

Math Refresher Course

Columbia University Department of Political Science

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Day 1

Prepared by Jessamyn Blau

1 Statistics in Political Science at Columbia

In the political science department, all students must take one course in statistics or formal modeling for the M.Phil. Many also choose to take a two-course sequence in statistics or formal modeling to fulfill the research tools requirement.

The first course in the statistics sequence at Columbia is POLS 4910 Quantitative Political Research. In this course, students use Stata to write short papers using basic statistical analysis techniques. This course does not assume any strong math background. This course is a requirement for both POLS 4911 Analysis of Political Data and POLS 4912 Multivariate Political Analysis. Students take *either* 4911 or 4912. 4911 is essentially a continuation of 4910, and requires no math background. 4912 covers the same material as 4911, but in a more detailed and math-heavy fashion. 4912 is appropriate for methods minors. Students who plan to minor in methods or take 4912 should take POLS 4360 Math Methods for Political Science concurrently with 4910 in the fall semester.

Following 4912, there is no set sequence of statistics courses required. However, methods minors are required to take at least four courses in methods (including game theory/formal modeling courses). Most students take POLS 4291 Limited Dependent Variables in the fall of their second year, and POLS 4292 Panel and Time-Series Data in the spring. Following the completion of 4912, students can also take POLS 8990/8991 Workshop in Quantitative Political Science, a two-course seminar that can be taken multiple times.

Further information is available at: <http://www.columbia.edu/cu/polisci/grad/main/intro/index.html>.

2 Basics

2.1 Stata

Stata is a statistical software package used in the introductory statistical sequences 4910/4911 or 4910/4912. Stata is installed on all of the computers in the computer lab in Lehman Library. You can also purchase a copy of Stata. Stata is definitely a must-have software package if you are planning on doing any kind of statistical work in the future. For the purposes of 4910, it is totally feasible to do your work in the computer lab. By the time students reach 4912, they tend to want to have Stata on their own computers, particularly since this course is intended for those planning to undertake advanced statistical research.

There are four flavors of Stata. Stata MP is designed for parallel processing but is really not necessary for most users. Stata SE is the full-sized version of Stata, and can handle over 32,000 variables, unlimited observations and a matrix size of 11,000. Intercooled Stata (Stata IC) can handle over 2,000 variables, unlimited observations and a matrix size of 800. Finally, Small Stata handles 99 variables, 1,000 observations and a matrix size of 40. For most users, Stata SE or IC are recommended. If you are planning to use large surveys, you might have a data set with more than 2,047 variables, in which case Stata SE would be preferable. Many students, however, find that Intercooled Stata is sufficient.

See <http://www.stata.com>.

2.2 L^AT_EX

2.2.1 What is L^AT_EX?

L^AT_EX is a document preparation system like Microsoft Word. However, it is not WYSIWYG (“what you see is what you get”), allows more complex formatting, and is therefore better compared to a set of markup commands like HTML. Political scientists (and many other social scientists) use this language to produce papers with specific formatting and, often, extensive mathematical notation. TeX is the name of the typesetting program, and L^AT_EX is the name of the document markup language. This document was prepared with L^AT_EX.

You are not expected, obviously, to produce papers for courses in L^AT_EX, but you may find it useful if you are producing a document with figures and

mathematical formulas. Moreover, John Huber requires students in Math Methods for Political Science (POLS 4360) to produce at least one typewritten homework assignment, which is a good time to learn \LaTeX .

You can prepare a \LaTeX document in any text editor, even Windows' Notepad or Mac's TextEdit. This is the source document, which consists of plain text and markups, which will instruct a compiler to present your document in a specific way. Most people, however, do not produce \LaTeX documents in a simple text editor, because a \LaTeX program is required to convert the source file (with the extension .tex) to a .pdf file.

It takes a while to learn the basic commands to produce a \LaTeX document, but it is worth it. You can find many tutorials on the Internet.

2.2.2 \LaTeX Programs

To design a \LaTeX document, you need a text editor *and* a compiler. Some programs provide text editing and compiling in the same place, while for other text editors you also need a compiler. However, all of the text editors listed below are specifically designed with \LaTeX -type coding in mind.

Windows

TeXnicCenter is a text editor and compiler for Windows. Available at: <http://www.toolscenter.org/>.

