Math Refresher Course

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Day 2

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6 Calculus CONT'D

6.9 Antiderivatives and Integration

Integration is the reverse of differentiation. Thus if $F'(x) = f(x)$ we can **integrate** $f(x)$ to get $F(x)$. The function $F(x)$ is the **antiderivative** or **integral** of the function $F(x)$. An integral is given by the following notation:

$$\int f(x)dx$$

where the elongated S (standing for "sum") is called the integral sign, $f(x)$ is the **integrand** and the dx shows that we are integrating with respect to x.

When we find an integral, we write that:

$$\int f(x)dx = F(x) + c$$

We must include a c to indicate that there are technically an infinite number of integrals of a particular integrand.

Note that the integral $\int f(x)dx$ is an **indefinite integral**. See below for a discussion of **definite integrals**.

6.9.1 Calculating Integrals

6.9.1.1 Power Rule This power rule is the counterpart to the power rule for derivatives:

$$x^n dx = \frac{1}{n+1}x^{n+1} + C, \qquad (n \neq -1)$$

6.9.1.2 Exponents The integral of e^x follows from the definition of its derivative:

$$\int e^x dx = e^x + C$$

6.9.1.3 Logarithmic Functions The **power rule** for integrals stipulates that it is not valid when given x^{-1}. The logarithmic function rule solves this problem:

$\int \frac{1}{x}dx = \ln x + C$

6.9.2 Calculating Integrals with Multiple Terms

6.9.2.1 Constant Factor Rule Where a is a constant:

$$\int af(x)dx = a\int f(x)dx$$

Note that a *cannot* be factored out if it is a variable term!

6.9.2.2 Sum Rule When the functions $f(x)$ and $g(x)$ are both integrable:

$$\int(f+g)dx = \int f(x)dx + \int g(x)dx$$

6.9.2.3 Integration by Substitution Integration by parts is the counterpart of the chain rule. It states that the integral of $f(u)\frac{du}{dx}$ with respect to x is the integral of $f(u)$ with respect to u. Thus:

$$\int f(u)\frac{du}{dx}dx = \int f(u)du = F(u) + c$$

This is clearer from an example:

Example

Given $\int 2x(x^2+1)dx$, you can simply multiply out the terms and find the derivative. If we do this, we find:

$$\int 2x(x^2+1)dx = \int 2x^3 + 2xdx$$
$$= \frac{1}{2}x^4 + x^2 + c$$

But we can also do it by substitution. Let $u = x^2 + 1$.

$$\frac{du}{dx} = 2x \rightarrow \frac{du}{2x} = x$$
$$\int 2x(x^2+1)dx = \int 2xu\frac{du}{2x} = \int udu$$
$$= \frac{1}{2}u^2 + c$$
$$= \frac{1}{2}(x^2+1) + c = \frac{1}{2}x^4 + x^2 + c$$

6.9.2.4 Integration by Parts

Integration by parts is the counterpart of the product rule. It states that:

$$\int udv = uv - \int vdu$$

This might be easier to understand in the following form:

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

Again, an example will help:

Example

$\int xe^x dx$
Let $u = x, \qquad du = dx$
Let $dv = e^x dx, \qquad v = e^x$
$\int xe^x = xe^x - \int e^x dx$
$= xe^x - e^x + C$

When you are choosing u and v, do so such that:

- u is easy to *differentiate*
- dv is easy to *integrate*
- $\int vdu$ is easier to compute than $\int udv$

Finally, note that you can always verify your result by ... differentiating!

6.9.3 Problems

1. Integrate:
   (a) $\int 9x^8 dx$
   (b) $\int (x^5 - 3x)dx$
   (c) $\int 3e^x dx$
2. Integrate by substitution:
   (a) $\int \frac{1}{(2x+3)^2} dx$

(b) $\int x^2(1+x^3)^100dx$

(c) $\int \frac{x+1}{x^2+2x+3}dx$

3. Integrate by parts:

(a) $\int ln(x)dx$

(b) $\int x^2e^{x^2}dx$

6.9.4 Fundamental Theorem of Calculus

You can think of a integral as the area under a curve. This is particularly useful with reference to **definite integrals**. A definite integral is an integral evaluated over a certain interval. Specifically, to calculate the integral for this interval $[a, b]$, divide the integral into N equal subintervals Δ:

$$\Delta = \frac{b-a}{N}$$

with endpoints: $x_0, x_1, x_2, ...x_n$ such that:

$$x_0 = a$$
$$x_1 = a + \Delta$$
$$x_2 = a + 2\Delta$$
$$\vdots$$
$$x_n = a + n\Delta$$

which yields the **Riemann sum**:

$$f(x_1)(x_1 - x_0) + f(x_2)(x_2 - x_1) + f(x_n(x_n - x_{n-1})$$
$$= \sum_{i-1}^{N} f(x_i)\Delta$$

