Approval Quorums Dominate Participation Quorums

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Abstract

We study direct democracy with population uncertainty. Voters' participation is often among the desiderata by the election designer. We show that with a participation quorum, i.e. a threshold on the fraction of participating voters below which the status quo is kept, the status quo may be kept in situations where the planner would prefer the reform, or the reform is passed when the planner prefers the status quo. On the other hand, using an approval quorum, i.e. a threshold on the number of voters expressing a ballot in favor of the reform below which the status quo is kept, we show that those drawbacks of participation quorums are avoided. Moreover, an electoral system with approval quorum performs better than one with participation quorum even when the planner wishes to implement the corresponding participation quorum social choice function.
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1 Introduction

Direct democracy, in the form of referenda and initiatives, are used in many countries for decision making. Beside Switzerland and the United States,

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their use has spread out to many European countries and Australia.[1] As mentioned by Casella and Gelman (2008), in US states the number of referenda has increased in every decade since 1970, at an average rate of seventy per cent per decade.

In all referendum or initiative electoral rules, there is some form of minimum (absolute or relative) support requirement. In some cases (take Switzerland for example) there is a minimum number of signatures to put an initiative to vote, but there is no minimum turnout requirement in the actual vote; in contrast, in many other cases there is also an additional "quorum" requirement at the time of the vote. Quorums are a simple way of protecting the status quo. The subset of voters who proposed the reform (or lobbyists) have had the time to think of it and to measure the gain they can draw from the reform. It may be the case that voters who are currently indifferent between the status quo and the reform would prefer the status quo if they were better informed or if the cost of voting were smaller. As stated in Qvortrup (2002), the rationale for a turnout requirement is that "a low turnout in referendums is seen as a threat to their legitimacy".[2]

Legitimacy, however, may have multiple meanings: the two most frequently used types of legitimacy turnout requirements are the so called "participation quorum" and "approval quorum." When a participation quorum is imposed, an electoral outcome is considered legitimate if enough voters turn out, hence legitimacy is due to a sufficiently large set of citizens who care enough and have clear enough preferences. On the other hand, the approval quorum is a minimum required number of votes in support of the proposal, and hence legitimacy is in terms of a minimal strength of the absolute support for a reform, regardless of whether the rest of the population care or not or whether they have clear preferences or not.[3]

In this paper, we aim to show the superiority of approval over participation quorums in the following sense: when an absolute minimum approval

[1] See e.g. Matsusaka (2005a, 2005b) for an account of the increasing use of direct democracy around the world.

[2] See also LeDuc (2003) for a discussion of the fear to have a minority of the population prevail over a passive majority.

[3] Among the examples of participation quorums used in reality, the Italian example is the most used, even if similar quorums exist in other countries. For the approval quorum type of rules, on the other hand, Germany is the most recognized example. See Corte-Real and Pereira (2004) for a description of the various types of turnout requirements used in the world and for an axiomatic discussion.

for the reform is part of the desiderata of a social planner, the approval quorum rules do implement efficiently those social preferences; but the same is not true for a social planner whose legitimacy concerns are in terms of participation: participation based social preferences are not well served by participation quorum rules. Moreover, an approval quorum dominates a participation quorum because it does better even in the latter "territory," i.e. leading to outcomes that are closer to what is recommended by social objectives defined in terms of participation.

The large incentive-to-abstain problem with a high participation quorum has already been discussed in other papers, especially with reference to Italian experiences.[4] We study the consequences of this incentive to abstain when voters are strategic and there is some population uncertainty, that is, the number of voters actually casting their ballot is a random variable. We confirm that it is rational for supporters of the status quo to abstain, with two potential types of "mistakes:" first, as it is well-known, there are profiles of voters' preferences under which the status quo is kept even though the majority of citizens would have favored the reform. In addition, surprisingly, there are other profiles of preferences in which the reform is passed even though the majority is in favor of the status quo.[5]

With an approval quorum, which requires a minimum number of votes in favor of the reform in order to pass it (on top, of course, of a majority of votes casted), all the incentives to abstain disappear. More precisely, we show that sincere voting is always rational and is the only rational way of voting under this type of rules. Consequently, approval quorum rules implement approval based social preferences. The same factor, namely the huge difference in terms of incentives to abstain between the two quorum rules, is responsible for making an approval quorum a better way of implementing even participation based social preferences with respect to a participation quorum.

Obtaining the outcome preferred by the majority of the population is only one of the possibilities when using a participation quorum, while other bad equilibria with low participation exist and are often voluntarily induced by strategic party leaders who want to inhibit an effective use of direct democ-

[4]See e.g. Herrera and Mattozzi (2010), Hizen and Shinmyo (2010), and Aguiar-Conraria and Magalhaes (2010a), who find evidence from cross country data from 1970 to 2007. See also Zwart (2009) for a discussion of how "high" a participation quorum should be, conditional on having chosen to have one.

[5]See Aguiar-Conraria and Magalhaes (2010b) for a different analysis of this second type of mistake.

racy. On the other hand, with an approval quorum no equilibrium exists without sincere behavior and efficient aggregation of preferences or information in our model. To reiterate, these comparative results hold when we view the elections as preference aggregation devices as well as when we view them as information aggregation devices, hence the results are very robust if one accepts the methodological focus on rational individual strategic voting.[6]

Studying strategic voting in large elections raises well known difficulties: any voting profile such that no voter is pivotal for the outcome of the election is an equilibrium, as no voter can profitably deviate. As a consequence, the set of Nash equilibria is large. It is therefore impossible to explain the regularities one observes in large elections, including the ones about large abstention in referendum with quorums, using standard assumptions. A long series of models have provided solutions to this problem, all introducing some ingredients that make some pivotal probabilities always positive. Such ingredients can be uncertainty about preferences of other voters (see e.g. Feddersen and Pesendorfer (1997)), perceived probabilities of tie events (see e.g. Myerson and Weber (1993)), or uncertainty about the actual number of voters. This population uncertainty has been modelled in two ways. Sometimes it is assumed that the number of players is fixed but each player has a fixed probability of not participating in the election, so that the number of actual voters is distributed according to some binomial distribution (see e.g. Feddersen and Pesendorfer (1996) and Laslier (2009)). Alternatively, the number of players is assumed to be Poisson distributed (see e.g. Feddersen and Pesendorfer (1999) and Myerson (2000)). We follow the latter assumption.

Myerson (1999) proved that such Poisson games are characterized by two properties: action independence, implying that the numbers of voters of each type choosing any action are independently distributed; and environmental equivalence, implying that a player of any type considers a probability distribution of types of the other players identical to the one for the game itself. These two properties will be used below and make the analysis of the voting game quite simple. The key mechanism we highlight thanks to the assumption of population uncertainty works as follows. Under a participa-

[6]If one believes that elections should be studied focusing on parties' "mobilization" efforts rather than on individual strategic voting, then the key difference between approval and participation quorums that we emphasize becomes undiscernable. This is why Herrera and Mattozzi (2010), using a mobilization model, found no substantial difference between participation and approval quorums.

tion quorum, a status quo supporter always faces the dilemma that her vote may be pivotal in reaching the quorum at the benefit of the reform, or her vote may be pivotal in favor of the status quo if one vote is needed to obtain a majority against the reform. We prove that the former tie event is more likely than the latter one even if the quorum is expected to be reached.

Several authors have recently used Myerson's theory of large Poisson games to analyse strategic voting in large elections. Though some authors have used it for positive objectives (see e.g. Castanheira (2003), Herrera and Morelli (2010) and Bouton (2009)), that theory has mainly been used to discuss voting rules normatively (see, for instance, Myerson (1998, 2002), Bouton and Castanheira (2009), and Goertz and Maniquet (2010)). Our contribution is mainly normative and the clear conclusion we draw is that approval quorum should be preferred to participation quorums.

