{C}u_e^c$ and sum over electron and proton spins. Choosing the four-vector $C=(0,\vec{p}\times\vec{q})$, we find that

$$\text{Im}B_\pm|\vec{p}\times\vec{q}|^2=(\vec{p}\times\vec{q})(\vec{p}\times\vec{b}_\pm) \tag{43}$$

or equivalently, using Eq. (37),

$$\text{Im}B_\pm=\frac{1}{32\pi}\left(1-\frac{4\mu^2}{s}\right)^{1/2}\int\frac{d\Omega_l}{4\pi}\mathfrak{M}^*(s;\hat{l}\cdot\hat{q})B_\pm(s;\hat{p}\cdot\hat{l})\times\left[\frac{(\hat{q}\cdot\hat{l})-(\hat{p}\cdot\hat{l})(\hat{q}\cdot\hat{p})}{1-(\hat{q}\cdot\hat{p})^2}\right]. \tag{44}$$

From a standard orthogonality relation[12] for P_J' it follows that the partial-wave expansion of $B_\pm$ is

$$B_\pm(s;z)=\sum_J\frac{(2J+1)}{[J(J+1)]^{1/2}}P_J'(z)g_\pm^J(s)\,, \tag{45}$$

and the unitarity constraint for the g_J is

$$\text{Im}g_\pm^J(s)=e^{-i\delta_J}\sin\delta_J g_\pm^J(s)\,. \tag{46}$$

The decay rate for $p\to\pi\pi e^+$ can be written as a sum of squares of the partial-wave amplitudes $f_\pm^J(s)$ and $g_\pm^J(s)$:

$$\Gamma=\frac{m}{2^8\pi^3}\int_{4\mu^2}^{m^2}ds\left(1-\frac{4\mu^2}{s}\right)^{1/2}\left(1-\frac{s}{m^2}\right)^2\sum_J(2J+1)[(|f_+^J|^2+|f_-^J|^2)+(s-4\mu^2)(|g_+^J|^2+|g_-^J|^2)]\,. \tag{47}$$

The partial-wave amplitudes which follow from the expressions for the (Born) decay amplitudes given in Sec. II are real on the positive real s axis, $s>0$. These partial-wave amplitudes we denote by $\bar{f}_\pm^J(s)$ and $\bar{g}_\pm^J(s)$. The bar signifies that these are not the same as the true partial-wave amplitudes $f_\pm^J(s)$ and $g_\pm^J(s)$ which have a cut for $s>4\mu^2$ and satisfy the unitarity constraints given in Eqs. (42) and (46). A simple form for the partial-wave amplitudes $f_\pm^J(s)$ and $g_\pm^J(s)$ that is consistent with the unitarity constraint is

$$f_\pm^J(s)=\left[\frac{D_J(\mu^2)}{D_J(s)}\right]\bar{f}_\pm^J(s) \tag{48}$$

and

$$g_\pm^J(s)=\left[\frac{D_J(\mu^2)}{D_J(s)}\right]\bar{g}_\pm^J(s)\,. \tag{49}$$

The quantities $\bar{f}_\pm^J$ and $\bar{g}_\pm^J$ can be deduced from the expressions for the decay amplitudes given in Sec. II. The Omnes D_J function is defined by[13]

$$D_J(s)=\exp\left[-\frac{1}{\pi}\int_{4\mu^2}^{\infty}ds'\,\frac{\delta_J(s')}{s'-s-i\epsilon}\right] \tag{50}$$

and takes into account the effects of final-state $\pi\pi$ interactions in the Jth partial wave. The amplitudes $f_\pm^J(s)$ and $g_\pm^J(s)$ defined in Eqs. (48) and (49) satisfy the unitarity constraints given in Eqs. (42) and (46) because $D_J(s)$ has a cut for $s>4\mu^2$ and equals $|D_J(s)|\exp(-i\delta_J)$ in this region. The normalization factor $D_J(\mu^2)$ was inserted in Eqs. (48) and (49) so that the decay amplitudes following from $f_\pm^J$ and $g_\pm^J$ will satisfy the current-algebra constraints which restrict the amplitude in the neighborhood of $s=\mu^2$. Note that since the functions $D_J(s)$ are real and slowly varying for $s<4\mu^2$, $\mathrm{Im}f_\pm^J(s)\approx\mathrm{Im}\bar{f}_\pm^J(s)$ and $\mathrm{Im}g_\pm^J(s)\approx\mathrm{Im}\bar{g}_\pm^J(s)$ on the left-hand cut.