The **Fundamental Theorem of Calculus States** that as we iterate this process, achieving smaller and smaller subintervals, we obtain the definite integral $\int_a^b f(x)dx$:

$$\lim_{\Delta \to 0} \sum_{i-1}^{N} f(x_i)\Delta = \int_a^b f(x)dx$$

Think of these small rectangles. The problem is that if we have a curve, there is a portion of each rectangle that sticks out above the curve. Thus, if we can get this portion to come close to zero, we will come infinitesimally close to correctly estimating the area under the curve – the definite integral.

The Fundamental Theorem of Calculus tells us two very important things:

1. If $f(x)$ is a continuous function and $F(x) = \int_0^x f(t)dt$, $F'(x) = f(x)$
2. $\int_a^b f(x)dx = F(b) - F(a)$ where $F' = f$

6.9.5 Problems

Evaluate:

1. $\int_1^3 \frac{1}{2}x^2 dx$
2. $\int_1^3 3\sqrt{(x)}dx$
3. $\int_0^1 x(x^2 + 6)dx$

6.10 Differentials

Let $y = f(x)$ be a differentiable function:

- the **differential of** x is dx, and is an independent variable
- the **differential of** y is $dy = f'(x)dx$

Conceptually, suppose we have some infinitesimal change in x, which we can call dx. There will be some change in y, which we can approximate using the derivative of x, $f'(x)$ and dx. This approximation is the differential of y, given above as $dy = f'(x)dx$.

dy is a dependent variable, dependent on x and dx. dx is an independent variable. Depending on the value of x, we get a different point on our curve, and depending on dx we get different distance from our curve in our estimation.

6.10.1 Relationship Between Differentials and Derivatives

"Differentiating" is often used as a synonym for finding the derivative of a function. In fact, it is the process of finding the differential dy – the change in y given the change in x – that should be referred to as "differentiation." Since a derivative, however, is just a quotient of the differentials, we can see why the terms are used interchangeably.

For precision, however, when we find a derivative $\frac{dy}{dx}$, we should say specifically that we are taking the derivative "with respect to x" (or the relevant independent variable).

6.10.2 Problems

Find the differential dy:

1. $y = \frac{x+1}{2x-1}$
2. $y = -x(x^2 + 3)$

6.11 Calculus of Several Variables

6.11.1 Functions of Several Variables

A function f of several variables has a set of elements A for which it is defined and a set of elements B, the set in which f takes its values. A is the domain of the function, while B is the domain. More generally:

$$f : R^n \to R^m$$

where

R^n : n-dimensional set of numbers, where n is the number of independent variables

R^m: m-dimensional set of numbers, where m is the number of dependent variables

Example

The investment function is given by:

$$z = A(1 + \frac{r}{n})^{nt}$$

where:

Dependent variable: z, return on investment
Independent variables:

1. A, initial investment
2. r, rate
3. n, number of times compounded per year
4. t, number of years until maturity

6.11.2 The Chain Rule Revisited

You can think of the chain rule as a type of function of two variables. Recall that the chain rule is given by:

$$h(x) = f(g(x))$$
$$h'(x) = f'(g(x)) \cdot g'(x)$$

An example for review:

Example

$$z = 3y^2, \text{ where } y = 2x + 5$$
$$z' = \frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = 6y(2) = 12y = 12(2x + 5)$$

You can extend the chain rule to more functions:

$$h(x) = f(g(j(x)))$$
$$h'(x) = f'(g(j(x))) \cdot g'(j(x)) \cdot j'(x)$$

In these cases, however, we are ultimately really considering a function of just one variable. In some cases, however, we must consider functions of multiple variables.

6.11.3 Partial Differentiation

6.11.3.1 Definition of Partial Derivative Given a function $y = f(x_1, x_2, \ldots, x_n)$, the partial derivative of f with respect to x_k describes how x_k impacts y while holding the other variables constant.

