J. Appl. Prob. **20**, 563–573 (1983)
Printed in Israel

THE PROPORTIONAL BETTOR'S RETURN ON INVESTMENT

S. N. ETHIER,* *Michigan State University*
S. TAVARÉ,** *Colorado State University*

Abstract

Suppose you repeatedly play a game of chance in which you have the advantage. Your return on investment is your net gain divided by the total amount that you have bet. It is shown that the ratio of your return on investment under optimal proportional betting to your return on investment under constant betting converges to an exponential distribution with mean $\frac{1}{2}$ as your advantage tends to 0. The case of non-optimal proportional betting is also treated.

CONVERGENCE IN DISTRIBUTION; GAMMA DISTRIBUTION; BETTING SYSTEMS

1. Introduction

Consider a game of chance that is played repeatedly and in which at each trial the bettor either wins or loses the amount of his bet. Suppose that the win probability p satisfies $\frac{1}{2} < p < 1$, so that the game is advantageous. Given $f \in (0, 1]$, one possible strategy is for the bettor to wager a proportion f of his current fortune at each trial. Letting $X_1, X_2, \cdots$ be independent and identically distributed (i.i.d.) with

$$X_1 = \begin{cases} 1 & \text{with probability } p \\ -1 & \text{with probability } 1-p, \end{cases} \tag{1.1}$$

the proportional bettor's fortune after n trials is

$$F_n = F_0 \prod_{i=1}^{n} (1 + fX_i), \tag{1.2}$$

Received 30 September 1982.
* Postal address: Department of Statistics and Probability, Michigan State University, East Lansing, MI 48824, U.S.A.
Supported in part by NSF Grant MCS-8102063.
** Postal address: Department of Statistics, Colorado State University, Fort Collins, CO 80523, U.S.A.

where F_0 is his initial fortune (a positive constant). The exponential rate of growth of his fortune is defined by

$$G_p(f) = \mathbb{E}\log(1 + fX_1), \tag{1.3}$$

since $\lim_{n\to\infty}(1/n)\log(F_n/F_0) = G_p(f)$ a.s. by the law of large numbers. In particular, if f is such that $G_p(f) > 0$, then $F_n \to \infty$ a.s. The choice f^* of f that maximizes $G_p(f)$ (namely, $f^* = 2p - 1$) results in an optimal betting system (Kelly (1956), Breiman (1961), Finkelstein and Whitley (1981)), which is often referred to as the Kelly system.

Our interest centers on the proportional bettor's return on investment (i.e., net gain divided by total amount bet), which after n trials is given by

$$R_n = \frac{F_n - F_0}{f\sum_{k=0}^{n-1} F_k}. \tag{1.4}$$

Assume that f is such that $G_p(f) > 0$. In Section 2 we show that as $n \to \infty$,

$$R_n \xrightarrow{\mathscr{D}} R \tag{1.5}$$

and

$$R_n \left| \left\{ \sum_{i=1}^{n} \chi_{\{X_i=1\}} = [np] \right\} \right. \xrightarrow{\mathscr{D}} R, \tag{1.6}$$

where $\xrightarrow{\mathscr{D}}$ denotes convergence in distribution and

$$R = \frac{1}{f\sum_{k=1}^{\infty}(F_0/F_k)}. \tag{1.7}$$

The random variable R, which by (1.5) represents the proportional bettor's (asymptotic as $n \to \infty$) return on investment, can be shown to satisfy $0 < R < 1$ almost surely and

$$\mathbb{E}R < \mathbb{E}X_1. \tag{1.8}$$

The latter result says that the expected return on investment under proportional betting is less than it is under constant betting. For fixed p and f, it is difficult to say by how much, but asymptotic results can be obtained by letting $p \to \frac{1}{2}+$ and $f \to 0+$.

Fix $\alpha \in (0, 2)$. Then $G_{(1+\varepsilon)/2}(\alpha\varepsilon) > 0$ for ε positive and sufficiently small, and for such ε we let $X_1(\varepsilon), X_2(\varepsilon), \cdots$ be i.i.d. with $X_1(\varepsilon)$ given by (1.1) with $p = (1+\varepsilon)/2$, and we define $R^\alpha(\varepsilon)$ by (1.7) and (1.2) with $X_i = X_i(\varepsilon)$ and $f = \alpha\varepsilon$. This corresponds to the case in which the betting proportion is α times the optimal betting proportion. In Section 3 we show that as $\varepsilon \to 0+$,