The paper is organized as follows. In Section 2 we describe the basic model, in which there is population uncertainty but no individual information problem, in the sense that each citizen knows exactly what alternative would be best for her. Section 3 contains the analysis of the model and Section 4 highlights the main result. In Section 5 we extend our result to the case in which there are independent citizens whose preferences for reform or status quo depend on the state of Nature, which is uncertain. In Section 6 we conclude with some brief remarks. All technical proofs are in the appendix.

2 Model

An electorate is called to decide whether to reform a status quo policy or not. The real voting population is uncertain. We assume it is Poisson distributed, with expected size n. For sufficiently large n, this assumption is close to assuming that each citizen in a population of size $\frac{n}{p}$ has an independent probability p of being selected by nature to go to vote.

A fraction θ^S of citizens is realized to strictly prefer the status quo, a fraction θ^R have opposite preferences, and the remaining fraction of citizens are indifferent between the two options. Consequently, the number of actual citizens preferring S to R (resp., R to S) is Poisson distributed with expected value $\theta^S n$ (resp., $\theta^R n$), with $\theta^S + \theta^R \leq 1$.

2.1 Conservative Social Preferences

The social planner does not know the exact distribution of preferences and is assumed to be biased in favor of the status quo. Hence her objective is to design rules that would make reforms pass *only when* the support for such reforms is *"sufficiently clear"* or *"sufficiently strong."*. There are at least two different "incarnations" of this conservative bias:[7]

1. *Participation Quorum Social Preferences (PQSP)*: R is the socially preferred outcome if and only if the realized numbers of supporters are $\theta^S n < \theta^R n$ and the total number of agents with strict preferences is above some threshold, $(\theta^S + \theta^R)n \geq qn$, for some $q \in [0,1]$ – see figure 1;

2. *Approval Quorum Social Preferences (AQSP)*: R is the socially preferred outcome if and only if $\theta^S n < \theta^R n$ and there is a sufficiently large absolute number of supporters of the reform, i.e., $\theta^R n \geq \widehat{q}n$, for some $\widehat{q} \in [0,1]$ – see figure 2.[8]

In summary, *sufficiently clear* can relate to the number of people who manifest their strict preferences or to the absolute support for the reform. Both these social preferences reflect a conservative bias. The difference is that the first type of preferences imposes that enough people in the electorate should have clear (strict) preferences; the second type of social preferences instead just require a minimum dimension of the class of people demanding the reform, regardless of the intensity of preferences of the rest of the population.

[7] We define social preferences in terms of expected vote shares. We could have defined them in terms of actual vote shares. Given that we concentrate on equilibria in sufficiently large populations, the difference between the two, that is, the probability that the recommendation of social preferences based on expected shares differs from that based on actual shares, is negligible.

[8] The quorums are expressed here in terms of absolute numbers of voters. They could not be expressed in fraction terms, as our population is potentially unbounded, due to our assumption of a Poisson distribution. This distribution, however, can be viewed as an approximation of a binomial distribution, where our n parameter corresponds to the expected number of interested citizens, pN, where N would be the actual size of the population and p the common probability of being called (by nature) to go to vote. With this interpretation, the quorums, in terms of a fraction of N, are pq and $p\widehat{q}$.

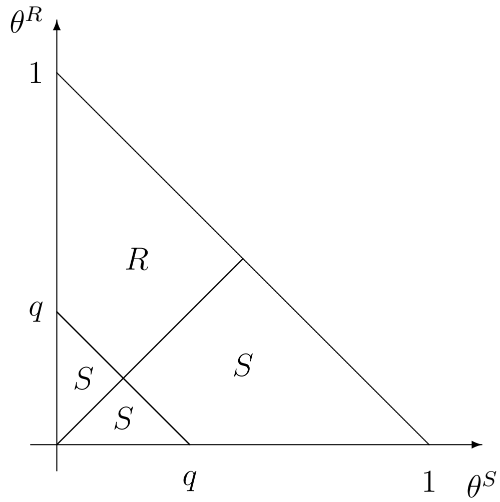

Figure 1: participation quorum social preferences

To clarify the logic behind these two types of social preferences, consider the example of $q = 0.5$ and $\hat{q} = 0.25$. If only 30 percent of the population eligible to vote has a strict preference but 99 percent of such voters are in favor of the reform, the preference for the reform would be considered sufficiently clear by a planner focused on the minimal absolute support requirement of AQSP, whereas the presence of 70 percent of indifferent citizens would induce the planner with PQSP to consider the status quo as the preferred option even though 99 percent of the voters with strict preferences are for the reform. Rather than disputing which of these types of social preferences are the most reasonable, we simply notice that both seem to exist in reality[9] and move to the evaluation of the way in which they are and/or should be implemented.

2.2 Voting Rules

What electoral rules should a planner design, as a function of her social preferences?

At the time of the referendum, all citizens who are selected by nature to have the opportunity to vote may choose to vote for S, for R, or to abstain. N^S will denote the number of voters who actually vote for the status quo,

[9]See Venice Commission (2005) for the underpinnings of the various existing rules.

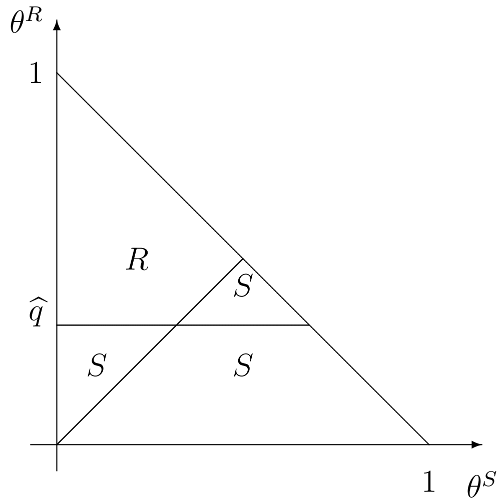

Figure 2: $\widehat{q}-$approval quorum social preferences

and N^R for the reform. We assume that indifferent citizens always choose to abstain.

Consider two existing electoral rules that constitute the most intuitive voting game form candidates for implementation of social preferences PQSP and AQSP respectively:

1. Under the participation quorum electoral rule (PQER), the outcome of the election is R if and only if $N^S < N^R$ and the total number of non-abstaining citizens, $N^S + N^R$, is larger than the threshold qn.

2. Under the approval quorum electoral rule (AQER), the outcome of the election is R if and only if $N^S < N^R$ and N^R is larger than the threshold $\widehat{q}n$.

One would think that PQER should be the best rule to implement PQSP, and AQER should be used to implement AQSP. Surprisingly, we will show that this intuitive connection is false, and in fact AQER *"dominates"* PQER, in a sense to be clarified below, even when social preferences are PQSP.

2.3 Strategies and Equilibrium Concept

Citizens know θ^S and θ^R but they are uncertain about the exact population of actual voters. For all population parameters, a strategy is a choice of voting behaviour for each of the two types of citizens with strict preferences (given that indifferent citizens are assumed to abstain). We assume everyone maximizes expected utility, and we look for stable (with respect to small perturbations of the strategies) Bayesian Nash equilibria involving non-dominated strategies for sufficiently large n.[10]

3 Analysis

We begin by studying the citizens' best response functions and the resulting equilibria for each of the two voting game forms described above.

The two game forms share a few characteristics. Voting for R (resp., for S) is a weakly dominated strategy for citizens preferring S (resp., R). Therefore, such citizens only consider abstaining versus voting for their preferred outcome.

