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REGULAR ARTICLE

Neighborhood structure and the evolution of cooperation

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Abstract This paper deals with the problem of explaining the survival of cooperative behavior in populations in which each person interacts only with a small set of social 'neighbors', and individuals adjust their behavior over time by myopically imitating more successful strategies within their own neighborhood. We identify two parameters—the interaction radius and the benefit–cost ratio—which jointly determine whether or not cooperation can survive. For each value of the interaction radius, there exists a critical value of the benefit–cost ratio which serves as the threshold below which cooperation cannot be sustained. This threshold itself declines as the interaction radius rises, so there is a precise sense in which dense networks are more conducive to the evolution of cooperation.

Keywords Local interaction · Evolution · Cooperation

JEL Classification C72 · D64

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1 Introduction

This paper deals with the problem of explaining the survival and stability of cooperative behavior in large unrelated populations in which each person interacts only with a small set of social 'neighbors', and individuals adjust their behavior over time by myopically imitating more successful strategies within their own neighborhood.

The idea that cooperative behavior can survive under evolutionary competition with self-interested behavior when interaction is local goes back at least to Eshel (1972) and has received attention more recently by Bergstrom and Stark (1993); Nowak and May (1992, 1993), and Eshel et al. (1998, 1999) among others. What makes local interaction with imitation conducive to the survival of cooperative behavior is that individuals expressing a particular behavior become increasingly likely to have neighbors who express the same behavior. This is similar to assortative matching among cooperators (Bergstrom 2002), although under local interaction such assortation arises *endogenously*. The dynamics of imitation give rise to cooperative and noncooperative clusters, with individuals in cooperative clusters earning significantly higher payoffs on average than individuals in noncooperative clusters. This makes it possible for small clusters of cooperation to survive and spread over time. Although opportunistic behaviors can spread within cooperative neighborhoods, this process itself creates inefficient opportunistic clusters, the poor performance of which limits their further expansion. This intuitive argument suggests not only that cooperation may be stable under local interaction but also that the stability of cooperation is likely to be quite sensitive to changes in neighborhood structure.

Most models of local interaction assume that the neighborhood structure is highly regular and exogenously fixed. For example, Eshel et al. (1998) consider individuals arrayed in a circle, with each person linked to their two immediate neighbors and Nowak and May (1992) consider individuals arrayed in a two-dimensional grid with each person linked to their eight geographically closest neighbors. The effects of *changes* in neighborhood structure on the viability of altruism has not, to our knowledge, been systematically explored. In this paper, while maintaining the strong symmetry assumptions that have been made in previous work, we examine the effects of changes in interaction radius on the survival of altruism. By increasing interaction radius we mean that the social benefits of an altruistic act are spread across a larger number of individuals (with the aggregate social benefit and private cost being held constant) and, consequently, that the evolution of behavior at a particular location is influenced by the payoffs obtained at a greater number of other locations within the neighborhood.

Our model extends that of Eshel et al. (1998) by allowing individuals on a circle to interact not just with on two immediate neighbors, but an arbitrary number of neighbors symmetrically in either direction. Cooperative behavior is privately costly but socially beneficial, and its survival depends on the ratio of benefits to costs. The main result of the paper is that for any interaction radius, we identify a bifurcation value of the benefit–cost ratio such that states with cooperative behaviors can be sustained in the steady state of the dynamics when the benefit–cost ratio exceeds this threshold. When the benefit–cost ratio is insufficiently high, cooperators deep within cooperative clusters switch to opportunistic behavior. This causes cooperative clusters to be punctured from within, leading to ever smaller cooperative clusters that eventually disappear in the presence of opportunist neighbors. If, instead, the benefit–cost ratio is greater than the threshold, cooperative clusters expand until the surrounding opportunist clusters are small enough to contain the growth of the opportunist clusters. From this point onward, the population enters a steady state or cycle. We also show that the threshold benefit–cost ratio above which cooperation can be sustained itself *falls* as the interaction radius increases. Specifically, the aggregate benefits of the cooperative act can be smaller when the interaction radius is larger in order for cooperation to survive under evolutionary competition with opportunistic behavior. An interesting implication is that there is a sense in which dense networks are more conducive to the evolution of cooperation than sparse networks.

2 Neighborhoods, behavior, and dynamics

Consider a finite population P of n individuals such that each individual $i \in P$ has a set of social "neighbors" $N(i) \subset P \setminus \{i\}$ with whom she interacts. Each individual is a neighbor to their neighbors, so that $j \in N(i)$ if and only if $i \in N(j)$. If i and j are neighbors, they are said to be *connected*. Let k_i denote the cardinality of $N(i)$, that is, the number of individuals with whom i interacts.

During any given period t each individual may take one of two actions. One action is altruistic, and any individual i taking it incurs a cost α. The action yields an aggregate benefit $\beta > \alpha$, shared equally by all individuals in $N(i)$.[1] The other action is egoistic and has no cost to oneself or benefit to others. Without loss of generality, we normalize $\alpha = 1$ and interpret β as the benefit–cost ratio. Since $\beta > 1$, efficiency requires the altruistic action to be taken by all players. From the perspective of any individual, however, the opportunistic action yields a higher payoff regardless of the actions taken by her neighbors. This is a multiplayer prisoner's dilemma with local interaction, of the kind studied by Bergstrom and Stark (1993); Eshel et al. (1998); Nowak and May (1992); Albin and Foley (2001). A central question in this literature is whether or not altruism can persist under evolutionary dynamics which are payoff monotonic in the sense that more highly rewarded actions are replicated at greater rates than are less highly rewarded actions within the neighborhood.