$$R^{\alpha}(\varepsilon)/\mathbb{E}X_1(\varepsilon) \xrightarrow{\mathcal{D}} \text{gamma}\left(\frac{2}{\alpha}-1, \frac{2}{\alpha}\right), \tag{1.9}$$

where $\text{gamma}(\theta, \lambda)$ denotes the distribution on $(0, \infty)$ with density $g(x) \equiv \Gamma(\theta)^{-1}\lambda^{\theta}x^{\theta-1}e^{-\lambda x}$. Moreover, we have uniform integrability, so

$$\lim_{\varepsilon \to 0+} \mathbb{E}R^{\alpha}(\varepsilon)/\mathbb{E}X_1(\varepsilon) = 1 - \frac{\alpha}{2}. \tag{1.10}$$

In the case of optimal proportional betting, $\alpha = 1$ and (1.9) becomes

$$R^{1}(\varepsilon)/\mathbb{E}X_1(\varepsilon) \xrightarrow{\mathcal{D}} \text{exponential}(2), \tag{1.11}$$

where $\text{exponential}(\lambda)$ denotes the distribution on $(0, \infty)$ with density $g(x) \equiv \lambda e^{-\lambda x}$, while (1.10) reduces to

$$\lim_{\varepsilon \to 0+} \mathbb{E}R^{1}(\varepsilon)/\mathbb{E}X_1(\varepsilon) = \tfrac{1}{2}. \tag{1.12}$$

In a recent paper, Wong (1981) argued that the optimal proportional bettor's return on investment is only about one-half of that of the constant bettor, at least when the actual number of wins is approximately the expected number. By (1.6), the latter condition is unnecessary. Ignoring it and assuming that Wong meant *expected* return on investment, we see that (1.12) can be viewed as a precise formulation of Wong's assertion.

Observe that

$$h(\alpha) \equiv \lim_{\varepsilon \to 0+} \mathbb{P}\{R^{\alpha}(\varepsilon)/\mathbb{E}X_1(\varepsilon) > 1\} \tag{1.13}$$

represents the (asymptotic as $\varepsilon \to 0+$) probability that proportional betting outperforms constant betting in terms of return on investment when the betting proportion is α times the optimal betting proportion. Using (1.9), it can be checked that $h(0+) = \frac{1}{2}$, $h(1) = e^{-2}$, and $h(2-) = 0$. (We believe that h is monotone decreasing on $(0, 2)$ but do not have a proof.) However, it would probably be a mistake to regard this or (1.10) as an indictment of proportional betting. As Wong (1981) put it (referring only to the optimal case), '... proportional betting costs you about half of your arithmetic expectation. You can think of this as being the premium you have to pay for the insurance against going broke that you get with proportional betting.'

Up until now, we have assumed that X_1, the bettor's net gain per unit bet, is $\{-1, 1\}$-valued. This assumption, however, is unnecessary. In particular, (1.5) and (1.8) hold with $X_1, X_2, \cdots$ taken to be i.i.d. non-degenerate $[-1, \infty)$-valued random variables with positive (finite) expectation (assuming of course that $\mathbb{E}\log(1 + fX_1) > 0$). As for (1.9) and (1.10), we require for some $M > 0$ that $X_1(\varepsilon), X_2(\varepsilon), \cdots$ be i.i.d. non-degenerate $[-1, M]$-valued random variables for each $\varepsilon \in [0, \varepsilon_0)$, that $X_1(\varepsilon) \xrightarrow{\mathcal{D}} X_1(0)$ as $\varepsilon \to 0+$, and that $\mathbb{E}X_1(\varepsilon) > 0 >$

$\mathbb{E}[X_1(\varepsilon)/(1+X_1(\varepsilon))]$ for $0<\varepsilon<\varepsilon_0$ and $\mathbb{E}X_1(0)=0$. (The boundedness assumption on $X_1(\varepsilon)$ is probably stronger than necessary but involves no real loss of applicability.) In this case, (1.9) and (1.10) hold with $R^\alpha(\varepsilon)$ defined for ε sufficiently small by (1.7) and (1.2) with $X_i = X_i(\varepsilon)$ and $f=\alpha f^*(\varepsilon)$, $f^*(\varepsilon)$ being the choice of f that maximizes $\mathbb{E}\log(1+fX_1(\varepsilon))$, the exponential rate of growth of the bettor's fortune. Clearly, these results generalize the previously stated ones.