Let any citizen who prefers S decide to actually vote with probability σ. Assume, for the sake of simplicity, that $\sigma\theta^S n$ is an integer. Then, the total number of votes for S is a Poisson of mean $\sigma\theta^S n$, that is,

$$\text{Prob}(N^S = k) = \frac{e^{-\sigma\theta^S n}(\sigma\theta^S n)^k}{k!}.$$

3.1 AQER game form

The difference in expected utility between voting for one's preferred outcome and abstaining comes from the probability of being pivotal. The key observation for a citizen preferring S (resp., R) is that the only possibility for her to affect the outcome of the election is making it switch from R to S (resp., from S to R), and these possibilities occur with probability

$$\text{Prob}(N^S = N^R - 1 \text{ and } N^R \geq \widehat{q}n)$$

(resp.,

$$\text{Prob}(N^S = N^R \text{ and } N^R \geq \widehat{q}n - 1)).$$

[10]Given that we have an unbounded number of potential players, describing individual strategies and considering asymmetric equilibria would be complex. In the symmetric equilibrium that we select all agents of any given type use the same voting strategy.

There is no swing voter's curse in such elections. Hence, the expected utility derived from voting for one's preferred outcome is either zero or strictly positive, as the probabilities above are either zero or strictly positive. Abstaining is then a weakly dominated strategy. As a result, a *sincere* profile, with every voter voting for her preferred outcome if she is called by nature to vote, is a stable Bayesian Nash equilibrium involving undominated strategies.

Given population uncertainty, it could happen that $\theta^S n > \theta^R n > \widehat{q}n$ whereas $N^R > N^S > \widehat{q}n$, so that the reform is passed when the planner strictly prefers the status quo. The most likely event, however, is that the planner's preferences are satisfied. Let us remember that as n becomes arbitrarily large, all the mass of probability is concentrated arbitrarily close to the most likely event. Therefore, as n goes to infinity, N^S (resp., N^R) is arbitrarily close to $\theta^S n$ (resp., $\theta^R n$) and the optimal outcome is implemented with a probability tending to 1.

3.2 PQER game form

Under PQER, citizens who prefer the reform can never influence the outcome in their favor by abstaining, hence abstaining is a weakly dominated strategy for them. All those citizens vote for R if they happen to vote.

For agents who prefer the status quo, on the other hand, there is the following strategic voting problem: adding one vote in favor of the status quo may change the outcome of the election from the status quo to the reform if this vote is pivotal in reaching the quorum and the reform has the majority of votes. On the other hand, adding one vote in favor of the status quo may also be pivotal for the status quo, if the quorum is already reached and the reform is one vote ahead of the status quo. Consequently, the expected utility of voting for S instead of abstaining depends on

$$\begin{aligned}&\text{Prob}(N^S = N^R - 1 \text{ and } N^S + N^R \geq qn)\\ -\ &\text{Prob}(N^S < N^R - 1 \text{ and } N^S + N^R = qn - 1),\end{aligned}$$

where N^R is Poisson distributed with mean $\theta^R n$.

Let us consider a citizen who prefers S. If all other citizens of her type have decided to abstain, then $\text{Prob}(N^S = N^R - 1 \text{ and } N^S + N^R \geq qn) = 0$ whereas the other probability is strictly positive. This proves that abstaining is not weakly dominated. Consequently, finding an equilibrium amounts to finding the equilibrium value of the probability σ with which citizens preferring S actually vote for S. From what has already been said, we can derive the following lemma.

Lemma 1 *Under the participation quorum electoral rules, for all θ^S, θ^R, the participation game admits an equilibrium where $\sigma = 0$, that is, where all citizens preferring the status quo abstain.*

Proof: Let θ^S, θ^R be given. Assume all citizens of type S (those who strictly prefer the status quo) but one choose $\sigma = 0$. Then, for all $n \in \mathbb{N}$,

$$\text{Prob}(N^S = N^R - 1 \text{ and } N^S + N^R \geq qn) = 0,$$

whereas

$$\begin{aligned} &\text{Prob}(N^S < N^R - 1 \text{ and } N^S + N^R = qn - 1) \\ &= \text{Prob}(N^R = qn - 1) = \frac{e^{-\theta^R n}(\theta^R n)^{qn-1}}{(qn-1)!} > 0, \end{aligned}$$

so that for all $n \in \mathbb{N}$: $\sigma = 0$ is a best response, proving the claim. □

Equipped with this lemma, we are now able to characterize the set of outcomes that, as a function of the parameters θ^S, θ^R of the population, are supported by an equilibrium.

Lemma 2 *Under the participation quorum electoral rules, the set of electoral outcomes supported by an equilibrium is the function of the parameters θ^S, θ^R illustrated below.*

Proof: See the appendix. □

This result reveals that there are two regions of parameters where the set of possible electoral outcomes does not coincide with what the planner would have chosen, had she known the preferences of the electorate. The first region (region 3 in the proof, where $\theta^R < q$, $\theta^R > \theta^S$, and $\theta^R + \theta^S > q$) is composed of the populations where the defenders of the reform are more numerous than the defenders of the status quo, but are not sufficient, by themselves, to reach the quorum. By abstaining, therefore, the defenders of the status quo succeed in preventing the reform from being voted. That is

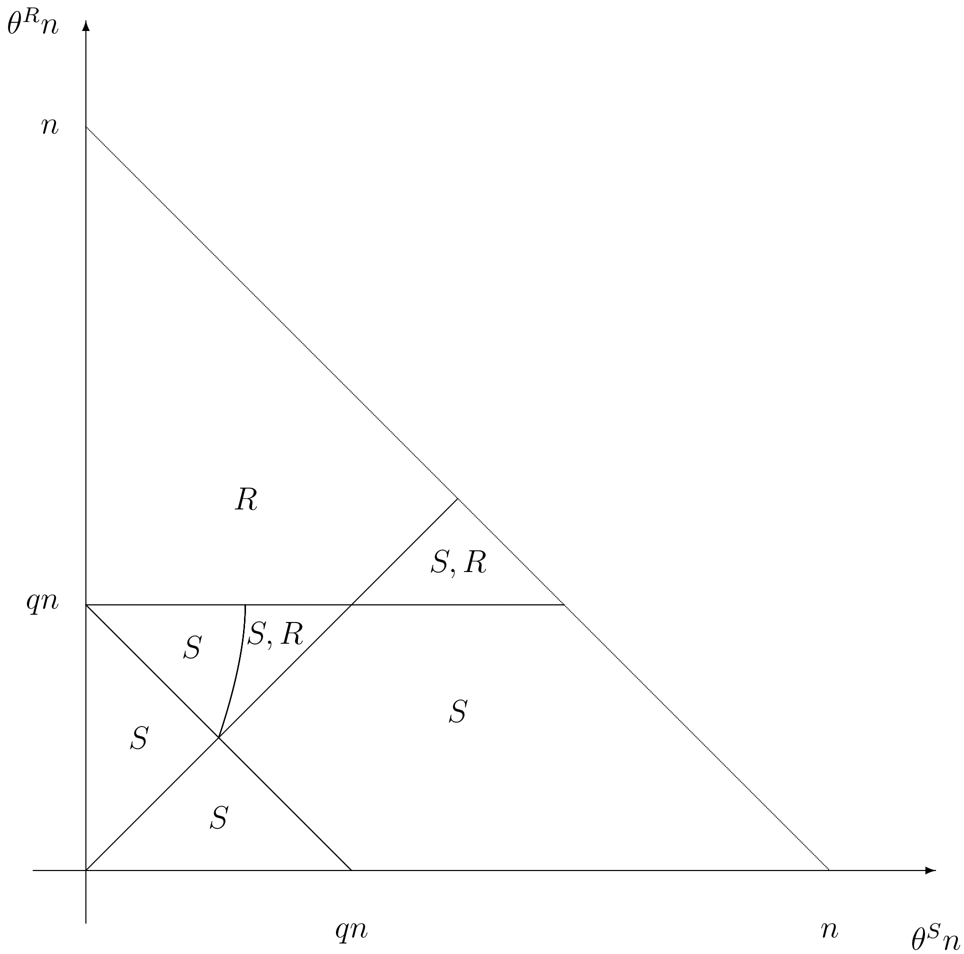

Figure 3: The possible electoral outcomes under a $q-$participation quorum

the already known effect of participation quorums, the Italian story. Observe that the undesired equilibrium is *the only one* for a large subset of parameters in this region.