Given a change δx_k in x_k, the change in y will be:

$$\frac{\delta y}{\delta x_1} = \frac{f(x_1+\delta x_1, x_2, \ldots, x_n) - f(x_1, x_2, \ldots, x_n)}{\delta x_1}$$

such that:

$$f_1 = \frac{\delta y}{\delta x_1} = \lim_{\delta x_1 \to 0} \frac{\delta y}{\delta x_1}$$

6.11.3.2 Calculating Partial Derivatives Partial differentiation uses the same tools we learned for single-variable differentiation, only we must remember to keep all but the variable of interest *constant*. Recall: what is the derivative of a constant?

Example

$$f(x, y) = 8x^3y^2 + 3xy^{-2}$$
$$\frac{\delta f}{\delta x} = 24x^2y^2 + 3y^{-2}$$
$$\frac{\delta f}{\delta y} = 16x^3 - 6xy^{-3}$$

One can also take **second partial derivatives**. Note however that the second partial derivative does not need to be taken with respect to the same variable as the first! Returning to the above example, we see that there are in fact *three* second partial derivatives.

Example

The second partial derivatives are:

- $\frac{\delta^2 f}{\delta x^2} = 48xy^2$
- $\frac{\delta^2 f}{\delta y^2} = 16x^3 + 18xy^{-4}$
- $\frac{\delta^2 f}{\delta x \delta y} = 48x^2 - 6y^{-3}$

The last partial derivative, above, is called a **cross-partial derivative**. Here, we have taken the derivative with respect to one variable, and then differentiated the second time by another variable. *Regardless of the order in which they are calculated, cross-partial derivatives are always identical.* This can be tested by finding $\frac{\delta^2 f}{\delta y \delta x}$.

6.11.4 Total Differentials

Partial differentiation assumes **independence** of our input variables. We cannot use partial differentiation when this is not true, for instance when one variable can affect the outcome both independently, and through its effect on another input variable. We must then use total differentiation instead.

Fortunately, we can expand the concept of differentials to multiple independent variables to find the **total differential**. Given the function:

$$U = U(x_1, x_2, \ldots, x_n)$$

we can write the total differential as:

$$dU = \frac{\delta U}{\delta x_1}dx_1 + \frac{\delta U}{\delta x_2}dx_2 + \ldots + \frac{\delta U}{\delta x_n}dx_n$$
$$= \sum_{i=1}^{n} \frac{\delta U}{\delta x_1} dx_i$$

6.11.5 Total Derivatives

A **total derivative** allows us to explain how x_n affects y without having to hold the other variables constant. One can imagine that there are many cases where a variable will have both a direct and an indirect effect on the output variable.

Given:

$$y = f(x, z) \text{ where } x = g(z)$$

To find $\frac{dy}{dz}$, or the effect of z on y, we:

1. Find the total differential dy
2. Multiply the total differential by $\frac{1}{dz}$

3. Obtain the sum of the direct and indirect effects of z on y

Thus:

$$dy = \frac{\delta y}{\delta x}dx + \frac{\delta y}{\delta z}dz$$
$$\frac{1}{dz}dy = \frac{1}{dz}\frac{\delta y}{\delta x}dx + \frac{1}{dz}\frac{\delta y}{\delta z}dz$$
$$\frac{dy}{dz} = \frac{\delta y}{\delta x}\frac{dx}{dz} + \frac{\delta y}{\delta z}\frac{dz}{dz}$$
$$\frac{dy}{dz} = \frac{\delta y}{\delta x}\frac{dx}{dz} + \frac{\delta y}{\delta z}$$

In the above equation:

1. $\frac{\delta y}{\delta x}\frac{dx}{dz}$ is the indirect effect of x on y
2. $\frac{\delta y}{\delta z}$ is the direct effect of z on y

6.11.6 Problems

1. Find the first and second partial derivatives:
   (a) $u = (x^2 + t^2)^2 + e^x$
   (b) $f(x) = xe^{y \cdot 2x} + y$
2. Find the total differential:
   (a) $y = 3x^2 + xz^2 - 3z$
   (b) $y = \frac{e^x}{(x-1)y}$
3. Find the total derivative $\frac{dz}{dy}$ of $z = f(x, y) = 5x + xy - y^2$, where $x = g(y) = 3y^2$