For the rest of the text editors listed below, you will need a compiler. A good one is MiKTeX, available at <http://miktex.org>.

The following are good choices for Windows-based text editors:

- emacs, available at: <http://www.gnu.org/software/emacs/>
- WinEdt, available at: <http://www.winedt.com/>
- LEd, available at: <http://www.latexeditor.org/>

Mac

TexShop is the equivalent of TeXnicCenter for Mac, in that it is a text editor and compiler in one. Available at: <http://www.uoregon.edu/~koch/texshop/>.

Another option is iTeXMac, available at: <http://itexmac.sourceforge.net/Presentation.html> (not updated since 9/2006).

- BBEdit, available at: <http://www.barebones.com/>

- emacs, available at: <http://www.gnu.org/software/emacs>

A great add-on to L^AT_EX for Mac users is BibDesk, a bibliography tool available at: <http://bibdesk.sourceforge.net/>. The equivalent for Windows is BibTex, available at: <http://www.bibtex.org/>. The JabRef reference manager allows you to manage bibliographical entries, and works on Mac OS X and Windows. Available at: <http://jabref.sourceforge.net/>.

2.2.3 L^AT_EX Tutorials

- “Getting Started with L^AT_EX,” available at: <http://www.maths.tcd.ie/dwilkins/LaTeXPrimer/>
- “Help With L^AT_EX,” available at: <http://www-h.eng.cam.ac.uk/help/tpl/textprocessing/TeX/latex/latex2e-html/ltx-2.html>
- Basic Mathematical Symbols: “L^AT_EX Symbols,” available at: <http://omega.albany.edu:8008/Symbols.html>
- “The Comprehensive L^AT_EX Symbol List,” available at: <http://alawi.csail.mit.edu/latex/symbols-letter.pdf>

2.3 Other

2.3.1 EndNote

EndNote is a bibliographical tool that integrates with Microsoft Word. It is available for Mac and Windows. You enter bibliographic into EndNote and then use the EndNote menu in Word to insert the references, which are automatically formatted in the journal style of your choice. The Columbia library offers EndNote training, which you will probably get e-mails about. It is available for free download through Columbia at: <http://www.columbia.edu/acis/software/endnote/>.

2.3.2 R

R is both a language (that is similar to/overlaps with the language S) and a statistical-analysis environment. It allows for more advanced programming than Stata. It is not required for the 4910/4911 or 4910/4912 sequences, but is required for upper-level statistics courses, including POLS 4291 and POLS 4292. It is available for free download at <http://www.r-project.org/>.

2.4 Programming Help

1. Hamilton, Lawrence C., *Statistics with STATA* 9th Edition (Belmont, CA: Duxbury, 2006)
2. Venables, W.N. et al, *An Introduction to R* (Bristol: Network Theory Limited, 2005)

3 Notation

3.1 Math

$\sum_{i=1}^n x_i$	the sum from $i = 1$ to n of x_i
$ m $	the absolute value of m
$n!$	n factorial
$\log_b x$	the logarithm of x to base b
$\ln x$ or $\log_e x$	the natural logarithm of x
e	the base of natural logarithms or ~ 2.718

3.2 Logic

\therefore	“therefore”
\exists	“there exists”
\forall	“for all”
iff or \leftrightarrow	“if and only if”
\ni or s.t.	“such that”
\Rightarrow or \rightarrow	“implies that”
\neg	“not”
\vee	“or”

3.3 Sets

$a \in S$	a is an element of (belongs to) set S
$a \notin S$	a is not an element of (does not belong to) set S
$S \subset T$	set S is a subset of set T
$A \cup B$	the union of set A and set B
$A \cap B$	the intersection of set A and set B
\tilde{S}	the complement of set S
$\{\}$ or \emptyset	the null set
a, b, c	the elements of a set
(a, b)	the open interval from a to b
$[a, b]$	the closed interval from a to b

3.4 Number Sets

\mathbb{P}	Prime Numbers: all numbers divisible only by 1 and themselves
\mathbb{Z}	Integers: all whole numbers (positive, negative and 0)
\mathbb{N}	Natural Numbers: all positive integers
\mathbb{Q}	Rational Numbers: all numbers that can be written as fractions
\mathbb{C}	Complex Numbers: all numbers including $i = \sqrt{-1}$
\mathbb{R}	Real Numbers: all rational and irrational numbers
\mathbb{R}^2	the two-dimensional set of all real numbers (the x-y plane)
\mathbb{R}^n	the n -dimensional set of all real numbers
\mathbb{R}^+	all positive real numbers
\mathbb{R}^-	all negative real numbers