There is a second region of parameters with an undesirable outcome (region 6 in the proof, where $\theta^S > \theta^R > q$): when supporters of the status quo are more numerous than the defenders of the reform, but the latter are sufficiently numerous to reach the quorum by themselves, there exists an equilibrium where all citizens preferring the status quo abstain, but the quorum is reached and the reform passes. Population uncertainty is the crucial ingredient yielding this result. It is indeed rational for status quo supporters to abstain, because, given the uncertainty about the actual voting population, the probability of being pivotal in favor of the reform by making the number of voters reach the quorum, even if the quorum *is expected* to be reached, is larger than the probability of being pivotal in favor of the status quo. The consequence is that the reform is passed, whereas sincere voting would have confirmed society's preference towards the status quo, in an electoral system protecting it.

In terms of plausibility of the "bad" equilibria we identified, the general abstention of region 6 by supporters of the status quo when the expected number of voters in favor of the reform is largely above the quorum is less plausible than general participation. But when the expected number of voters in favor of the reform is only slightly above the quorum, then it seems extremely plausible that supporters of the status quo try to enforce it by abstaining.

4 Main Comparative Result

Figure 1 tells us what the electoral designer exhibiting participation quorum social preferences would decide if she were completely informed about the citizens' preferences. Figure 3 tells us what this planner actually implements by introducing a participation quorum. As pointed out in the previous section, the two figures do not coincide. Figure 2 tells us what this planner would have implemented had she introduced an *approval* quorum in the electoral system. Here comes our central result: the set of population parameters for which the outcome of the elections coincides with the preference of the planner is larger under an approval than under a participation quorum.

Theorem 1 *The set of population parameters θ^S, θ^R for which the outcome of the election coincides with the q-participation quorum social preferences is larger under a $\hat{q}$-approval quorum electoral rule with $\hat{q} = q$ than under a q-participation quorum electoral rule.*

Proof: Let q denote the value of the parameter describing the planner's q-participation quorum social preferences. The following statements follow from Lemma 2 and the fact that citizens have a dominant strategy to vote sincerely in approval quorum elections:
1) If

$$\begin{aligned} \text{either} \quad & \theta^S \leq \theta^R, \text{ and } \theta^R \geq q, \\ \text{or} \quad & \theta^S + \theta^R \leq q \\ \text{or} \quad & \theta^S + \theta^R \geq q, \theta^S \geq \theta^R \text{ and } \theta^R \leq q, \end{aligned}$$

then the outcome of the elections under either a q-participation or a $\hat{q}$-approval quorum with $\hat{q} = q$ coincides with the q-participation quorum social preferences;
2) If

$$\begin{aligned} & \theta^S + \theta^R \geq q, \\ & \theta^S \leq \theta^R, \text{ and} \\ & \theta^R \leq q, \end{aligned}$$

then neither the outcome of the elections under a q-participation nor a $\hat{q}$-approval quorum with $\hat{q} = q$ coincides with the q-participation quorum social preferences;
3) If

$$\begin{aligned} & \theta^S \geq \theta^R, \text{ and} \\ & \theta^R \geq q, \end{aligned}$$

then the outcome of the elections under a $\hat{q}$-approval quorum with $\hat{q} = q$, that is, R, coincides with the q-participation quorum social preferences, whereas the outcome of the elections under a q-participation quorum does not (as it is either R or S). □

As a consequence, independently of whether the planner has participation or approval quorum *social preferences*, she *should* introduce approval quorum *in the electoral rules*. What is also surprising is the fact that a, say, 5 per cent participation quorum needs to be replaced with a no less than 5 per cent approval quorum.

The above theorem is based on the existence of regions of population parameters in which the set of outcomes of the participation quorum elections differs from the set of outcomes that are preferred by participation quorum social preferences. In one of these regions, region 6 in the proof of lemma 2 and case 3 in the proof of Theorem 1, there are equilibria that support the correct outcome, the status quo, but there are other equilibria supporting the undesired outcome, the reform, whereas all the equilibria under the approval quorum voting game support the correct outcome. A planner that would be satisfied whenever her preferred outcome is supported by *some* equilibrium could be fine with participation quorums. If, on the contrary, she demands her preferred outcome to be supported by *all* equilibria, then she should prefer approval quorum.

The third case analysed in the proof of the theorem corresponds to a region of parameters (recall that it is region 6 in the proof of Lemma 2) that disappears when $q = 0.5$ (if $\theta^R \geq q = 0.5$, then it is impossible to have $\theta^S \geq \theta^R$). Hence with $q = 0.5$ the two types of quorum rules are equivalent in terms of implementation of PQSP. In all other cases a participation quorum is *strictly* dominated ex ante by setting an approval quorum with the same threshold.

5 Information Aggregation Extension

All the analysis so far rests on the assumption that there is no uncertainty about the distribution of preferences. It is rather natural to think that the partisans of the reform, who have had the time to study the issue and compute their costs and benefits, know their preferences with certainty, but it is not clear whether all the other agents know what is in their best interest. Moreover, we can even think that some citizens may be convinced that the decision will affect their welfare but may be uncertain about the direction of change in their welfare. In this section we show that, in this case as well, an approval quorum voting rule is better than a participation one, and for exactly the same reasons as above.

We change the model in the following manner: there is a fraction θ^R of partisans of the reform, and a fraction θ^I of independent citizens, $\theta^R+\theta^I \leq 1$. The others are indifferent between the status quo and the reform. We assume that they abstain. Independent citizens have the same preferences: they prefer the reform in state r, and the status quo in state s. The two states of nature have prior probability π^r and π^s respectively, $\pi^r+\pi^s = 1$. Among the independent citizens, a fraction γ are informed about the state of nature, but the remaining $(1-\gamma)$ do not receive any information. All these parameters are common knowledge. This model is an extension of Feddersen and Pesendorfer (1996)'s model.

The two social preferences now depend on the state of nature. They are described in the following figures. According to the $q-$participation quorum social preferences, in state r the reform should be the outcome, except if less than q percent of the population are concerned by the issue. In state s, the outcome should be the same as in the previous section. According to the $\widehat{q}-$approval quorum social preferences, in state r the reform should be the outcome except if less than $\widehat{q}$ percent of the population is supporting the reform, exactly like with the other social preferences. In state s, the outcome should be the same as in the previous section.

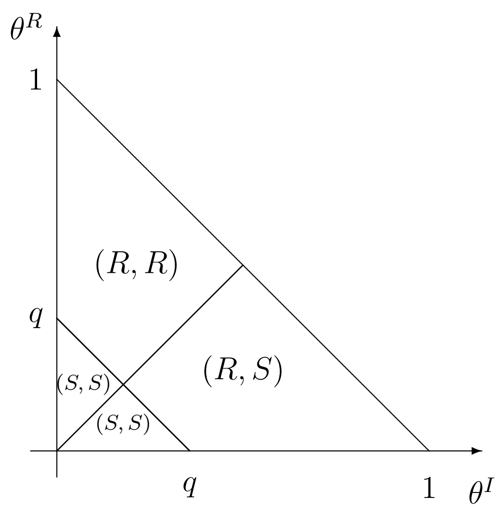

Figure 4: The $q-$participation quorum social preferences, as a function of the state of nature

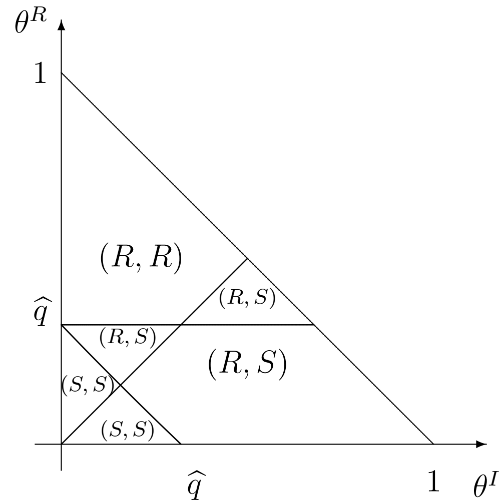

Figure 5: The $\widehat{q}-$approval quorum social preferences, as a function of the state of nature

Let us analyse the approval voting game first. Partisans of the reform have, as before, a dominant strategy to vote for the reform. Informed independent citizens can condition their vote on the state of nature, which they observe. In state r, they can only gain by voting for R, and in state s they can only gain by voting for S (abstaining is a weakly dominated strategy). Their dominant strategy is therefore (R, S), which reads: vote for R in r, for S in s.