One of the best-known applications of proportional betting occurs in the game of blackjack (Griffin (1981)). While it seems likely that the above results hold at least qualitatively in this context, it should be recognized that several of our assumptions fail to hold here. First, successive blackjack hands are not independent unless separated by a shuffle. Second, successive hands are not identically distributed from the card-counter's point of view; in particular, his advantage fluctuates. Third, our requirement that $X_1 \geqq -1$ (i.e., the bettor cannot lose more than the amount of his bet) seems to preclude the blackjack options of pair splitting, doubling down, and insurance; however, by suitably rescaling, it it easy to generalize the above results, replacing -1 by an arbitrary negative constant. Finally, we have assumed implicitly that money is infinitely divisible, which of course is not the case in a gambling casino.

2. Asymptotic distribution of R_n as $n \to \infty$

Our first result concerns the random variable R defined by (1.7), though here we do not assume (1.1).

Proposition 2.1. Let $X_1, X_2, \cdots$ be i.i.d. non-degenerate $[-1,\infty)$-valued random variables with $0<\mathbb{E}X_1<\infty$. Fix $f\in(0,1]$ such that

$$\mathbb{E}\log(1+fX_1)>0, \tag{2.1}$$

and let F_0 be a positive constant. Define $F_1, F_2, \cdots$ by (1.2), $R_1, R_2, \cdots$ by (1.4), and R by (1.7).

(a) $0<R<\operatorname{ess\,sup} X_1$ a.s.

(b) $R_n \xrightarrow{\mathscr{D}} R$ as $n\to\infty$.

(c) There exists a random variable R' (defined on the same probability space that $X_1, X_2, \cdots$ are defined on) such that $R' \overset{\mathscr{D}}{=} R$, R' is independent of X_1, and

$$R=(1+fX_1)R'/(1+fR'). \tag{2.2}$$

(d) $\mathbb{E}R<\mathbb{E}X_1$.

Proof. (a) By the law of large numbers and (2.1),

$$\begin{aligned}\lim_{k\to\infty}(F_0/F_k)^{1/k} &= \lim_{k\to\infty}\exp\{-(1/k)\log(F_k/F_0)\}\\ &= \exp\{-\mathbb{E}\log(1+fX_1)\}<1 \quad \text{a.s.,}\end{aligned}$$

so the series in (1.7) converges almost surely by the root test. Hence $R > 0$ almost surely. For the second inequality, if $\operatorname{ess\,sup} X_1 = \infty$, there is nothing to prove. If $M = \operatorname{ess\,sup} X_1 < \infty$, then

$$1/R \geqq f \sum_{k=1}^{\infty} (1+fM)^{-k} = 1/M \qquad \text{a.s.}$$

with equality only if $X_i = M$ for each $i \geqq 1$. Hence the non-degeneracy of X_1 implies that $R < M$ almost surely.

(b) Observe that

$$\begin{aligned} R_n &= \frac{1-(F_0/F_n)}{f\sum_{k=0}^{n-1}(F_k/F_n)} \overset{\mathscr{D}}{=} \frac{1-(F_0/F_n)}{f\sum_{k=0}^{n-1}(F_0/F_{n-k})} \\ &= \frac{1-(F_0/F_n)}{f\sum_{k=1}^{n}(F_0/F_k)} \xrightarrow{\text{a.s.}} R \end{aligned} \tag{2.3}$$

as $n \to \infty$, where the equality in distribution follows by reversing the order of $X_1, X_2, \cdots, X_n$; we have also used $F_n \to \infty$ a.s. as $n \to \infty$, valid by (2.1).

(c) Define R' in terms of $X_2, X_3, \cdots$ exactly in the same way that R is defined in terms of $X_1, X_2, \cdots$. Then

$$\frac{1}{R} = \frac{1}{1+fX_1}\left(f + \frac{1}{R'}\right), \tag{2.4}$$

which is equivalent to (2.2).

(d) By (2.2), $R \leqq (1+fX_1)/f$, and thus R has finite expectation. Now the function $\Psi(r) \equiv r/(1+fr)$ is strictly concave on $(0,\infty)$, so by (c) and Jensen's inequality (using the non-degeneracy of X_1 and therefore R),

$$\mathbb{E}R = \mathbb{E}[1+fX_1]\mathbb{E}[R/(1+fR)] < (1+f\mathbb{E}X_1)\mathbb{E}R/(1+f\mathbb{E}R),$$

from which the desired result follows.

The next result will allow us to prove a generalization of (1.6).