3.5 Calculus

$\lim_{x \rightarrow \infty} f(x)$	the limit as x goes to infinity of $f(x)$
$\frac{dy}{dx}$ or $f'(x)$	the first derivative of the function $f(x)$
$\frac{dy}{dx} _{x=x_0}$ or $f'(x_0)$	the first derivative of the function evaluated at $x = x_0$
$\frac{d^2y}{dx^2}$ or $f''(x)$	the second derivative of the function $f(x)$
$\lim_{x \rightarrow \infty} f(x)$	the limit as x goes to infinity of $f(x)$
$\int f(x)dx$	the indefinite integral of the function
$\int_a^b f(x)dx$	the definite integral of the function (evaluated from a to b)

3.6 Matrix Algebra

A	matrix A
A' or A^T	the transpose of matrix A
A^{-1}	the inverse of matrix A
$ A $	the determinant of matrix A
$r(A)$	the rank of matrix A
trA	the trace of matrix A

4 Basic Math

4.1 Order of Operations

The order of operations is the order in which expressions are evaluated.

- Parentheses: parentheses override the order of operations.
- Exponents and roots.
- Multiplication and division, from left to right.
- Addition and subtraction, from left to right.

4.2 Fractions

4.2.1 Reduced Form

We can reduce the form a fraction by factoring the numerator and denominator and then canceling common factors between them (by dividing by the same factor provided it is not zero).

4.2.2 Signs and Fractions

- $-\frac{a}{b} = \frac{-a}{b}$
- $\frac{-a}{-b} = \frac{a}{b}$

4.2.3 Addition

Adding (or subtracting) fractions requires that the denominator of all fractions being added together be the same. Thus, in some cases, the top and bottom of some of the terms must be multiplied to get a common denominator.

- $a + \frac{b}{c} = \frac{a \cdot c + b}{c}$
- $\frac{a}{b} + \frac{c}{d} = \frac{a \cdot d + c \cdot b}{b \cdot d}$

4.2.4 Multiplication

In contrast with addition (or subtraction), multiplication (or division) of fractions does not require a common denominator. Simply multiply across. Note that to divide fractions, you simply take the inverse of the second fraction, and multiply.

- $a \cdot \frac{b}{c} = \frac{a \cdot b}{c}$
- $\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$
- $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{a \cdot d}{b \cdot c}$

4.2.5 Problems

1. Reduce $\frac{16a^2b^3}{4ab^3}$
2. Simplify $\frac{4}{2x} + \frac{4}{x} + \frac{8}{2x}$
3. Simplify $\frac{1}{1-a} + \frac{1}{2} + \frac{1}{1+a}$
4. Simplify $5x^2y \cdot \frac{4x}{20xy^2}$
5. Simplify $\frac{\frac{1}{x} + \frac{1}{y}}{\frac{1}{xy}}$

4.3 Exponents

An exponent is represented by:

$$a^n \quad \text{the } n^{\text{th}} \text{ power of } a$$

where:

- a is the base
- n is the power

4.3.1 Properties of Exponents:

Let $a \in \mathbb{R}$, then:

- $a^n = a \times a \times a \times a \dots$ a multiplied by itself n times

- $a^1 = a$
- $a^0 = 1, \quad a \neq 0$
- $a^{-1} = \frac{1}{a}$
- $a^{-n} = \frac{1}{a^n}$
- $a^n \cdot a^m = a^{n+m}$
- $\frac{a^n}{a^m} = a^n \div a^m = a^{n-m}$
- $(a^n)^m = a^{n \cdot m}$
- $(a \cdot b)^n = a^n \cdot b^n$
- $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

4.3.2 Problems

1. Show that $(-a)^2 \neq -(a^2)$
2. Simplify $\frac{x^{12} \cdot x^{-3}}{x^2 \cdot x^{-3} \cdot x^4}$
3. Simplify $\left(\frac{a^3}{b^2}\right)^3$

4.4 Roots

4.4.1 Square Root

A square root (\sqrt{a}) is a number when multiplied with itself gives a . A square root can also be expressed in exponential form, such that: $a^{\frac{1}{2}}$.