The situation is different for uninformed independent citizens. They have to choose how to vote without knowing the state. On the other hand, they know that informed citizens vote as a function of their information, so that the uninformed citizens' strategy should consist in maximising the probability that informed citizens be pivotal. Their dilemma is that their vote for R can be needed in state r if partisans and informed citizens are not sufficient to meet the quorum, but their vote for S can be needed in state s if the informed citizens are not sufficient to out-balance the partisans, in case these ones reach the threshold.

Let σ_R and σ_S denote the probability that an uninformed independent citizen votes for the reform and the status quo, respectively. In state r, the expected fraction of the population voting for R is $\lambda^{R|r} = \theta^R + (\sigma_R(1-\gamma)+\gamma)\theta^I$, whereas a fraction $\lambda^{S|r} = \sigma_S(1-\gamma)\theta^I$ is expected to vote for S.

Similarly, $\lambda^{R|s} = \theta^R + \sigma_R(1-\gamma)\theta^I$, and $\lambda^{S|s} = (\sigma_S(1-\gamma)+\gamma)\theta^I$. Let us call $Piv1$ the event that a vote for S makes the outcome of the election change from R to S, and $Piv2$ the event that a vote for R leads the quorum to be reached and the outcome to change from S to R, that is (assuming qn is an integer),

$$\text{Prob}(Piv1) = \sum_{qn}^{\infty} \text{Prob}(N^S = k-1 \text{ and } N^R = k),$$

and

$$\text{Prob}(Piv2) = \sum_{k=0}^{qn-1} \text{Prob}(N^S = k \text{ and } N^R = qn-1).$$

From the analysis and the comparison of these crucial events, we can show that approval quorum rules determine no information inefficiency:

Lemma 3 *Under the approval quorum electoral rules, the outcome of the election coincides with the approval quorum social preferences for all population θ^I, θ^R.*

Proof: See the Appendix. □

Rational voting under an approval quorum voting rule is not as simple with preference uncertainty as without. But the lemma proves that an approval quorum surprisingly does not prevent efficient information aggregation: the outcome is always the same as what it would be if uninformed independent citizens were actually informed.

Here is the intuition of this result. Given that there is no way of making a mistake under approval quorum when a citizen knows her preferences, the partisans of the reform have a dominant strategy to vote for it (as previously) and the informed independent citizens to vote for R in r and S in s. The only delicate question is for the uninformed independent citizens.

Let us illustrate their optimal strategy numerically.[11] Assume $\widehat{q} = 0.20$, $\theta^R = 0.10$, $\gamma\theta^I = 0.5$ and $(1-\gamma)\theta^R$=0.40. The dilemma of the uninformed independents is that they should vote for R in r, as the other voters are not numerous enough to make R reach the quorum, but they should not vote "too much" for R, as they still like S better in state s. This is achieved, for

[11] All the percentage of votes in these examples are expressed as ratios of the expected total number of potential voters in the population, n.

instance, if they choose the following mixed strategy: with probability 0.20, they vote for R, otherwise they abstain. As a result, R is expected to obtain 23% of votes in r, so that the quorum is reached, and only 18% in s, which guarantees the election of S.

Assume now that $\widehat{q} = 0.20$, $\theta^R = 0.25$, $\gamma\theta^I = 0.5$ and $(1-\gamma)\theta^R$=0.40. The dilemma is now that they need to vote for S, otherwise R is likely to be elected in s, but if too many independent citizens vote for S, R could fail to be elected in r. The optimal strategy looks like this: with probability 0.625, they vote for S, otherwise they abstain. As a result, R is expected to receive 30% of votes, and S 25%, in r, and the expected outcome is reversed in s, the desired outcome. As a result, the outcome always coincides with the planner's preferred candidate.

This will no longer be the case with the participation quorum, as can be appreciated from the next theorem.

Theorem 2 *The set of population parameters for which the outcome of the election coincides with the q-participation quorum social preferences is larger under a $\hat{q}$-approval quorum with $\hat{q} = q$ than a q-participation quorum even when independent citizens' preferences depend on an uncertain state of Nature.*

Proof: See the Appendix. □

Again, let us illustrate the reasoning numerically. First, there is a region of parameters where S is elected in s whereas the planner prefers R. Assume that $q = 0.20$, $\theta^R = 0.15$, $\gamma\theta^I = 0.3$ and $(1-\gamma)\theta^R$=0.06. What is the best strategy of the informed independents? Let us begin by assuming that all the uninformed independents vote for S. Then, the quorum is expected to be reached in both states of nature, and R is expected to win in both cases. In r, there is no ambiguity, informed citizens vote for R. In s, however, given the large victory margin of R over S, whereas the quorum is not passed by much, the most likely tie is that an additional vote makes the quorum be reached. Consequently, informed independents decide to abstain in s. Knowing that, it is not rational for the uninformed independents to vote for S with certainty. By voting for S with only, say, 0.67% of probability, they guarantee that R is elected in r (the quorum is likely to be reached, as 22% of citizens show up), and, simultaneously, they keep the probability that the quorum is reached in s sufficiently low, as only 19% of the citizens are expected to participate.

Second, there is a region where R is elected in s whereas the planner prefers S. Assume that $q = 0.20$, $\theta^R = 0.22$, $\gamma\theta^I = 0.5$ and $(1-\gamma)\theta^R$=0.40. As above, it is likely that informed citizens decide to vote for R in r and abstain in s. This is especially made rational if all uninformed citizens decide to abstain. They all know, indeed, that R is expected to win in r, as it obtains 27% of votes. Abstention is rational even in s, though, as the most likely tie is that an additional vote makes the quorum be reached.

These two regions of parameters are similar to the ones identified in the previous section. In each of them an undesired equilibrium exists, supporting the outcome that the planner would like to avoid.

6 Concluding remarks

We have proven that approval quorums, by giving citizens the incentive to vote sincerely, protect the status quo in a better way than participation quorums even if the preferences of the planner are consistent with what sincere voting would yield under participation quorums. Whether the objective of elections is to aggregate preferences or to aggregate information, we have shown that the result is unchanged.

Our dominance result is strict when the participation quorum is less than 50 percent. In such cases (Azerbaijan has a 25% participation quorum) the policy conclusion of this paper is clear: switch towards an approval quorum. However, even if the alternative is a participation quorum greater than or equal than 50 percent, the weak dominance should, in our view, be considered a strong enough argument to favor approval quorum rules, given that it is unlikely that everybody will always share the view that the appropriate legitimacy concern should be the one embedded in participation quorum social preferences.

Our starting point was that protecting the status quo is a common objective for the electoral designer. However, we have simply assumed such an objective, without trying to derive it endogenously. Also, the paper does not design the electoral rules that would implement the participation quorum social preferences fully. We don't have a solution to that question, but our conjecture is that no electoral rule can implement these preferences as it can never be profitable for a citizen to show up at the ballot when the reform has collected strictly more votes than the status quo and only one vote is missing to reach the quota.