Proposition 2.2. Let $X_1, X_2, \cdots, f, F_0, F_1, F_2, \cdots$, and R be as in Proposition 2.1. For each $n \geqq 1$, let $X_{1n}, \cdots, X_{nn}$ be $[-1,\infty)$-valued random variables, and define

$$F_{mn} = F_0 \prod_{i=1}^{m} (1+fX_{in}), \quad R_{mn} = \frac{F_{mn} - F_0}{f\sum_{k=0}^{m-1} F_{kn}},$$

for $1 \leqq m \leqq n$. Assume that

$$(X_{1n}, \cdots, X_{nn}) \overset{\mathscr{D}}{=} (X_{nn}, \cdots, X_{1n}), \qquad n \geqq 1, \tag{2.5}$$

$$(2.6) \qquad (X_{1n}, \cdots, X_{mn}) \xrightarrow{\mathscr{D}} (X_1, \cdots, X_m) \qquad \text{as } n \to \infty,\ m \geqq 1,$$

$$(2.7) \qquad \mathbb{E}(F_0/F_{mn})^u \leqq \mathbb{E}(F_0/F_m)^u, \qquad 1 \leqq m \leqq n,\ 0 < u < 1.$$

Then $R_{nn} \xrightarrow{\mathscr{D}} R$ as $n \to \infty$.

Proof. As in (2.3), we infer from (2.5) that

$$R_{nn} \stackrel{\mathscr{D}}{=} \frac{1-(F_0/F_{nn})}{f \sum_{k=1}^{n} (F_0/F_{kn})},$$

so it will suffice to show that $F_0/F_{nn} \xrightarrow{\mathrm{P}} 0$ and

$$\sum_{k=1}^{n} (F_0/F_{kn}) \xrightarrow{\mathscr{D}} \sum_{k=1}^{\infty} (F_0/F_k).$$

By (2.6), the latter will follow from Theorem 4.2 of Billingsley (1968), provided we can show that

$$(2.8) \qquad \lim_{m\to\infty} \sup_{n \geqq m+1} \mathbb{P}\left\{ \sum_{k=m+1}^{n} (F_0/F_{kn}) \geqq \delta \right\} = 0$$

for every $\delta > 0$; the former will also be a consequence of (2.8). Now $\phi(t) \equiv \mathbb{E}\exp(-t\log(1+fX_1))$ satisfies $\phi(0)=1$ and $\phi'(0)<0$, so there exists $u \in (0,1)$ with $\phi(u)<1$. Thus, the probability in (2.8) is bounded by

$$\begin{aligned}
\delta^{-u}\mathbb{E}\left(\sum_{k=m+1}^{n} (F_0/F_{kn})\right)^u &\leqq \delta^{-u} \sum_{k=m+1}^{n} \mathbb{E}(F_0/F_{kn})^u \\
&\leqq \delta^{-u} \sum_{k=m+1}^{n} \mathbb{E}(F_0/F_k)^u = \delta^{-u} \sum_{k=m+1}^{n} \phi(u)^k \\
&\leqq \delta^{-u}\phi(u)^{m+1}/(1-\phi(u))
\end{aligned}$$

for each $n > m \geqq 1$, where we have used (2.7). This implies (2.8) and completes the proof.

Corollary 2.3. Let $X_1, X_2, \cdots, f, F_0, R_1, R_2, \cdots$ and R be as in Proposition 2.1. Suppose in addition that $l \geqq 2$ and there exist distinct $\xi_1, \cdots, \xi_l \in [-1, \infty)$ such that $p_j \equiv \mathbb{P}\{X_1 = \xi_j\} > 0$ for $j = 1, \cdots, l$ and $\Sigma_{j=1}^{l} p_j = 1$. For each $n \geqq 1$, let $m_{1n}, \cdots, m_{ln}$ be non-negative integers summing to n, and put

$$A_n = \left\{ \sum_{i=1}^{n} \chi_{\{X_i = \xi_j\}} = m_{jn},\ j = 1, \cdots, l \right\}.$$

Assume that $\lim_{n\to\infty} m_{jn}/n = p_j$ for $j = 1, \cdots, l$. Then $R_n \mid A_n \xrightarrow{\mathscr{D}} R$ as $n \to \infty$.

Proof. For each $n \geqq 1$, let $(X_{1n}, \cdots, X_{nn})$ have the conditional distribution of $(X_1, \cdots, X_n)$ given A_n, and apply Proposition 2.2. Conditions (2.5) and (2.6) are easily checked, while (2.7) follows from Theorem 4 of Hoeffding (1963).