Note:

- $\sqrt{a} \cdot \sqrt{a} = a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a^{\frac{1}{2} + \frac{1}{2}} = a^1 = a$
- $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$

4.4.2 n^{th} Root

An n^{th} root ($\sqrt[n]{a}$) is the number b such that $b^n = a$.

Note:

- $\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}, \quad a \geq 0, b \geq 0$
- $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}, \quad a \geq 0, b \geq 0$
- $\sqrt[n]{a^m} = (\sqrt[n]{a})^m = (a^{\frac{1}{n}})^m = a^{\frac{m}{n}}$

Note: to rationalize an equation is to remove the roots.

4.4.3 Problems

1. $\sqrt{9 \cdot 49}$
2. $\sqrt{9^3}$
3. Simplify $\sqrt{75} + \sqrt{3} - 2\sqrt{27}$
4. Rationalize the denominator:
 - (a) $\frac{(a-1)}{\sqrt{a-1}}$
 - (b) $\frac{a^2}{a^3\sqrt{a}}$

5 Algebra

5.1 Basic Rules of Algebra

- $a + b = b + a$
- $(a + b) = (b + a)$
- $a + 0 = a$
- $a + (-a) = 0$
- $ab = ba$
- $(ab)c = a(bc)$

- $1 \cdot a = a$
- $a \cdot a^{-1} = 1$ (for all $a \neq 0$)
- $(-a)b = a(-b)$
- $(-a)(-b) = ab$
- $-(a + b) = -a - b$
- $a(b + c) = ab + ac$
- $(a + b)(c + d) = ac + ad + bc + bd$

5.2 Special Identities

- $(a + b)^2 = a^2 + 2ab + b^2$
 - $(a - b)^2 = a^2 - 2ab + b^2$
 - $(a - b)(a + b) = a^2 - b^2$
1. Expand $(x^2 + 5)^2$
 2. Expand $(\sqrt{a-1} + \sqrt{a} - 1)^2$
 3. Simplify $(ab - 3b)(a + 3b)$

5.3 Factoring

A quadratic equation is one in the form of $ax^2 + bx + c$. To factor an equation in this form, you must rewrite the equation in the form $(dx + e)(fx + g)$ – see *Basic Rules of Algebra*.

In the simplest case (where $a = 1$) you must find two numbers (e and g) that multiply to the constant term c and add up to the constant term b .

Example

Factor: $x^2 + 5x + 6$.

Solution: $(x + 3)(x + 2)$

There is no pattern for factoring. The best way to proceed is usually to focus on finding the factors of c and then determining which ones will add up to b . There are some tips:

Given that c is positive:

1. The factors must be either both positive or both negative.
2. If b is positive, the factors are positive; otherwise, the factors are negative.

Given that c is negative:

1. The factors are of alternating signs.
2. If b is positive, then the larger factor is positive; otherwise, the smaller factor is.

1. Factor $21abc - 14ab^2$
2. Factor $x^2 + 8x + 15$
3. Factor $2x^2 + 7x + 3$
4. Factor $6x^2 + 11x - 10$

5.4 Equations With a Single Unknown

The easiest way to solve an equation with a single unknown is to:

1. Isolate all of the terms containing the unknown on one side of the equation.
2. Isolate all of the terms that do *not* contain the unknown on the other side.
3. Combine all of the terms containing the unknown, and all those not containing the unknown.
4. Calculate the unknown.

5.5 Problems

1. Evaluate $2x^2 + 3x$ for $x = -3$
2. $(x + 4) - (x - 2) = 6x$
3. $(x + 2)^2 - 5x = (x - 1)^2$
4. $\frac{x-1}{x+1} + 17 = \frac{4}{x+1}$
5. Solve $4(3x - 2) = \frac{1}{3}(x - 1) + 4$

6 Calculus

6.1 Functions

A linear function is a rule that assigns a number in R^1 to each number in R^1 . It is essentially a machine: you input one number, and a rule is used to output a corresponding number. For instance:

$$\begin{aligned} f(x) &= x + 1 \\ \text{or (equivalently)} \\ y &= x + 1 \end{aligned}$$

If you input 4 as x , your output is 5.

In this function, x is called the **input**, **independent** or **exogenous** variable. y or $f(x)$ is the **output**, **dependent** or **endogenous** variable.