Appendix

Proof of Lemma 2:

Let us begin with a complete description of the best reply correspondence of a citizen of type S. For this exercise, let $\theta^S n$ denote the expected size of the population of other citizens of type S. Let us assume that the symmetric strategy of these other citizens is σ, so that the number of actual votes in favor of S, N^S, is Poisson distributed with mean $\sigma\theta^S n$. We already know that the number of actual votes in favor of R, N^R, is Poisson distributed with mean $\theta^R n$. By voting for S, a citizen can be pivotal in favor of S if $N^S = N^R - 1$ and $N^S + N^R \geq qn$, and she may be pivotal in favor of R if $N^S < N^R - 1$ and $N^S + N^R = qn - 1$. Let us refer to the former case as $Piv1$, and to the latter as $Piv2$. We have

$$\begin{aligned}\text{Prob}(Piv1) &= \sum_{k=int\left(\frac{qn}{2}\right)}^{\infty} \text{Prob}(N^S = k \text{ and } N^R = k+1) \\ &= \sum_{k=int\left(\frac{qn}{2}\right)}^{\infty} \frac{e^{-\sigma\theta^S n}(\sigma\theta^S n)^k}{k!} \frac{e^{-\theta^R n}(\theta^R n)^{k+1}}{(k+1)!}\end{aligned}$$

and

$$\begin{aligned}\text{Prob}(Piv2) &= \sum_{k=0}^{int\left(\frac{qn-1}{2}\right)-1} \text{Prob}(N^S = k \text{ and } N^R = qn-k-1) \\ &= \sum_{k=0}^{int\left(\frac{qn-1}{2}\right)-1} \frac{e^{-\sigma\theta^S n}(\sigma\theta^S n)^k}{k!} \frac{e^{-\theta^R n}(\theta^R n)^{qn-k-1}}{(qn-k-1)!}\end{aligned}$$

where $int(x)$ stands for the integer value of x. We look at equilibrium for n sufficiently large. As n becomes larger, the probabilities of $Piv1$ and $Piv2$ tend to 0. Let μ_1, μ_2 denote the magnitude of these events, that is, the speed at which they tend to zero, that is, for $i \in \{1, 2\}$,

$$\mu_i = \lim_{n\to\infty} \frac{\ln(\text{Prob}(Pivi))}{n}.$$

The event with the largest magnitude will necessarily be more likely than the other one for n sufficiently large (that is precisely the meaning of n being sufficiently large). Let us compute these magnitudes. Using Theorem 1 in Myerson (2000), we know that the magnitude of such an event is identical to the magnitude of the most likely subevent, that is, of the exact sequence of numbers $N^S = k$ and $N^R = k+1$ that maximizes

$$\sigma\theta^S\psi\left(\frac{k}{\sigma\theta^S n}\right) + \theta^R\psi\left(\frac{k+1}{\theta^R n}\right)$$

under the constraint that $k \geq int\left(\frac{qn}{2}\right)$, and the exact sequence of numbers $N^S = k$ and $N^R = qn - k - 1$ that maximizes

$$\sigma\theta^S\psi\left(\frac{k}{\sigma\theta^S n}\right) + \theta^R\psi\left(\frac{qn-k-1}{\theta^R n}\right)$$

under the constraint that $k \leq int\left(\frac{qn-1}{2}\right) - 1$, respectively, where $\psi(x) = x(1-\ln x) - 1$. Let k_1 and k_2 denote the arguments maximizing the above expressions, respectively. Simple derivation leads to

$$k_1 = \max\left\{int\left(\frac{qn}{2}\right), \sqrt{\frac{1}{4} + \sigma\theta^S\theta^R n^2} - \frac{1}{2}\right\},$$

and

$$k_2 = \min\left\{int\left(\frac{qn-1}{2}\right) - 1, \frac{\sigma\theta^S}{\sigma\theta^S + \theta^R}(qn-1)\right\}.$$

Observe that the second critical value of k_1 can be approximated by its limit value $\sqrt{\sigma\theta^S\theta^R}n$. In the case where $\sqrt{\sigma\theta^S\theta^R} \leq \frac{q}{2}$, k_1 tends towards its first critical value $\frac{qn}{2}$, assuming $\frac{qn}{2}$ is an integer. Event $Piv1$, that is, a tie between R and S, is more likely for values $N^S = \frac{qn}{2} - 1$ and $N^R = \frac{qn}{2}$. We compute that

$$\begin{aligned}\mu_1 &= \lim_{n\to\infty} n^{-1}\ln\left(\frac{e^{-\sigma\theta^S n}(\sigma\theta^S n)^{\frac{qn}{2}-1}}{(\frac{qn}{2}-1)!}\frac{e^{-\theta^R n}(\theta^R n)^{\frac{qn}{2}}}{(\frac{qn}{2})!}\right)\\ &= \lim_{n\to\infty} n^{-1}\ln\left(\frac{e^{-(\sigma\theta^S+\theta^R)n}(\sigma\theta^S\theta^R n^2)^{\frac{qn}{2}}}{\left((\frac{qn}{2})!\right)^2}\frac{q}{2\sigma\theta^S}\right),\end{aligned}$$

which, using the Stirling formula (according to which $k!$ can be approximated by $\sqrt{2\pi k}\left(\frac{k}{e}\right)^k$), yields

$$\mu_1 = \lim_{n\to\infty} n^{-1}\ln\left(\frac{e^{-(\sigma\theta^S+\theta^R-q)n}}{\pi qn}\left(\frac{4\sigma\theta^S\theta^R n^2}{(qn)^2}\right)^{\frac{qn}{2}}\frac{q}{2\sigma\theta^S}\right),$$
$$= \lim_{n\to\infty} -(\sigma\theta^S+\theta^R-q)+\frac{q}{2}\ln\frac{4\sigma\theta^S\theta^R}{q^2}-\frac{\ln\pi qn}{n}+\frac{\ln\frac{q}{2\sigma\theta^S}}{n},$$

which gives

$$\mu_1 = q - q\ln q + q\ln 2\sqrt{\sigma\theta^S\theta^R} - (\sigma\theta^S+\theta^R). \tag{1}$$

Similar computations lead to the following magnitude equations. If $\sqrt{\sigma\theta^S\theta^R} > \frac{q}{2}$, k_1 tends towards its second critical value, $n\sqrt{\sigma\theta^S\theta^R}$, and

$$\mu_1 = 2\sqrt{\sigma\theta^S\theta^R} - (\sigma\theta^S+\theta^R). \tag{2}$$

If k_2 tends to $\frac{qn}{2}-1$, then

$$\mu_2 = q - q\ln q + q\ln 2\sqrt{\sigma\theta^S\theta^R} - (\sigma\theta^S+\theta^R). \tag{3}$$

If k_2 tends to $\frac{\sigma\theta^S qn}{\sigma\theta^S+\theta^R}$, then

$$\mu_2 = q - q\ln q + q\ln(\sigma\theta^S+\theta^R) - (\sigma\theta^S+\theta^R). \tag{4}$$

Region 1. $\theta^S < \theta^R$ and $\theta^S+\theta^R < q$. In this region, k_1 tends to $\frac{qn}{2}$ and k_2 tends to $\frac{\sigma\theta^S qn}{\sigma\theta^S+\theta^R}$. Consequently, $\mu_1 < \mu_2$ (as the geometric mean $\sqrt{\sigma\theta^S\theta^R}$ is always smaller than the arithmetic mean $\frac{\sigma\theta^S+\theta^R}{2}$). Independently of σ, a citizen has incentive to abstain. The only equilibrium, therefore, is $\sigma = 0$, and the expected outcome is S.