3. Asymptotic distribution of $R/\mathbb{E}X_1$

The main result of this section specifies the asymptotic distribution of $R/\mathbb{E}X_1$ as the bettor's advantage tends to 0.

Lemma 3.1. Let M and ε_0 be positive constants. Let $X_1(\varepsilon)$ be a non-degenerate, $[-1, M]$-valued random variable for each $\varepsilon \in [0, \varepsilon_0)$. Suppose that $X_1(\varepsilon) \xrightarrow{\mathcal{D}} X_1(0)$ as $\varepsilon \to 0+$,

$$\mathbb{E}X_1(\varepsilon) > 0 > \mathbb{E}[X_1(\varepsilon)/(1 + X_1(\varepsilon))], \qquad 0 < \varepsilon < \varepsilon_0, \tag{3.1}$$

where $-1/0 \equiv -\infty$, and $\mathbb{E}X_1(0) = 0$. For $0 < \varepsilon < \varepsilon_0$, define

$$G_\varepsilon(f) = \mathbb{E}\log(1 + fX_1(\varepsilon)), \qquad 0 \leqq f \leqq 1,$$

and $m_1(\varepsilon) = \mathbb{E}X_1(\varepsilon)$, $m_2(\varepsilon) = \mathbb{E}X_1(\varepsilon)^2$.

(a) For each $\varepsilon \in (0, \varepsilon_0)$, $G_\varepsilon(f)$ is strictly concave in f and has a unique maximum at $f = f^*(\varepsilon)$, say, in $(0, 1)$.

(b) $f^*(\varepsilon) = m_1(\varepsilon)m_2(\varepsilon)^{-1}(1 + o(1))$ as $\varepsilon \to 0+$.

(c) For each $\alpha \in (0, 2)$, as $\varepsilon \to 0+$,

$$G_\varepsilon(\alpha f^*(\varepsilon)) = \alpha f^*(\varepsilon)m_1(\varepsilon)(1 - \tfrac{1}{2}\alpha + o(1)).$$

Proof. (a) Fix $\varepsilon \in (0, \varepsilon_0)$. Since

$$G'_\varepsilon(f) = \mathbb{E}\frac{X_1(\varepsilon)}{1 + fX_1(\varepsilon)} \quad \text{and} \quad G''_\varepsilon(f) = -\mathbb{E}\frac{X_1(\varepsilon)^2}{(1 + fX_1(\varepsilon))^2},$$

$G_\varepsilon(f)$ is strictly concave in f, $G'_\varepsilon(f)$ is strictly decreasing in f, and $G'_\varepsilon(0) > 0 > G'_\varepsilon(1-)$ by (3.1). Hence there exists a unique $f^*(\varepsilon) \in (0, 1)$ with $G'_\varepsilon(f^*(\varepsilon)) = 0$; clearly, this choice of f maximizes $G_\varepsilon(f)$.

(b) First we claim that $f^*(\varepsilon) \to 0$ as $\varepsilon \to 0+$. Suppose not. Then there exists $\varepsilon_n \to 0+$ with $f^*(\varepsilon_n) \to f_0 \in (0, 1]$. If $f_0 < 1$, then

$$0 = \lim_{n\to\infty} \mathbb{E}\frac{X_1(\varepsilon_n)}{1 + f^*(\varepsilon_n)X_1(\varepsilon_n)} = \mathbb{E}\frac{X_1(0)}{1 + f_0X_1(0)} < 0 \tag{3.2}$$

by Jensen's inequality, a contradiction. If $f_0 = 1$ the second equality in (3.2) becomes an inequality $(\leqq)$ and $\lim_{n\to\infty}$ is replaced by $\limsup_{n\to\infty}$. Here we are using Theorem 5.3 of Billingsley (1968) and the assumption that $X_1(\varepsilon)$ is bounded above by M. This establishes the claim. Now since $G'_\varepsilon(f^*(\varepsilon)) = 0$, we have

$$m_1(\varepsilon) - f^*(\varepsilon)m_2(\varepsilon) + f^*(\varepsilon)^2\mathbb{E}\frac{X_1(\varepsilon)^3}{1 + f^*(\varepsilon)X_1(\varepsilon)} = 0$$

and hence

$$f^*(\varepsilon) = \frac{m_1(\varepsilon)}{m_2(\varepsilon)} + \frac{f^*(\varepsilon)^2}{m_2(\varepsilon)}\mathbb{E}\frac{X_1(\varepsilon)^3}{1 + f^*(\varepsilon)X_1(\varepsilon)} \tag{3.3}$$

for $0 < \varepsilon < \varepsilon_0$. The desired conclusion now follows from (3.3) and the initial claim.

