Experiment 1-1:

Uncertainty and Error

1. Introduction

There is no such thing as a perfect measurement! All measurements have errors and uncertainties, no matter how hard we might try to minimize them. Understanding possible errors is an important issue in any experimental science. The conclusions we draw from the data, and especially the strength of those conclusions, will depend on how well we control the uncertainties.

Lets look at an *example*:
You measure two values 2.5 and 1.5. From theory, the expected value is 2.3, so the value 2.5 almost agrees, whereas 1.5 is far off. But if you take into account the uncertainties (i.e. the interval in which your result is expected to lie), neither may be far off. For experimental uncertainties of 0.1 and 1.0, respectively, your two measured values may be expressed 2.5 ± 0.1 and 1.5 ± 1.0. The expected value falls within the range of the second measurement but not of the first!

This first lab deals exclusively with this important subject. The techniques studied here will be essential for the rest of this two-semester lab course. The issues are important in order to arrive at good judgements in any field (like medicine) in which it is necessary to understand, not just numerical results, but the uncertainties associated with those results.

2. Theory

2.1 Types of Uncertainties

Uncertainty in a measurement can arise from three possible origins: the measuring device, the procedure of how you measure, and the observed quantity itself. Usually the largest of these will determine the uncertainty in your data.

There are two basically different types of uncertainties: systematic and random uncertainties.

2.1.1 Systematic Uncertainties

Systematic uncertainties or systematic errors always bias results in one specific direction. Your result will consistently be too high or too low.

An *example* of a systematic error follows. Assume you want to measure the length of table in cm using a meter stick. But suppose the meter stick has been manufactured incorrectly or the stick is made of metal that has contracted due to the temperature in the room, so that the stick is less than one meter long. Clearly all the calibrations on the stick are smaller than they should be. Your numerical value for the length of the table will then always be too large no matter how often or how carefully you measure. Another example might be reading temperature from a mercury thermometer in which a bubble is present in the mercury column.

Systematical errors are usually due to imperfections in the equipment, improper or biased observation, or by the presence of additional physical effects you did not take into account. (An example might be an experiment on forces and acceleration in which friction in the setup and is not taken into account!)

In performing experiments, try to estimate the effects of as many systematic errors as you can, and then remove or correct for the most important. By being aware of the sources of systematic error beforehand, it is often possible to perform experiments with sufficient care to compensate for weaknesses in the equipment.

2.1.2 Random Uncertainties

In contrast to systematic uncertainties, random uncertainties are unbiased -- meaning it is equally likely that an individual measurement is too high or too low. Random uncertainty means that several measurements of a quantity will not always come out the same but will spread around a mean value. The mean value will be much closer to the “real” value than any individual measurement.

From your everyday experience you might now say, "Stop! Whenever I measure the length of a table with a meter stick I get exactly the same value no matter how often I measure it!" This may happen if your meter stick is insensitive to random measurements, because you use a coarse scale (like mm) and you always read the length to the nearest mm. But if you would use a meter stick with a finer scale, or if you interpolate to fractions of a mm, you would definitely see the spread. As a general rule, if you do not get a spread in values, you can improve your measurements by using a finer scale or by interpolating between the finest scale marks on the ruler.

How can one reduce the effect of random uncertainties? Consider the following *example*. Ten people measure the time of a sprinter using stopwatches. It is very unlikely that each of the ten stopwatches will show exactly the same result. You will observe a spread in the results. Even if each started their watch at exactly the same time (unlikely) some persons will have stopped the watch early, some of them late. But if you *average* the times of the ten stop watches, the *mean* value will be a better estimate of the true value than any individual measurement, since the effects of the people who stop early will compensate for those who stop late. In general, making multiple measurements and averaging can reduce the effect of random uncertainty.

Remarks:
We usually specify any measurement by including an estimate of the random uncertainty. (Since the random uncertainty is unbiased we note it with a ± sign). So if we measure a time of 7.6 seconds, but we expect a spread of about 0.2 seconds, we write as a result:

$$t = (7.6 \pm 0.2)\ \text{s}$$

indicating the uncertainty of this measurement is 0.2 s or about 3%.