Region 2. $\theta^S \geq \theta^R$ and $\theta^S+\theta^R < q$: in this region, k_1 tends to $\frac{qn}{2}$, and k_2 tends to either value. If it tends to $\frac{\sigma\theta^S qn}{\sigma\theta^S+\theta^R}$, then the same reasoning as above holds, and $\sigma = 0$ is the only equilibrium. If it tends to $\frac{qn}{2}-1$, then $\mu_1 = \mu_2$. The probabilities of $Piv1$ and $Piv2$ tend to zero at the same speed, but that does not mean that they are equal. Actually, the most likely subevent of $Piv1$ and $Piv2$ are when $N^S = N^R - 1 = int(\frac{qn}{2})$ and $N^S = int(\frac{qn+1}{2}) - 2, N^R = int(\frac{qn}{2}) + 1$ respectively. In both cases, N^R takes the same value, so that Prob($Piv1$) > Prob($Piv2$) $\Leftrightarrow$

Prob($N^S = int(\frac{qn}{2})$) $>$ Prob($N^S = int(\frac{qn+1}{2}) - 2$). Consequently, Prob($Piv1$) $>$ Prob($Piv2$) $\Leftrightarrow \sigma\theta^S n > \frac{qn}{2} - 1$. That shows that there is an equilibrium with $\sigma = \frac{qn-2}{2\theta^S n}$. But this equilibrium is unstable: for any slight decrease (resp., increase) in σ, $\mu_1 < \mu_2$ (resp., $\mu_1 > \mu_2$) and abstaining (resp., voting for S) is a best reply. So we have two stable symmetric equilibria in this region, namely $\sigma = 0$ and $\sigma = 1$. In both cases, the expected outcome is S, as the number of voters in favor of R is expected to be below the quorum, and the total expected number of votes for S if all S supporters vote for S is larger than the expected number of votes for R.

Region 3. $\theta^S < \theta^R < q < \theta^S + \theta^R$: in this region, k_2 tends to $\frac{\sigma\theta^S qn}{\sigma\theta^S + \theta^R}$. If $\sqrt{\sigma\theta^S\theta^R} < \frac{q}{2}$, then k_1 tends to $\frac{qn}{2}$ and $\mu_1 < \mu_2$ and a citizen maximizes her utility by abstaining. If $\sqrt{\sigma\theta^S\theta^R} \geq \frac{q}{2}$, then k_1 tends to $n\sqrt{\sigma\theta^S\theta^R}$. We may, again, have a mixed strategy equilibrium with $\mu_1 = \mu_2$ and

$$2\sqrt{\sigma\theta^S\theta^R} = q - q\ln q + q\ln(\sigma\theta^S + \theta^R).$$

But, again, such an equilibrium cannot be stable. Indeed,

$$\frac{\partial(\mu_1 - \mu_2)}{\partial\sigma} = \sqrt{\frac{\theta^S\theta^R}{\sigma}} - \frac{q\theta^S}{\sigma\theta^S + \theta^R},$$

and, as $q < 2\sqrt{\sigma\theta^S\theta^R}$, we can deduce, by replacing q with its upper bound,

$$\frac{\partial(\mu_1 - \mu_2)}{\partial\sigma} > \sqrt{\frac{\theta^S\theta^R}{\sigma}}\frac{\theta^R - \sigma\theta^S}{\sigma\theta^S + \theta^R} > 0,$$

where the last inequality comes from $\theta^S < \theta^R$ and $\sigma \leq 1$. So, only $\sigma = 0$ and $\sigma = 1$ are equilibrium candidates. If $\sigma = 0$, then $\mu_1 < \mu_2$ and abstaining is an equilibrium in the whole region, with outcome S. If $\sigma = 1$, then $\mu_1 > \mu_2$ if and only if $\sqrt{\theta^S\theta^R} \geq \frac{q}{2}$ and $2\sqrt{\theta^S\theta^R} > q - q\ln q + q\ln(\theta^S + \theta^R)$. In this subregion, voting for S is also an equilibrium, and the expected outcome is R. To sum up, in this region, where the planner always prefers R, there is always an equilibrium with outcome S and in one subregion it is the only equilibrium outcome.

Region 4. $\theta^S > \theta^R$, $\theta^R < q < \theta^S + \theta^R$: in this region, both k_1 and k_2 can converge towards any of their respective values. By the same argument as above, we can prove that there are two equilibria, $\sigma = 0$ and $\sigma = 1$, but the expected outcomes associated to these equilibria are both S, as either the quorum is not reached in equilibrium (if all S supporters abstain) or S gets more votes than R (if all S supporters actually vote). There is also a mixed strategy equilibrium, which, for the same reason as above, is unstable.

Region 5. $\theta^S < \theta^R$ and $\theta^R > q$: independently of the optimal strategy of citizens of type S, the expected outcome is R, as citizens of type R are numerous enough to reach the quorum and they are more numerous than citizens of type S.

Region 6. $\theta^S > \theta^R > q$. Let us look immediately at the two extreme equilibrium candidates, $\sigma = 0$ and $\sigma = 1$. In the former case, k_1 and k_2 converge towards $\frac{qn}{2}$ and 0 respectively, so that unambiguously $\mu_1 < \mu_2$, and abstaining is a best reply. This is, therefore, an equilibrium, with expected outcome R, as $\theta^R n > qn$. This is the most surprising equilibrium of this game form. The reform is passed, whereas more citizens strictly prefer S to R than R to S. If $\sigma = 1$, then k_1 and k_2 converge to $n\sqrt{\theta^S\theta^R}$ and $\frac{qn}{2} - 1$ respectively. Then, $\mu_1 > \mu_2$ if and only if

$$2\sqrt{\sigma\theta^S\theta^R} - q + q\ln q - q\ln(\sigma\theta^S + \theta^R) > 0,$$

but this is always the case, as the inequality holds for $\theta^S = \theta^R = q$ (the smallests values of these parameters in region 6) and the expression is increasing in both θ^S and θ^R. This proves that $\sigma = 1$ is an equilibrium, and the equilibrium outcome is S.

Proof of lemma 3: The same kind of computations as in the above proof reveal that in state $i \in \{r, s\}$ if $\sqrt{\lambda^{R|i}\lambda^{S|i}} \geq q$, then the most likely subevent of $Piv1$ occurs when k tends to $\sqrt{\lambda^{R|i}\lambda^{S|i}}n$, and

$$\mu_{1|i} = 2\sqrt{\lambda^{R|i}\lambda^{S|i}} - (\lambda^{R|i} + \lambda^{S|i}), \tag{5}$$

whereas, if $\sqrt{\lambda^{R|i}\lambda^{S|i}} \leq q$, then the most likely subevent of $Piv1$ occurs when k tends to qn, and

$$\mu_{1|i} = 2q - 2q\ln q + q\ln\lambda^{R|i}\lambda^{S|i} - (\lambda^{R|i} + \lambda^{S|i}). \tag{6}$$

If $\lambda^{S|i} \leq q$, then the most likely subevent of $Piv2$ occurs when k tends to $\lambda^{S|i}n$ (its most likely value), and

$$\mu_{2|i} = q - q\ln q + q\ln\lambda^{R|i} - \lambda^{R|i}, \tag{7}$$

whereas, if $\lambda^{S|i} \geq q$, then the most likely subevent of $Piv2$ occurs when k tends to $qn-1$, and

$$\mu_{2|i} = 2q - 2q\ln q + q\ln\lambda^{R|i}\lambda^{S|i} - (\lambda^{R|i} + \lambda^{S|i}). \tag{8}$$

Having computed the magnitudes, we prove first that it is impossible that S wins in state r when $\theta^R + \theta^I \geq q$. Note that this only happens if $\sigma_R < 1$. We have to distinguish between two cases.