(c) By a Taylor expansion and (b), as $\varepsilon \to 0+$,

$$\begin{aligned} G_\varepsilon(\alpha f^*(\varepsilon)) &= \mathbb{E}\log(1+\alpha f^*(\varepsilon)X_1(\varepsilon)) \\ &= \alpha f^*(\varepsilon)m_1(\varepsilon) - \tfrac{1}{2}\alpha^2 f^*(\varepsilon)^2 m_2(\varepsilon) + O(f^*(\varepsilon)^3) \\ &= \alpha f^*(\varepsilon)m_1(\varepsilon)\{1 - \tfrac{1}{2}\alpha f^*(\varepsilon)m_1(\varepsilon)^{-1}m_2(\varepsilon) + o(1)\} \\ &= \alpha f^*(\varepsilon)m_1(\varepsilon)(1 - \tfrac{1}{2}\alpha + o(1)). \end{aligned}$$

Theorem 3.2. Let M, ε_0, and F_0 be positive constants. Let $X_1(\varepsilon), X_2(\varepsilon), \cdots$ be i.i.d. random variables for each $\varepsilon \in [0, \varepsilon_0)$ with $X_1(\varepsilon)$, $0 \leqq \varepsilon < \varepsilon_0$, satisfying the conditions of Lemma 3.1. Fix $\alpha \in (0, 2)$, and, using the notation of Lemma 3.1, define $R^\alpha(\varepsilon)$ by (1.7) and (1.2) with $X_i = X_i(\varepsilon)$ and $f = \alpha f^*(\varepsilon)$ for each $\varepsilon \in (0, \varepsilon_0)$ for which $G_\varepsilon(\alpha f^*(\varepsilon)) > 0$. (This is possible by Lemma 3.1 (c).) Then, as $\varepsilon \to 0+$, (1.9) holds. Moreover, $R^\alpha(\varepsilon)/\mathbb{E}X_1(\varepsilon)$ is uniformly integrable in ε, so (1.10) also holds.

Proof. Let $\varepsilon > 0$ be such that $G_\varepsilon(\alpha f^*(\varepsilon)) > 0$, and denote the nth moment of $R^\alpha(\varepsilon)$ by $\mu_n(\varepsilon)$ for each $n \geqq 1$. Let $n \geqq 1$. By Proposition 2.1(c),

$$\mu_n(\varepsilon) = \mathbb{E}(1+\alpha f^*(\varepsilon)X_1(\varepsilon))^n \mathbb{E}[R^\alpha(\varepsilon)^n/(1+\alpha f^*(\varepsilon)R^\alpha(\varepsilon))^n].$$

Since $1 - nx \leqq (1+x)^{-n} \leqq 1 - nx + \binom{n+1}{2}x^2$ for all $x > 0$, and since $\mu_{n+2} \leqq M\mu_{n+1}$, we have

$$(1+\eta)(\mu_n - n\alpha f^*\mu_{n+1}) \leqq \mu_n \leqq (1+\eta)\left\{\mu_n - n\alpha f^*\left(1 - \frac{n+1}{2}M\alpha f^*\right)\mu_{n+1}\right\},$$

where the dependence on ε is implicit and where $\eta = \mathbb{E}(1+\alpha f^* X_1)^n - 1$. Rearranging,

$$\beta_1\mu_n/m_1^n \leqq \mu_{n+1}/m_1^{n+1} \leqq \beta_2\mu_n/m_1^n, \tag{3.4}$$

where

$$\begin{aligned} \beta_1 &= \eta/n\alpha f^* m_1(1+\eta), \\ \beta_2 &= \eta\Big/n\alpha f^* m_1\left(1 - \frac{n+1}{2}M\alpha f^*\right)(1+\eta). \end{aligned}$$

Letting $\varepsilon \to 0+$ and noting that

$$\eta = n\alpha f^* m_1 + \binom{n}{2}\alpha^2 f^{*2} m_2 + O(f^{*3}),$$

we conclude from Lemma 3.1(b) that both β_1 and β_2 converge to $1 + ((n-1)/2)\alpha$. By induction applied to (3.4) (using Proposition 2.1(d) for the initial step),