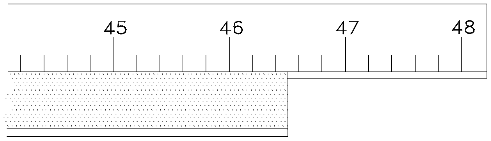
From hereon, we use the term “uncertainty” to refer to random uncertainty, whereas systematic uncertainty will be specified as “error”.

2.2 Numerical Estimates of Uncertainties

For this laboratory, we will estimate uncertainties with three approximation techniques, which we describe below. You should note which technique you are using in a particular experiment.

2.2.1 Upper Bound

Most of our measuring devices in this lab have scales that are coarser than the ability of our eyes to measure.



For example in the figure above, we can definitely say that our result is somewhere between 46.4 cm and 46.6 cm. We assume as an *upper* bound of our uncertainty, an amount equal to half this width (in this case 0.1cm). The final result can be written

$$l = (46.5 \pm 0.1)\ \text{cm}.$$

2.2.2 Estimation from the Spread (2/3 method)

For data in which there is random uncertainty, we usually observe individual measurements to cluster around the mean and drop in frequency as the values get further from the mean (in both directions).[*] Find the interval around the mean that contains about 2/3 of

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[*] There is a precise mathematical procedure to obtain uncertainties (standard deviations) from a number of measured values. Here we will apply a simple “rule of thumb” that avoids the more complicated mathematics of that technique. The uncertainty using the standard deviation for the group of values in our example below is 0.2.

the measured points: half the size of this interval is a good estimate of the uncertainty in each measurement.

Example:
You measure the following values of a specific quantity:

9.7, 9.8, 10, 10.1, 10.1, 10.3

The mean of these six values is 10.0. The interval from 9.75 to 10.2 includes 4 of the 6 values; we therefore estimate the uncertainty to be 0.225. The result is that the best estimate of the quantity is 10.0 and the uncertainty of a single measurement is 0.2.[*]

2.2.3 Square-Root Estimation in Counting

For inherently random phenomena that involve counting individual events or occurrences, we measure only a single number N. This kind of measurement is relevant to counting the number of radioactive decays in a specific time interval from a sample of material. It is also relevant to counting the number of Lutherans in a random sample of the population. The (absolute) uncertainty of such a single measurement, N, is estimated as the square root of N. As example, if we measure 50 radioactive decays in 1 second we should present the result as 50±7 decays per second. (The quoted uncertainty indicates that a subsequent measurement performed identically could easily result in numbers differing by 7 from 50.)

2.3 Number of Significant Digits

The number of significant digits in a result refers to the number of digits that are relevant. The digits may occur after a string of zeros. For example, the measurement of 2.3mm has two significant digits. This does not change if you express the result in meters as 0.0023m. The number 100.10, by contrast, has 5 significant digits!

When you record a result, how many significant digits should you keep? Let's illustrate the procedure with the following *example*. Assume you measure the diameter of a circle to be $d = 1.6232$cm, with an uncertainty of 0.102 cm. You now round your uncertainty to one or two significant digits (up to you). So (using one significant digit) we initially quote $d = (1.6232 \pm 0.1)$cm. Now we compare the mean value with the uncertainty, and keep only those digits that the uncertainty indicates are relevant. Finally, we quote the result as $d = (1.6 \pm 0.1)$cm for our measurement.

Suppose further that we wish to use this measurement to calculate the circumference c of the circle with the relation $c = \pi \cdot d$. If we use a standard calculator, we might get a 10 digit display indicating

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[*] Note that about 5% of the measured values will lie *outside* ± twice the uncertainty.

$$c = 5.099433195 \pm 0.3204424507 \text{ cm.}$$

This is not a reasonable way to write the result! The uncertainty in the diameter had only one significant digit, so the uncertainty of the circumference calculated from the diameter cannot be substantially better. Therefore we should record the final result as

$$c = 5.1 \pm 0.3 \text{ cm.}$$

(If you do intermediate calculations, it is a good idea to keep as many figures as your calculator can store. The above argument applies when you <u>record</u> your results!)