Note that a function assigns only one value of y to any x . Thus, for any value of x , there is one and only one y value that corresponds. However, any given value of y can correspond to multiple values of x . For instance:

$$y = x^2$$

Each value of x produces one value of y . However, each value of y (except $y = 0$) corresponds to two values of x : for instance, when $y = 4$, $x = 2$ or $x = -2$.

6.2 Types of Functions

6.2.1 Constant

A constant function is given by: $f(x) = k \forall x$.

$$\text{e.g. } f(x) = 4$$

6.2.2 Monomial

$$f(x) = \alpha x^k$$

where:

- k the degree of the monomial
- α the coefficient of the monomial

6.2.3 Polynomial

A polynomial is a function of two or more monomials.

$$\text{e.g. } h(x) = -x^7 + 3x^4 - 10x^2$$

6.2.4 Rational Function

A rational function is a ratio of two polynomials.

$$\text{e.g. } y = \frac{x^2+1}{x-1}$$

6.2.5 Exponential Function

In an exponential function, the independent variable appears as an exponent. It is given by: $f(x) = k^x$

$$\text{e.g. } f(x) = 10^x$$

6.3 Domain

6.3.1 Definition of the Domain

The **domain** is the set of numbers for which a function is defined, i.e. the set of x for which the function can assign a set of corresponding y values.

$$\text{e.g. For } y = x, \text{ the domain is } R^1.$$

The domain of a function can be limited for many reasons. For example, you cannot divide by zero or calculate the square root of a negative number. A domain can also be limited when certain **intervals** are specified.

6.3.2 Open Intervals and Closed Intervals

An **open interval** is one that does not include its endpoints, such that the interval is given by $\{x : a < x < b\}$. On a graph, the endpoints of an open interval are not filled in (open circles). It is generally denoted as: (a, b) .

A **closed interval** is one that does include its endpoints, such that the interval is given by $\{x : a \leq x \leq b\}$. On a graph, the endpoints of a closed interval are filled in. It is generally denoted as: $[a, b]$.

Intervals can also be half-open and half-closed.

6.4 Limits and Continuity

6.4.1 Limits

The **limit** is the basis of calculus. It tells us how a given function is behaving at certain key points. Essentially, a sequence $f(x)$ has a limit l as x approaches c if $f(x)$ grows closer and closer to l as x approaches c .

An example will help. Take for instance the function $f(x) = \frac{1}{x}$. What is the limit of this function at some key points? As $x \rightarrow +\infty$, the y value seems to get closer and closer to zero without actually being zero. This makes sense: $\frac{1}{100}$ is a small number, $\frac{1}{10000}$ is even smaller. The same is true as $x \rightarrow -\infty$: x converges to zero. The limit of this function in both of these cases is zero. What about as $x \rightarrow 0$? By looking at a graph of these functions, we can see that $f(x)$ does not have a limit at $x = 0$. See *Continuity*.

A function $f(x)$ converges to a limit l iff for every $\epsilon > 0 \exists \omega$ s.t. $\forall x > \omega$, $|f(x) - l| < \epsilon$.

To evaluate a limit at infinity, the definition is essentially the same:

- A function $f(x)$ has a limit of l as x approaches $+\infty$ iff for every $\epsilon > 0 \exists \omega$ s.t. if $x > \omega |f(x) - l| < \epsilon$.
- A function $f(x)$ has a limit of l as x approaches $-\infty$ iff for every $\epsilon > 0 \exists \omega$ s.t. if $x < \omega |f(x) - l| < \epsilon$.

Here, we are trying to get at the question of "growing closer and closer to l ". This definition says that within an interval of ϵ from l , there is an x within ω of c that achieves this value.

Note that ω must always be expressed in terms of ϵ . Choosing ω can be complicated, Basically, you want to work backwards from the equation you want to prove, substituting ω for x . Knowing what δ to choose is not something you need to particularly worry about.

Example

What is ω when $\epsilon = .01$ for $\lim_{x \rightarrow 8} \frac{x}{4} = 2$?

To determine the value of ω , substitute in the given values for $f(x)$ and ω :

$$\begin{aligned} \left| \frac{x}{4} - 2 \right| &< .01 \\ 7.96 &< x - 8 < .04 \end{aligned}$$

Solving, we find that $\omega = .04$.