$\sup_\varepsilon \mu_n/m_1^n < \infty$, and hence $(R^\alpha/\mathbb{E}X_1)^n$ is uniformly integrable in ε for each $n \geqq 1$ (Billingsley (1968), p. 32). Also from (3.4), if $\lim_{\varepsilon\to0+}\mu_n/m_1^n$ exists, then $\lim_{\varepsilon\to0+}\mu_{n+1}/m_1^{n+1}$ exists, and

$$\lim_{\varepsilon\to0+}\mu_{n+1}/m_1^{n+1} = \left(1+\frac{n-1}{2}\alpha\right)\lim_{\varepsilon\to0+}\mu_n/m_1^n. \tag{3.5}$$

Since $R^\alpha/\mathbb{E}X_1$ is uniformly integrable in ε, it is tight (Billingsley (1968), p. 41), hence relatively compact. Let U be any weak limit as $\varepsilon\to0+$, so that $R^\alpha/\mathbb{E}X_1 \xrightarrow{\mathscr{D}} U$ as $\varepsilon\to0+$ through some subsequence. Let ν_n denote the nth moment of U for each $n\geqq1$. By the uniform integrability proved above, $\mu_n/m_1^n\to\nu_n$ as $\varepsilon\to0+$ through the subsequence for each $n\geqq1$. By (3.5),

$$\nu_{n+1} = \left(1+\frac{n-1}{2}\alpha\right)\nu_n, \qquad n\geqq1. \tag{3.6}$$

Define $\theta = 2/\alpha - 1$ and $\lambda = 2/\alpha$. Using (3.6), we find that U has Laplace transform

$$\begin{aligned}\mathbb{E}e^{-tU} &= 1+\sum_{n=1}^{\infty}(-t)^n\nu_n/n!\\ &= 1+\sum_{n=1}^{\infty}(-t)^n(\theta+n-1)\cdots(\theta+1)\nu_1/\lambda^{n-1}n!\\ &= \left\{\sum_{n=0}^{\infty}\left(\frac{t}{\lambda}\right)^n\binom{-\theta}{n}\right\}\frac{\lambda}{\theta}\nu_1+1-\frac{\lambda}{\theta}\nu_1\\ &= \left(1+\frac{t}{\lambda}\right)^{-\theta}\frac{\lambda}{\theta}+1-\frac{\lambda}{\theta}\nu_1.\end{aligned}$$

As $t\to\infty$, this tends to $1-(\lambda/\theta)\nu_1$, which must therefore be non-negative. Hence U is a mixture of gamma(θ,λ) and δ_0, the unit mass concentrated at 0.

Let $0<\delta<\min(1,\theta)$. We claim that

$$\limsup_{\varepsilon\to0+}\mathbb{E}(R^\alpha(\varepsilon)/\mathbb{E}X_1(\varepsilon))^{-\delta}<\infty. \tag{3.7}$$

Granting this for the moment, it follows that U must be purely gamma(θ,λ). Since U was an arbitrary weak limit as $\varepsilon\to0+$ of $R^\alpha(\varepsilon)/\mathbb{E}X_1(\varepsilon)$, we conclude that (1.9) holds as $\varepsilon\to0+$.

We turn to the proof of (3.7). As $\varepsilon\to0+$,

$$\begin{aligned}\mathbb{E}(1+\alpha f^*X_1)^{-\delta} &= 1-\delta\alpha f^*m_1+\tfrac12\delta(\delta+1)\alpha^2f^{*2}m_2+O(f^{*3})\\ &= 1-\delta\alpha f^*m_1(1-\tfrac12(\delta+1)\alpha f^*m_1^{-1}m_2+o(1))\\ &= 1-\delta\alpha f^*m_1(1-\tfrac12(\delta+1)\alpha+o(1))\end{aligned} \tag{3.8}$$

by Lemma 3.1(b). Since $0 < \delta < \theta$, this is less than 1 for all ε sufficiently small. In particular, for such ε,

$$\begin{aligned}\mathbb{E}(R^\alpha)^{-\delta} &= (\alpha f^*)^\delta \mathbb{E}\left(\sum_{k=1}^\infty (F_0/F_k)\right)^\delta \\ &\leqq (\alpha f^*)^\delta \sum_{k=1}^\infty \mathbb{E}(F_0/F_k)^\delta \\ &= (\alpha f^*)^\delta \sum_{k=1}^\infty \{\mathbb{E}(1+\alpha f^* X_1)^{-\delta}\}^k \\ &< \infty.\end{aligned}$$

Now by Proposition 2.1(c) (in particular, (2.4)),

$$\mathbb{E}\left(\frac{m_1}{R^\alpha}\right)^\delta = \mathbb{E}(1+\alpha f^* X_1)^{-\delta}\mathbb{E}\left(\alpha f^* m_1 + \frac{m_1}{R^\alpha}\right)^\delta.$$