2.4 Relative and Absolute Uncertainty

There are two ways to record uncertainties: the absolute value of the uncertainty or the uncertainty relative to the mean value. So in the example above, you can write
$c = (5.1 \pm 0.3)\text{cm}$ or equally well $c = 5.1\text{cm}(1.00 \pm 0.06)$. The second form may look a bit odd, but it tells you immediately that the uncertainty is 6% of the measured value. The number 0.3cm is the absolute uncertainty and has the same units as the mean value (cm). The 0.06 (or 6%) is the relative uncertainty and has no units since it is the ratio of two lengths.

2.5 Propagation of Uncertainties

Often, we are not directly interested in a measured value, but we want to use it in a formula to calculate another quantity. In many cases, we measure many of the quantities in the formula and each has an associated uncertainty. We deal here with how to propagate uncertainties to obtain a well-defined uncertainty on a computed quantity.

2.5.1 Adding/Subtracting Quantities

Whether we <u>add or subtract</u> quantities, the uncertainties must always be <u>added</u> (never subtracted) to obtain the <u>absolute uncertainty</u> on the computed quantity.*

Take as example measuring the length of a dog. We measure the distance between the left wall and the tail of the dog and subtract the distance from the wall to the dog's nose.

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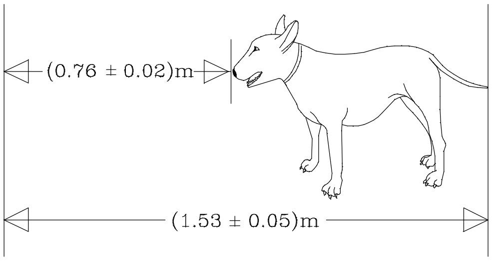
* The propagation of random uncertainties is actually slightly more complicated, but the procedure outlined here usually represents a good approximation, and it never underestimates the uncertainty. See "Remarks for Experts" at the end of section 2.5.2.



So the total length of the dog is

$$\begin{aligned} \text{length} &= (1.53 \pm 0.05)\text{ m} - (0.76 \pm 0.02)\text{ m} \\ &= (1.53 - 0.76) \pm (0.05 + 0.02)\text{ m} \\ &= (0.77 \pm 0.07)\text{ m}. \end{aligned}$$

2.5.2 Multiplying/Dividing Quantities

When we multiply or divide quantities, we add (never subtract!) the relative uncertainties to obtain the relative uncertainty on the computed quantity.

Take as example the area of a rectangle, whose individual sides are measured to be

$$a = 25.0 \pm 0.5\text{ cm} = 25.0\text{ cm }(1.00 \pm 0.02)$$
$$b = 10.0 \pm 0.3\text{ cm} = 10.0\text{ cm }(1.00 \pm 0.03).$$

The area is obtained as follows

$$\begin{aligned} area &= (25.0 \pm 0.5\text{ cm})\cdot(10.0 \pm 0.3\text{ cm}) \\ &= 25.0\text{ cm }(1.00 \pm 0.02)\cdot 10.0\text{ cm }(1.00 \pm 0.03) \\ &= (25.0\text{ cm}\cdot 10.0\text{ cm})(1.00 \pm (0.02 + 0.03)) \\ &= 250.0\text{ cm}^2\,(1.00 \pm 0.05) \\ &= 250.0 \pm 12.5\text{ cm}^2 \\ &= 250 \pm 10\text{ cm}^2. \end{aligned}$$

Note that the final step has rounded both the result and the uncertainty to an appropriate number of significant digits.

Remarks:

Note that uncertainties on quantities used in a mathematical relationship always increase the uncertainty on the result. The quantity with the biggest uncertainty usually dominates the final result. Often one quantity will have a much bigger uncertainty than all the others. In such cases, we can simply use this main contribution.

Remarks for Experts:
Our calculation of the uncertainty actually overestimates it. The correct method does not add the absolute/relative uncertainty, but rather involves evaluating the square root of the sum of the squares. For this lab, the simpler procedure described here will be adequate.

2.5.3 Square Roots

A quantity proportional to the square root of a quantity has a relative uncertainty that is smaller by a factor of 2. It follows that if you know the area of a square to be

$$\text{area} = 100 \pm 8\ \text{m}^2 = 100\ \text{m}^2(1.00 \pm 0.08)$$

then it follows that the side of the square is

$$\text{side} = 10\ \text{m}(1.00 \pm 0.04) = 10.0 \pm 0.4\ \text{m}.$$

You can convince yourself that this is true by checking it backwards using the rules described in section 2.5.2.