Example

What is the limit of $f(x) = x + 7$ as x approaches 4?

To solve this problem, we not only need to figure out what the answer is, but we need to show that this is true. We can deduce that as $x \rightarrow 4$, $f(x) \rightarrow 7$. We must now prove this by showing that for any value of ϵ specifies, there exists an ω such that:

$$|f(x) - 11| < \epsilon \text{ whenever } |x - 4| < \omega.$$

If we choose $\omega = \epsilon$, we now have $|x - 4| \leq \omega = \epsilon$. Since $|x - 4| < \epsilon$, we now have

$$|f(x) - 11| = |x + 7 - 11| = |x - 4| < \epsilon.$$

Note: In theory, it is important to understand the formal definition of the limit. In practice, one does not spend much time choosing and checking ω s.

6.4.2 Continuity

Essentially, a continuous function is one that has no “breaks.” More formally:

A function is **continuous** at x_n if $\lim_{x \rightarrow x_n} f(x) = f(x_n)$ for both the right- and left-side limits.

A few specifications:

- A polynomial or monomial function is continuous everywhere.
- A rational function is continuous except where the denominator equals zero.

6.5 Derivatives

The derivative of a function is the point at which:

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

The derivative is denoted by $f'(x)$.

Example

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{\frac{1}{2}(x+h) - \frac{3}{5} - (\frac{1}{2}x - \frac{3}{5})}{h} \\ f'(x) &= \lim_{h \rightarrow 0} \frac{\frac{1}{2}x + \frac{1}{2}h - \frac{3}{5} - \frac{1}{2}x + \frac{3}{5}}{h} \\ f'(x) &= \lim_{h \rightarrow 0} \frac{\frac{1}{2}h}{h} \\ f'(x) &= \lim_{h \rightarrow 0} \frac{1}{2} \\ f'(x) &= \frac{1}{2} \end{aligned}$$

Note: continuity is a necessary but not sufficient condition for differentiability.

You can think of a derivative as the slope of a function at any given point. Some people say that it is the rate of change of a function at any given point: thus, if a function measures speed, its derivative measures velocity, and the derivative of the velocity function measures acceleration.

6.5.1 Slope of a Line

Given this information, one can see that finding the derivative of a linear function simply means finding the slope. Given the linear function:

$$y = ax + b$$

the slope is given by the rate at which the function increases (or decreases), or the change in y divided by the change in x .

The slope is given by:

$$\frac{y-b}{x-0}$$

6.6 Computing Derivatives

Naturally, one does not always want to find the derivative of a linear function. To find the derivative of a non-linear function, you must find the slope of the line tangent to the point at which you wish to find the derivative. That is essentially what is done in the linear case, only there the tangent is equivalent to the function itself, and doesn't change.

6.6.1 Derivative of a Basic Monomial or Polynomial

The derivative of a monomial is given by the following, where $f'(x)$ is the derivative:

$$\begin{aligned}f(x) &= x^n \\f'(x) &= nx^{n-1}\end{aligned}$$

For a polynomial, simply compute the derivative of each term separately.

6.6.2 Product Rule

When two functions are both differentiable, the derivative of their product is given by the following:

$$\begin{aligned}F(x) &= f(x)g(x) \\F'(x) &= f(x)g'(x) + f'(x)g(x)\end{aligned}$$

6.6.3 Quotient Rule

When two functions are both differentiable, the derivative of their quotient is given by the following:

$$\begin{aligned}F(x) &= \frac{f(x)}{g(x)} \\F'(x) &= \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{(g^2)}\end{aligned}$$

6.6.4 Chain Rule

When two functions are differentiable, the following is true:

$$\begin{aligned}F(x) &= f(g(x)) \\F'(x) &= f'(g(x)) \cdot g'(x)\end{aligned}$$

6.6.5 Derivative of a Logarithmic Function

For a logarithmic function:

$$\begin{aligned}f(x) &= \ln x \\f'(x) &= \frac{1}{x}\end{aligned}$$

6.6.6 Derivative of an Exponential Function

For an exponential function:

$$\begin{aligned}f(x) &= e^x \\f'(x) &= e^x\end{aligned}$$

6.6.7 Higher-Order Derivatives

One can continue to take derivatives as long as the new function is continuously differentiable. The second derivative is denoted by $f''(x)$, the third by $f'''(x)$, and so on. The second derivative is especially useful.