The isospin-zero amplitude $a_\pm^{(I=0)}$ only gets contributions from even partial waves. The form for $a_\pm^{(I=0)}$ given in Sec. II [see Eq. (14)] can be cast into the form of Eq. (35). The resulting form factors $\bar{A}_\pm^{(I=0)}$ and $\bar{B}_\pm^{(I=0)}$ are

$$\bar{A}_\pm^{(I=0)}=-\sqrt{3}E_\pm\frac{g_r m}{2}\left(\frac{1}{2p\cdot p_2}+\frac{1}{2p\cdot p_3}-\frac{2}{m^2}C_0\right) \tag{51}$$

and

$$\bar{B}_\pm^{(I=0)}=\frac{\sqrt{3}}{2}E_\pm g_r\left(\frac{1}{2p\cdot p_2}-\frac{1}{2p\cdot p_3}\right). \tag{52}$$

In Eqs. (51) and (52) and hereafter *the pion mass is neglected*. Again we use a bar to denote that final-state interactions have not been included. The s-wave amplitude $\bar{f}^0(s)$ following from these form factors can be derived by inverting Eq. (41). We find that

$$\bar{f}_\pm^0(s)=\sqrt{3}E_\pm\frac{g_r}{m}\left\{C_0-\frac{2m^2}{m^2-s}\left[1-\frac{s}{m^2-s}\ln\left(\frac{m^2}{s}\right)\right]\right\}. \tag{53}$$

The second term in the square brackets depends on s and arises from the Born diagrams in Fig. 1. The first term is a constant independent of s and was added to make the amplitude consistent with current algebra.

The s-wave contribution dominates the rate for $p\to\pi\pi(I=0)e^+$ so we shall neglect higher partial waves.[14] The s-wave isospin-zero π-π phase shift δ_0 is consistent with the presence of a broad resonance of mass 700 MeV and width $\simeq$ 500 MeV. Therefore, we assume that in the physical region $0\leq s\leq m^2$ the function $D_0(s)$ has the form

$$\frac{D_0(s)}{D_0(0)}=\left(1-\frac{s}{s_0}\right)-i\gamma_0\sqrt{s}\,. \tag{54}$$

The parameters s_0 and γ_0 are related to the s-wave phase shift. Using

$$e^{i\delta_0}\sin\delta_0=\frac{1}{2i}\left[\frac{D_0^*(s)-D_0(s)}{D_0(s)}\right] \tag{55}$$

$$=\gamma_0 s_0\sqrt{s}\left[(s-s_0)-i\gamma_0 s_0\sqrt{s}\,\right]^{-1}, \tag{56}$$

it is evident that s_0 can be identified with the mass of the s-wave "resonance" and γ_0 controls its width. Therefore, the values $s_0\cong 0.5$ GeV^2 and $\gamma_0\cong 0.8$ GeV^{-1} are adopted. Performing the required integration [cf. Eq. (46)] we find that $\pi\pi$ final-state interactions enhance the rate for $p\to\pi\pi(I=0)e^+$ by about a factor of 1.5 so that $\Gamma(p\to\pi\pi(I=0)e^+)\approx 0.2\Gamma(p\to\pi^0e^+)$.

The isospin-one amplitude $a_\pm^{(I=1)}$ gets contributions only from odd partial waves. The expression for $a_\pm^{(I=1)}$ in Eq. (29) of Sec. II can be put in the form of Eq. (35). Then

$$\bar{A}_\pm^{(I=1)}=\sqrt{2}E_\pm\left[\frac{g_r m}{4}\left(-\frac{1}{p\cdot p_2}+\frac{1}{p\cdot p_3}\right)+\frac{g_r D_1}{m^3}4p\cdot q\right] \tag{57}$$

and

$$\bar{B}_\pm^{(I=1)}=\sqrt{2}E_\pm g_r\left[\frac{1}{4}\left(\frac{1}{p\cdot p_2}+\frac{1}{p\cdot p_3}\right)+\frac{1}{m^2}C_1\right]\,. \tag{58}$$

As in the isospin-zero case, the rate for $p\to\pi\pi(I=1)e^+$ is dominated by the contribution of the lowest partial wave.[14] Consequently we shall restrict our attention to the p-wave amplitudes $f_\pm^1(s)$ and $g_\pm^1(s)$, ignoring the rescattering corrections to the (<1%) contributions of higher partial waves. Inverting Eqs. (41) and (45) we find that[15]

$$\bar{f}_\pm^1(s)=\frac{\sqrt{2}}{m}E_\pm g_r\left\{\frac{2sm^2}{(m^2-s)^2}\left[\frac{m^2+s}{m^2-s}\ln\left(\frac{m^2}{s}\right)-2\right]-\frac{1}{6}\left(1-\frac{1}{g_A}\right)\left(1-b\,\frac{s}{m^2}\right)\right\} \tag{59}$$

and

$$\bar{g}_\pm^1(s)=-E_\pm\frac{g_r}{m^2}\left\{\frac{m^4}{(m^2-s)^2}\left[\frac{4s}{m^2-s}\ln\left(\frac{m^2}{s}\right)-\frac{2(m^2+s)}{m^2}\right]-\frac{1}{3}\left(1-\frac{1}{g_A}\right)(1-b)\right\}\,. \tag{60}$$

The last terms in Eqs. (59) and (60) were added to the Born amplitudes to comply with current-algebra restrictions.