3. Description of Experiments

You are to perform three experiments to practice the estimation of uncertainty and the propagation of errors. These involve measuring the period of a pendulum, measuring the reaction time of a human being, and measuring the human lung volume.

3.1 Pendulum

In the first part, we measure the time it takes a pendulum to oscillate through a full cycle and compare this to the theoretical prediction. Two methods of recording the time will be used to illustrate that different ways of making a measurement can result in very different uncertainties. In each case, many measurements will be taken to demonstrate the frequency of differing values.

Since a number of measurements need to be made, you should perform this experiment in teams of two students: one student measures the time values, the other records the results.

In the first series, start and stop the clock as the pendulum gets to its maximum position. After 25 measurements, sort the measurements in bins of 50 ms. (Therefore all measurements with times between e.g. 1.25s and 1.29s count in the same bin.) Make a graph of the frequency of measurements in the bins vs. the times of the bins. Calculate the average and estimate the uncertainty using the 2/3 rule.

In the second series, observe the time as the pendulum passes a fixed characteristic mark in the background. For this you might use a chair or the leg of a table. Perform the same experiment, but order the measurements in bins of size 10 ms. Again graph your result and get the average and the uncertainty.

See if there is a substantial difference between the uncertainties of these two measurements and also if the measured values coincide within uncertainty. In addition, measure the length *l* of the pendulum from the pivot point to the center of the mass. This allows you to determine the acceleration due to gravity at the earth's surface, *g*, using the formula

$$\frac{2 \cdot p}{t} = \sqrt{\frac{g}{l}}$$

where *t* is the measured period. Finally, compare your value of *g* to the accepted value

$$g = 9.8 \text{ m/s}^2.$$

3.2 Reaction Time

Next measure your reaction time to a specific stimulus. Perform this experiment with a partner, but every student should obtain his/her own data!

One student holds a ruler at the upper end and the other student places two fingers around (but not touching) the 50cm mark of the ruler. The first student (quietly) releases the ruler and the second student tries to grab it as soon as possible after he/she sees it released. Measure the distance *s* the ruler falls. After performing this experiment 10 times (for each student), look at the spread in the data and calculate the resulting uncertainty (using the 2/3 estimate). Use the relation

$$s = \frac{1}{2} g \cdot t^2$$

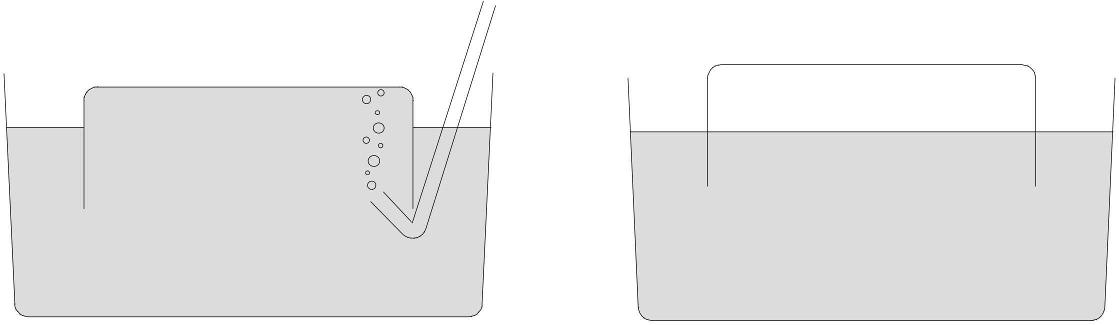
to calculate your average reaction time and the corresponding uncertainty. This experiment is an example of an indirect measurement. You measure a quantity (here the distance the ruler is falling) that you do not care about directly, but is necessary in order to calculate the quantity you want to know (the reaction time).

3.3 Lung Volume

The last experiment is to measure your lung volume. There are no detailed instructions for how you should perform the experiment, so this is your opportunity to be inventive! But, with the equipment provided, you should be able to get an estimate of the lung volume and an estimate of the uncertainty in your estimate.