Case 1: $\lambda^{S|r} \geq \lambda^{R|r} \geq q$. That clearly requires $\sigma_S > 0$. Given the dominant strategy of the informed independent citizens, we have $\lambda^{S|s} > \lambda^{S|r} \geq \lambda^{R|r} > \lambda^{R|s}$. Magnitude $\mu_{1|r}$ is given by Eq. (5), $\mu_{2|r}$ by Eq. (8). Proving that $\mu_{1|r} > \mu_{2|r}$ amounts to proving that

$$2\sqrt{\lambda^{R|r}\lambda^{S|r}} > 2q - 2q\ln q + q\ln\lambda^{R|r}\lambda^{S|r},$$

or

$$\sqrt{\lambda^{R|r}\lambda^{S|r}} - q\ln\sqrt{\lambda^{R|r}\lambda^{S|r}} > q - q\ln q,$$

which follows from the fact that function $x - q\ln x$ is increasing for $x > q$. Magnitude $\mu_{1|s}$ may be given either by equation (5), in which case $\mu_{1|r} > \mu_{1|s}$ follows from $\lambda^{S|r}\lambda^{R|r} > \lambda^{S|s}\lambda^{R|s}$ (remember that $\lambda^{S|r} + \lambda^{R|r} = \lambda^{S|s} + \lambda^{R|s}$), or by (6), in which case $\mu_{1|r} > \mu_{1|s}$ follows from the same argument as for $\mu_{2|r}$ above. Obviously, $\mu_{2|r} > \mu_{2|s}$, so that it is clear that $\mu_{1|r} > \mu_{2|s}$. Consequently, conditional on her vote being pivotal, an uninformed independent citizen is sure to be in state r, so that $\sigma_S > 0$ is not a best reply, a contradiction.

Case 2: $\lambda^{R|r} < q$. Again, it is crucial that $\lambda^{S|s} = \lambda^{S|r} + \gamma\theta^I$, and $\lambda^{R|r} = \lambda^{R|s} + \gamma\theta^I$. If $\lambda^{S|r}$ is such that $\sqrt{\lambda^{S|r}\lambda^{R|r}} \geq q$, then we are back to a case similar to the one above, and $\sigma_S > 0$ cannot be a best reply. If $q \leq \lambda^{S|r} \leq \frac{q^2}{\lambda^{R|r}}$, then $\mu_{1|r} = \mu_{2|r}$ (given by Eqs. (6) and (8)). Magnitudes $\mu_{1|s}$ and $\mu_{2|s}$ are given by the same Eqs., so that $\mu_{1|r} > \mu_{1|s}, \mu_{2|s}$ follows from $\lambda^{S|r}\lambda^{R|r} > \lambda^{S|s}\lambda^{R|s}$. If $\lambda^{S|r} < q$, then $\mu_{2|r}$ is given by Eq. (7); $\mu_{2|r} > \mu_{2|s}$ follows from $\lambda^{R|s} < \lambda^{R|r}$ and the fact that function $x - q\ln x$ is decreasing for $x < q$. The fact that $Piv1$ are less likely than $Piv2$ is immediate and comes from the fact that $Piv1$ occurs when the votes for R just reaches the quorum (same requirement as $Piv2$) *and* there is a tie between R and S. Consequently, conditional on her vote being pivotal, an uninformed independent voter is sure to be in state r, proving that $\sigma_R < 1$ is not a best reply, a contradiction. Note that this also shows that if $\lambda^{R|r} < q$, the best reply is $\sigma_S = 0$: uninformed citizens don't vote for S, (but may vote for R), using the quorum as a guarantee that S will win the election in s.

Second, we prove that it is impossible that R wins in state s when $\theta^R < \theta^I$. That occurs if $\lambda^{R|s} > q, \lambda^{S|s}$, which implies that $\sigma_S < 1$. Note that $\lambda^{R|r} = \lambda^{R|s} + \gamma\theta^I > \lambda^{R|s} > q$, so that clearly $\mu_{2|s} > \mu_{2|r}$. Also, $\lambda^{R|r} > \lambda^{R|s} > q, \lambda^{S|s} > \lambda^{S|r}$ makes it clear that $\mu_{1|s} > \mu_{1|r}$. Consequently (independently on how $Piv1$ and $Piv2$ are ranked), state s is infinitely more likely than r conditional on her vote being pivotal, so an uninformed independent citizen votes for S, so that $\sigma_S < 1$ is not a best reply.

Proof of theorem 2:

Lemma 3 shows that a $\hat{q}$-approval quorum always gives the outcome that coincides with the $\hat{q}$-approval social preferences. Consequently, in state r, it also coincides with the q-participation quorum social preferences, with $\hat{q} = q$, as they are the same in that state. We simply need to show that approval quorum does better than participation quorum in state s. Let us restrict ourselves to proving that the undesired equilibria (leading to S being chosen in Region 3 and R being chosen in Region 6) highlighted in Theorem 1 still prevails under the current assumptions.

Claim 1: an equilibrium exists such that S is elected in s, whereas $\theta^R < q$, $\theta^R > \theta^I$ and $\theta^R + \theta^I \leq q$. Let $\sigma_S^* \geq 0$ be defined by $\theta^R + \sigma_S^*(1-\gamma)\theta^I < q < \theta^R + (\sigma_S^*(1-\gamma) + \gamma)\theta^I$, and

$$\text{Prob}(N^R + N^S = qn - 1|R) = \text{Prob}(N^R + N^S = qn - 1|S),$$

that is,

$$\frac{e^{-[\theta^R + (\sigma_S^*(1-\gamma)+\gamma)\theta^I]n}([\theta^R + (\sigma_S^*(1-\gamma) + \gamma)\theta^I]n)^{qn-1}}{(qn-1)!}$$
$$= \frac{e^{-[\theta^R + (\sigma_S^*(1-\gamma))\theta^I]n}([\theta^R + (\sigma_S^*(1-\gamma))\theta^I]n)^{qn-1}}{(qn-1)!},$$

or,

$$e^{\frac{-\gamma\theta^I}{qn-1}}[\theta^R + (\sigma_S^*(1-\gamma) + \gamma)\theta^I] = [\theta^R + (\sigma_S^*(1-\gamma))\theta^I]$$

which yields

$$\sigma_S^* = \frac{\theta^R}{(1-\gamma)\theta^I}\frac{1-\gamma\theta^I e^{\frac{-\gamma\theta^I}{qn-1}}}{\theta^R e^{\frac{-\gamma\theta^I}{qn-1}}-1}.$$

We claim that informed independents playing $(R, \emptyset)$, and the uninformed independents playing σ_S^* is an equilibrium. By definition of σ_S^*, $\text{Prob}(Piv2|r) = \text{Prob}(Piv2|s)$. Clearly, $\text{Prob}(Piv1|i) < \text{Prob}(Piv2|i)$, all $i \in \{r, s\}$, as a $Piv1$ event requires a tie between R and S *and* that the quorum be reached. When informed citizens observe the state is r, they clearly vote for R. When they observe s, given that $\text{Prob}(Piv1|s) < \text{Prob}(Piv2|s)$, they prefer abstaining. Uninformed citizens do not observe the state. Conditional on their vote being pivotal, they are sure that they face a $Piv2$ event, so that, as $n \to \infty$, they tend to be indifferent between voting for S and abstaining. That equilibrium is stable: if $\sigma_S < \sigma_S^*$, (resp., $\sigma_S > \sigma_S^*$) then $\text{Prob}(Piv2|r) > \text{Prob}(Piv2|s)$ (resp., $\text{Prob}(Piv2|r) < \text{Prob}(Piv2|s)$) so that citizens prefer voting for S (resp. abstaining).

Claim 2: an equilibrium exists such that R is elected in s, whereas $\theta^R < \theta^I$. Assume $q < \theta^R < \theta^I$. We claim that, like in the public information framework of the previous sections, we have an equilibrium where independent uninformed citizens prefer S but abstain, as voting for S could make R win the election. Let us prove that informed independents playing $(R, \emptyset)$, and the uninformed independents playing $\emptyset$ is an equilibrium. As $\lambda^{R|r} = \lambda^{R|s} + \gamma\theta^I > \lambda^{R|s} > q$ and $\lambda^{S|r} = \lambda^{S|s} = 0$, we have $\mu_{2|s} > \mu_{2|r} > \mu_{1|s}, \mu_{1|r}$. In state s, given that $\mu_{2|s} > \mu_{1|s}$, any independent citizen prefers to abstain. As informed citizens observe the state of nature, voting for R in r and abstaining in s is a best reply. Conditional on her vote being pivotal, an uninformed independent citizen is sure to be in state s, so that abstaining is her best reply.

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