The final-state interactions of two pions in a p wave are dominated by the ρ resonance. Therefore, we assume that in the physical region $0 < s < m^2$ the function $D_1(s)$ has the form

$$\frac{D_1(s)}{D_1(0)} = \left(1 - \frac{s}{s_1}\right) - i\gamma_1 s^{3/2} \,. \tag{61}$$

Fitting s_1 and γ_1 to the mass and width of the ρ resonance gives $s_1 = 0.59$ GeV^2 and $\gamma_1 = 0.41$ GeV^{-3}. Performing the required integration we find the final-state interactions enhance the rate for $p \to \pi\pi(I=1)e^+$ by about a factor of 6 and hence $\Gamma(p \to \pi\pi(I=1)e^+) \approx 1.5\Gamma(p \to \pi^0 e^+)$ for both $b = 0$ and 1. Note that for $b = 0$ the piece added to the amplitudes $f_\pm^1(s)$ and $g_\pm^1(s)$ to satisfy the current-algebra constraints can be interpreted as arising from a bare ρ-nucleon-positron coupling of the form

$$\langle e^+\rho \,|\, \mathcal{H}_\pm \,|\, p\rangle = F_\pm \bar{v}_e^c (1 \pm \gamma_5) \not{\epsilon} u_p \,, \tag{62}$$

where ϵ is the ρ-polarization vector. Neglecting the off-shell dependence of the form factor $F_\pm$ we have that

$$F_\pm = g_r E_\pm \left(\frac{m_\rho}{m}\right)^2 \frac{1}{2f_{\rho\pi\pi}} (1 - 1/g_A) \,, \tag{63}$$

where m_ρ is the ρ-meson mass and $f_{\rho\pi\pi}$ is the ρ-two-pion coupling constant.

IV. DISCUSSION AND CONCLUSIONS

In this paper we have attempted to make a simple estimate of the ratio of two-pion to one-pion final states in proton decay without assuming a particular grand unified model. We found that for the isospin-zero two-pion final state the pole or Born contribution gave a ratio around one but when PCAC was imposed the ratio was reduced by almost an order of magnitude. In the isospin-one case the Born diagrams gave a small ratio of about one fifth. However, in this case the current-algebra constraints caused only a slight reduction in the ratio of two-pion to one-pion final states. Finally the effects of final-state strong interactions in the lowest partial waves were estimated using familiar dispersion-relation techniques (whose validity it would be inappropriate to discuss here) and were found to enhance the two-pion rates substantially. This oscillatory history is shown in Table I where the rates include the Born contributions to the higher partial waves. Note that the same results would have been achieved if the Born amplitude was first corrected for the effects of final-state interactions and then consistency with the results of current algebra was imposed.

The imposition of the PCAC condition is unique if one adds only constants (no growth in s) to the lowest possible partial-wave amplitudes.[16] In the $I = 1$ case this corresponds to the choice $b = 0$. However, because of the additional s dependence in the rate associated with $g_\pm^1$ [cf. Eq. (47)], we do not consider the choice $b = 0$ to be compelling. Fortunately PCAC has little effect on this amplitude and the rate is insensitive to the value of b.

The large rate for the isospin-one two-pion final state is more or less in qualitative agreement with bag-model estimates of $p \to \rho e^+$. We have also found a significant rate for isospin-zero two-pion final states. Because of the large amount of phase space available to the pions one should be suspect of the dramatic cancellation which occurred when the Born amplitude was adjusted to satisfy the current-algebra constraints. The rate for $p \to \pi\pi(I=0)e^+$ may be somewhat larger than we have calculated.[17]

Finally we note that other three-body modes, such as $n \to \pi^0\pi^- e^+$, $p \to \pi^+\pi^0\bar{\nu}$, and $n \to \pi\pi\bar{\nu}$ follow from our estimates by simple arguments [e.g., from isospin $\Gamma(n \to \pi\pi(I=1)e^+) = 2\Gamma(p \to \pi\pi(I=1)e^+)$].

ACKNOWLEDGMENTS

This work was supported by the Department of Energy under Contracts Nos. DE-AC03-76SF00515 (R.B.) and E(11-1)3230 (L.F.A.) and by the Harvard University Society of Fellows (M.B.W.). We are grateful to T. Tsao for assistance with some of the numerical work.