The experimental equipment includes a tank filled with water in which you place a plastic bowl that can collect air bubbles. You exhale one lungful of air through a straw so the air collects in the bowl. From observations about the air in the bowl, calculate the amount of air you exhaled.

Filling the bowl (left) and measuring the volume (right)



Safety Remarks:
If you may have trouble with asthma or hyperventilation please let the TA know so that you can be exempted from this part of the lab! If this occurs, you should copy the data from another student. (Please give student's name!)

You should in any case not perform this experiment more than 2-3 times, otherwise you may feel some drowsiness. It is also a good idea to rest a few minutes between measurements. (This period can be used to work on your lab report or help your lab partner take data.)

4. Specifics of the Experiments

4.1 Pendulum

- Measure the length of the pendulum from the pivot point to the center of the mass. Estimate the uncertainty in this measurement. (You are probably limited in this measurement because you need to guess somewhat on the location of the center of the mass.)
- Measure 25 times the time for the period of pendulum to swings from a maximum through one complete cycle (that is, to get back where it started).
- Locate your measured values in bins of 50ms.
- Calculate the average and determine the uncertainty using the 2/3 estimate.
- Graph the frequency of the bins vs. their assigned time.
- Measure 25 times the time for the period of the pendulum as it passes an arbitrary fixed point where it is moving rapidly. Make sure that you always start and stop the watch as the pendulum passes this fixed point going in the same direction!
- Locate the measured values in bins of 50ms.
- Again calculate the mean and uncertainty as before and graph it.

- Is there a difference between these two curves? If so, can you explain it?
- Are the two values equal within their uncertainties?
- From where does the difference in uncertainty arise?
- Calculate g using the second series. (You must propagate the error from t and l!)
- Does your accepted value of g = 9.8 m/s^2 fall within the uncertainty of your value?
- What are the main sources of error?

4.2 Reaction Time

- Take the 10 values for the falling distance. Each trial should begin with one student releasing the ruler from the upper end and the student catching the ruler after initially placing two fingers at the 50cm mark.
- If you have a few values that are far off from all the other values (e.g. because you were irritated or distracted) you may decide to ignore them when you calculate your reaction time. But note in your lab report which measured points were not used.
- Use the 2/3 estimate to obtain the uncertainty.
- Give your reaction time in sec. in your lab report.
- Why is it not very reasonable to give your measured values with a precision measured in millimeters?
- How well could you get the same initial conditions for each try?
- Note the major sources of error!
- Suggest improvements to the lab or shortly describe a better experiment!

4.3 Lung Volume

- Make a guess for your lung volume in liters! (A standard soda bottle has 2 liters, a can has 1/3 of a liter.)
- Decide what you are going to measure and how this will provide your lung volume.
- Take appropriate readings. You should work in teams of two students in this part. One exhales into the straw, while the other controls the bowl as it rises, accumulating air bubbles. Try to control the bowl so that the level of water inside and outside remains at the same height. This is of particular importance when you read the scale!
- Make a few comments on how well you think you could get identical initial conditions for the independent readings.
- Was your initial guess good or bad?
- Comment on the spread in your data!
- What are the major sources of error?
- How could you improve the lab?

5. Sample Data Sheet

5.1 Pendulum

Length of the pendulum = ± cm

From Max to Max

Using Reference Point

5.2 Reaction Time

1		5		9	
2		6		10	
3		7			
4		8			

5.3 Lung Volume

Value you guess for your lung volume (in liters = 1000 cm^3): l

Your measured values:

1	
2	
3	

6. Applications (for those interested in everyday relevance!)

You read of a certain test intended to indicate a particular kind of cancer. The test gives you a positive result for (80 ± 10)% of all persons tested who really have this kind of cancer (true positives). But the test also gives you a positive result for (2 ± 1)% of all healthy persons (false positives). Now you read a publication where the author performed this test on 10000 workers that deal with a certain chemical. The author got 400 positive samples from these workers and claims that this is strong evidence that this particular chemical enhances the development of this kind of cancer since it is known from literature that only (1 ± 0.5)% of the population are expected to have this kind of cancer. How reliable is the claim of the author?