Using the inequality $(x+c)^\delta \leqq x^\delta + \delta c x^{\delta-1}$, valid for all $x > 0$ and $c > 0$ (since $0 < \delta \leqq 1$), and Proposition 2.1(d) together with Jensen's inequality, we find that

$$\begin{aligned}\mathbb{E}\left(\frac{m_1}{R^\alpha}\right)^\delta &\leqq \mathbb{E}(1+\alpha f^* X_1)^{-\delta}\left\{\mathbb{E}\left(\frac{m_1}{R^\alpha}\right)^\delta + \delta\alpha f^* m_1 \mathbb{E}\left(\frac{m_1}{R^\alpha}\right)^{\delta-1}\right\} \\ &\leqq \mathbb{E}(1+\alpha f^* X_1)^{-\delta}\left\{\mathbb{E}\left(\frac{m_1}{R^\alpha}\right)^\delta + \delta\alpha f^* m_1\right\},\end{aligned}$$

and hence

$$\mathbb{E}(R^\alpha/m_1)^{-\delta} \leqq \delta\alpha f^* m_1\{1-\mathbb{E}(1+\alpha f^* X_1)^{-\delta}\}^{-1}. \tag{3.9}$$

By (3.8), the right side of (3.9) tends to $(1-\frac{1}{2}(\delta+1)\alpha)^{-1}$ as $\varepsilon \to 0+$, proving (3.7) and completing the proof.

Remark 3.3. We outline an alternative proof of (1.9), valid when $X_1(\varepsilon)$ is given by (1.1) with $p = (1+\varepsilon)/2$ and somewhat more generally. By (3.7), $(R^\alpha(\varepsilon)/\mathbb{E}X_1(\varepsilon))^{-1}$ (defined for ε positive and sufficiently small) is uniformly integrable in ε, hence tight, and therefore relatively compact. Let V be an arbitrary weak limit as $\varepsilon \to 0+$, and denote its Laplace transform by $\phi(t)$, $t \geqq 0$. Using Proposition 2.1(c) (in particular, (2.4)), a Taylor expansion, and Lemma 3.1(b), we find that ϕ satisfies the differential equation

$$\tfrac{1}{2}\alpha t\phi''(t) + (\alpha-1)\phi'(t) - \phi(t) = 0, \qquad t > 0. \tag{3.10}$$

Moreover, ϕ is monotone decreasing and $\phi(0+) = 1$. It follows from Gradshteyn and Ryzhik (1980), Equation 8.494(5), that

$$\phi(t) = 2(\lambda t)^{\theta/2} K_\theta(2\sqrt{\lambda t})/\Gamma(\theta), \qquad t > 0, \tag{3.11}$$

where $\theta = 2/\alpha - 1$, $\lambda = 2/\alpha$, and K_θ is the (decreasing) modified Bessel function of order θ. We conclude from Gradshteyn and Ryzhik (1980), Equation 3.471(9), that $1/V$ is gamma(θ, λ). Since V was an arbitrary weak limit of $(R^\alpha(\varepsilon)/\mathbb{E}X_1(\varepsilon))^{-1}$ as $\varepsilon \to 0+$, the desired result follows.

Acknowledgements

We are grateful to Peter A. Griffin for valuable correspondence and to Thomas G. Kurtz and Stanford Wong for helpful suggestions.

References

BILLINGSLEY, P. (1968) *Convergence of Probability Measures.* Wiley, New York.

BREIMAN, L. (1961) Optimal gambling systems for favorable games. *Proc. 4th Berkeley Symp. Math. Statist. Prob.* **1**, 65–78.

FINKELSTEIN, M. AND WHITLEY, R. (1981) Optimal strategies for repeated games. *Adv. Appl. Prob.* **13**, 415–428.

GRADSHTEYN, I. S. AND RYZHIK, I. M. (1980) *Table of Integrals, Series, and Products.* Academic Press, New York.

GRIFFIN, P. A. (1981) *The Theory of Blackjack*, 2nd edn. GBC Press, Las Vegas.

HOEFFDING, W. (1963) Probability inequalities for sums of bounded random variables. *J. Amer. Statist. Assoc.* **58**, 13–30.

KELLY, J. L., JR. (1956) A new interpretation of information rate. *Bell System Tech. J.* **35**, 917–926.

WONG, S. (1981) What proportional betting does to your win rate. *Blackjack World* **3**, 162–168.