[1]See, for example, H. Georgi and S. L. Glashow, Phys. Rev. Lett. 32, 438 (1974); A. J. Buras *et al.*, Nucl. Phys. B135, 66 (1978) where the SU(5) grand unified model is discussed.
[2]H. Georgi, H. R. Quinn, and S. Weinberg, Phys. Rev. Lett. 33, 451 (1974); W. J. Marciano, Phys. Rev. D 20, 274 (1979); D. A. Ross, Nucl. Phys. B140, 1 (1978); T. J. Goldman and D. A. Ross, Phys. Lett. 84B, 208 (1979); L. Hall, Harvard University Report No. HUTP-80/A024, 1980 (unpublished).
[3]S. Weinberg, Phys. Rev. Lett. 43, 1566 (1979); F. Wilczek and A. Zee, *ibid.* 43, 1571 (1979); Phys. Lett. 88B, 311 (1979).
[4]See, also, L. F. Abbott and M. B. Wise, Phys. Rev. D

22, 2208 (1980).

[5]We make this approximation for the remainder of the paper.

[6]M. Machacek, Nucl. Phys. B159, 37 (1979); C. Jarlskog and F. Yndurain, *ibid.* B149, 29 (1979); J. F. Donoghue, Phys. Lett. 92B, 99 (1980); E. Golowich, Phys. Rev. D 22, 1148 (1980); G. Karl and H. J. Lipkin, Phys. Rev. Lett. 45, 1223 (1980); M. B. Gavela *et al.*, Orsay Report No. LPTHE 80/15 (unpublished); Phys. Rev. D 23, 1580 (1981); Y. Tomozawa, Phys. Rev. Lett. 46, 463 (1981); H. M. Din, G. Girardi, and P. Sorba, Phys. Lett. 91B, 77 (1980).

[7]We compute these Born diagrams using a pion-nucleon coupling $g_r(\bar{N}\vec{\tau}\cdot\vec{\pi}\gamma_5 N)$. If a derivate pion-nucleon coupling were used a different result would be obtained. When these estimates are improved by demanding consistency with current algebra the results will be independent of the type of pion-nucleon coupling used in the computation of the Born-diagram amplitudes.

[8]For reviews of current-algebra techniques see S. B. Treiman, in *Lectures on Current Algebra and Its Applications* (Princeton University Press, Princeton, N.J., 1970); S. Coleman, in *Hadrons and Their Interactions*, edited by A. Zichichi (Academic, New York, 1968); S. L. Adler and R. F. Dashen, in *Current Algebras and Applications to Particle Physics* (Benjamin, New York, 1968).

[9]M. L. Goldberger and S. B. Treiman, Phys. Rev. D 11, 1178 (1958).

[10]We use the convention $\pi^+ = -\partial^\mu A_\mu^{(-)}/f_\pi \mu^2$ which gives pion fields with the same phase convention that is usually adopted for the spherical harmonics.

[11]For a review of the techniques used to derive such relations see G. Barton, in *Dispersion Techniques in Field Theory* (Benjamin, New York, 1965). We use the same symbol for the $p \to \pi\pi e^+$ and $ep \to \pi\pi$ amplitudes since they are related by crossing.

[12]That is,

$$\int \frac{d\Omega_l}{4\pi} P_J(\hat{l}\cdot\hat{q}) P'_L(\hat{p}\cdot\hat{l}) \left[\frac{(\hat{q}\cdot\hat{l}) - (\hat{p}\cdot\hat{l})(\hat{q}\cdot\hat{p})}{1-(\hat{p}\cdot\hat{q})^2}\right] = \frac{\delta_{JL}}{(2J+1)} P'_J(\hat{p}\cdot\hat{q}) \, .$$

[13]R. Omnes, Nuovo Cimento 8, 316 (1958); N. I. Muskelishvili, *Singular Integral Equations* (Nordhoof, Groningen, 1958).

[14]We have verified this by explicit computation.

[15]To invert Eq. (46) we used the orthogonality relation

$$\int_{-1}^{+1} \frac{dz}{2} P'_J(z) P'_L(z)(1-z^2) = \frac{J(J+1)}{2J+1}\delta_{JL} \, .$$

[16]In the case $p \to \pi\pi(I=1)e^+$ a term $p(p_2 - p_3)/[(p_2+p_3)^2 - m^2]$ in the form factor $\bar{A}_\pm^{(I=1)}$ contributes a constant to the partial-wave amplitude $\bar{f}_\pm^1$. However, such a term is unacceptable since it gives a form factor $A_\pm^{(I=1)}$ with a pole at $s = m^2$ for fixed t.

[17]The uncertainties are not too great because the $I=0$ amplitude is dominated by contributions from small s. However, for $p \to \pi\pi(I=1)e^+$ large-s contributions are important and our results have dubious validity.