Numerical Answer:

If one assumes that the 10000 workers would mirror the average population, then there should be

$$10000\cdot(0.010\pm0.005) = 100 \pm 50$$

persons having this cancer. Of them, the test gives

$$(0.8\pm0.1)\cdot(100\pm50) = 80 \pm 50$$

positive results (true positives).

There are then 9900±50 persons expected not to have this kind of cancer. Of them

$$(0.02\pm0.01)\cdot(9900\pm50) \approx 200 \pm 100$$

give positive results (false positives).

The total number of positives in the average population is therefore 280±150.

So how do you judge the author's conclusion of "strong evidence"?

If you wanted to design a new test using the same procedure but to arrive at a stronger conclusion, and you could either increase the rate of true positives or decrease the rate of wrong negatives, which would you choose?

Reference:

Paul Cutler: Problem Solving in Clinical Medicine, Chapter 5, Problem 5 (but changed)

7. Lab Preparation Examples

Below are some questions to help you prepare. You will not be expected to answer all these questions as part of your lab report.

Estimation of Uncertainty:

1. You have the following distribution of measured values

0			
1	I		
2	III		
3	IIIII	I	
4	IIIII	IIII	
5	IIIII	IIIII	IIIII
6	IIIII	IIIII	I
7	IIIII	III	
8	IIII		
9	II		
10	I		
	5	10	15

Estimate the uncertainty using the 2/3 estimate!

2. Estimate the mean and uncertainty of the following group of values:
   1.6s, 1.3s, 1.7s, 1.4s

3. In a radioactive decay you get 16 counts. What is the absolute uncertainty of the number of counts? What is the relative uncertainty?

4. In a radioactive decay you get 1600 counts. What is the absolute uncertainty of the number of counts? What is the relative uncertainty?

5. How many counts should you get so that the relative uncertainty is 1% or less?

Significant digits:

6. How many significant digits has $l = 0.0254$ m?

7. Write $t = 1.25578 \pm 0.1247$ s with two significant digits (in the uncertainty)!

Propagation of Uncertainty:

8. For a pendulum with $1 = 1.0 \pm 0.1$m you measure a period of $t = 2.0 \pm 0.2$ s.
   What is the value of the earth acceleration g?

9. You measure the volume of a box by measuring the length of the single sides. For the length of the single sides you get

a = 10.0 ± 0.1 cm b = 5.0 ± 0.2 cm c = 7.5 ± 0.3 cm.
What is the volume of the box (including uncertainty and units) in cm^3?
What is the volume of the box in m^3?

10. You measure the following quantities:
A = 1.0 ± 0.2 m B = 2.0 ± 0.2 m C = 2.5 ± 0.5 m/s
D = 0.10 ± 0.01 s E = 100 ± 10 m/s^2
Calculate the mean and uncertainty of
a) A+B =
b) A-B =
c) C·D =
d) C/D =
e) C·D + A =
f) ½ E·D + C =
g) A·B/(A-B) =
Include units! For e)-g) perform it step by step.

Relative and Absolute Uncertainty:

11. What is the relative uncertainty for v = 12.25 ± 0.25 m/s?

12. What is the absolute uncertainty if the mean value is 120 s and the relative uncertainty is 5%?

13. Given the following measurements which one has the highest absolute uncertainty and which one has the highest relative uncertainty?
l = 10.0 ± 0.2 m l = 10.0 m (1.00 ± 0.03) l = 12.5 ± 0.25 m
l = 7.24 m (1.00 ± 0.04)

14. Given the following measurements which one has the highest absolute uncertainty and which one has the highest relative uncertainty?
l =10.0 ± 0.2 m t = 7.5 ± 0.2 s d = 5.6 cm (1.00 ± 0.04)
v = 6.4 ·10^6 m/s (1.00 ± 0.03)
Caution: Don't get tricked!

Explanations:

15. Explain, using your own words, why the uncertainty decreases as you average over several measurements!

16. Explain, using your own words, the difference between uncertainty and error as you perform several measurements and average!

17. You measure the speed of light and get as a result c = (2.25 ± 0.25) 10^8 m/s. The value you find in books is c = 299 792 458 m/s. Using these values explain the difference between the uncertainty of your measurement and its error!