6.6.8 Problems

Find the derivative:

1. $f(x) = x^2 + 2x + 2$
2. $f(x) = x^4 - 3x^3$
3. $f(x) = 5x^2 \cdot 4x^2 + 3x$
4. $f(x) = 3x \cdot \frac{3}{4}x^2$
5. $f(x) = \frac{6x^2}{2x}$
6. $f(x) = (2x^2) \cdot e^x$
7. $f(x) = e^{(4x)}$
8. $f(x) = 2x(5x^2)$
9. $f(x) = \ln(9x^2)$

6.7 Maxima and Minima

Some functions are increasing for some values of x and decreasing for others. The point at which they change from one to the other is called a **stationary point**. Functions can have multiple stationary points.

A function's maxima and minima are found at its critical values. However, a critical value is a necessary but not sufficient condition for a maximum or minimum. A critical value of a function exists at any of the following:

1. where its derivative is equal to zero ($f'(x) = 0$),
2. where the derivative is not defined, or
3. at the boundary points or points of discontinuity of a function

6.7.1 Maximum

If at (x_n, y_m) a function changes from increasing to decreasing, the function reaches a **maximum** at (x_n, y_m) .

If the function $f(x)$ always lies below the point (x_n, y_m) – that is, if $f(x_i) < f(x_n) \forall i \neq n$, then (x_n, y_m) is a **global maximum**. Otherwise, it is a **relative** or **local maximum**.

6.7.2 Minimum

If at (x_n, y_m) a function changes from decreasing to increasing, the function reaches a **minimum** at (x_n, y_m) .

If the function $f(x)$ always lies above the point (x_n, y_m) – that is, if $f(x_i) > f(x_n) \forall i \neq n$, then (x_n, y_m) is a **global minimum**. Otherwise, it is a **relative** or **local minimum**.

6.7.3 Finding Maxima and Minima

Maxima and minima exist where $f'(x) = 0$, and:

1. if $f'(x_n) = 0$ & $f''(x_n) < 0$, then x_n is a maximum
2. if $f'(x_n) = 0$ & $f''(x_n) > 0$, then x_n is a **minimum**
3. if $f'(x_n) = 0$ & $f''(x_n) = 0$, find the first non-vanishing (non-zero) derivative:

- (a) if the n^{th} derivative > 0 and n is even \Rightarrow minimum
- (b) if the n^{th} derivative < 0 and n is even \Rightarrow maximum
- (c) if the n^{th} derivative is odd \Rightarrow point of inflection

6.7.4 Problems

Find all maxima and minima:

1. $f(x) = x^2 - 6x + 4$
2. $f(x) = \ln(x) - x$ for $x > 0$

6.8 Increasing and Decreasing Functions

6.8.1 Increasing Functions

A function $f(x)$ is said to be an **increasing** function of x if its output **increases** with every **increase** in the values of the input. Thus, a function is increasing if $f(x_n) < f(x_{n+1})$.

e.g. $f(x) = x + 1$

6.8.2 Decreasing Functions

A function $f(x)$ is said to be a **decreasing** function of x if its output **decreases** with every **increase** in the values of the input. Thus, a function is decreasing if $f(x_n) > f(x_{n+1})$.

e.g. $f(x) = -x^7$

6.8.3 Determining Whether a Function is Increasing or Decreasing

To determine whether a function is increasing or decreasing at a given point, take the derivative.

- If $f'(x) > 0$, $f(x)$ is increasing
- If $f'(x) < 0$, $f(x)$ is decreasing

By taking the second derivative as well, you can determine the *rate* at which the function is increasing or decreasing.

- If $f'(x) > 0$ and $f''(x) > 0$, $f(x)$ is increasing at an increasing rate
- If $f'(x) > 0$ and $f''(x) < 0$, $f(x)$ is increasing at a decreasing rate
- If $f'(x) < 0$ and $f''(x) > 0$, $f(x)$ is decreasing at an increasing rate
- If $f'(x) < 0$ and $f''(x) < 0$, $f(x)$ is decreasing at a decreasing rate

6.8.4 Problems

On all relevant intervals, find whether the following functions are increasing or decreasing:

1. $f(x) = 2x - 5$
2. $f(x) = \ln(x)$ on $(0, \infty]$
3. $f(x) = x^4 - 8x^2$