[1] The assumption that the altruist herself does not receive a share of the benefits is made for convenience; any such benefit can be accommodated by interpreting α as a net cost.

Let $s_i(t) = 1$ if individual i takes the altruistic action at time t and $s_i(t) = 0$ otherwise. The vector $s(t) = (s_1(t), ..., s_n(t))$ is the state of the system at time t. Let $\mathcal{S} \equiv \{0, 1\}^n$ denote the set of all states. The payoff to player i at time t is

$$\pi_i(t) = -s_i(t) + \beta \sum_{j \in N(i)} \frac{s_j(t)}{k_j}, \tag{1}$$

since $1/k_j$ is the benefit conferred by neighbor j onto i if $s_j(t) = 1$. The *neighborhood* of individual i is defined as the set of i's neighbors together with i herself, that is, $N(i) \cup \{i\}$. Consider any individual i with at least one neighbor taking a different action than i herself does. The total number of i's neighbors who take egoistic action at time t is simply

$$\sum_{j \in N(i) \cup \{i\}} \left(1 - s_j(t)\right).$$

The sum of the payoffs of these individuals may be expressed conveniently as

$$\sum_{j \in N(i) \cup \{i\}} \left(1 - s_j(t)\right) \pi_j(t).$$

This expression adds together all payoffs $\pi_j(t)$ of all (and only) those individuals choosing $s_j(t) = 0$. Using this, and Eq. 1, the average payoff of "egoists" in the neighborhood of individual i is therefore

$$\bar{\pi}_i^e(t) = \frac{\sum_{j \in N(i) \cup \{i\}} \left(1 - s_j(t)\right) \pi_j(t)}{\sum_{j \in N(i) \cup \{i\}} \left(1 - s_j(t)\right)} = \frac{\beta \sum_{j \in N(i) \cup \{i\}} \sum_{l \in N(j)} \left(1 - s_j(t)\right) s_l(t)/k_l}{\sum_{j \in N(i) \cup \{i\}} \left(1 - s_j(t)\right)}. \tag{2}$$

We interpret $\bar{\pi}_i^e(t)$ as the average "fitness" of egoists in the neighborhood of individual i at time t.

By similar reasoning, the average payoff of altruists in the neighborhood of individual i is

$$\bar{\pi}_i^a(t) = \frac{\sum_{j \in N(i) \cup \{i\}} s_j(t) \pi_j(t)}{\sum_{j \in N(i) \cup \{i\}} s_j(t)} = \frac{\beta \sum_{j \in N(i) \cup \{i\}} \sum_{l \in N(j)} s_j(t) s_l(t)/k_l}{\sum_{j \in N(i) \cup \{i\}} s_j(t)} - 1, \tag{3}$$

and $\bar{\pi}_i^e(t)$ may be interpreted as the average fitness of altruists in the neighborhood of individual i at time t. Finally, let $\rho(t) = \sum_{i \in P} s_i(t)/n$ be the share of altruists in the population at time t. This summary measures proportion of altruists in the population, and so its time evolution will be used to examine the survival and stability of cooperation.

The distribution of actions $s(t)$ evolves according to the deterministic dynamics described in Eshel et al. (1998). Specifically, each individual selects in period t whichever action resulted in the highest average payoff in her neighborhood in period $t-1$. For any player i in a heterogeneous neighborhood, choice of action is determined as follows.[2]

$$s_i(t) = \begin{cases} 0 \text{ if } \bar{\pi}_i^e(t-1) \geq \bar{\pi}_i^a(t-1), \\ 1 \text{ if } \bar{\pi}_i^e(t-1) < \bar{\pi}_i^a(t-1). \end{cases} \quad (4)$$

Players whose neighborhood is homogeneous observe only one action and hence continue to adopt that action: for any such individual i, $s_i(t) = s_i\,(t-1)$ if $s_i\,(t-1) = s_j\,(t-1)$ for all $j \in N_i$.

A sequence $\{s(t), s(t+1), ..., s(t+T)\}$ of successive states is an *absorbing cycle of period* T if $s(t+T) = s(t)$ and $s(t+i) \neq s(t)$ for all $i \in \{1, ..., T-1\}$. An absorbing cycle of period 1 is an *absorbing state*. Since the population size n is finite, trajectories from all initial states must reach an absorbing state or an absorbing cycle within a finite number of periods. For any given neighborhood structure, the absorbing set reached will depend on the initial state. The main questions with which the present study is concerned is the likelihood of reaching an absorbing set with a large proportion of altruists from some randomly given initial state, and how this changes with neighborhood structure.

3 Network structure

In this paper, we consider a special class of regular networks in which individuals are arrayed in a circle, and each one is connected to its closest r neighbors on either side. Such networks are *ring lattices* , and r is the *interaction radius*. Formally, we assume that each individual i is connected to individuals $i \pm k \pmod n$, for $1 \leq k \leq r$. That is, $N(i) = \{i \pm k \pmod n \mid 1 \leq k \leq r\}$. Figure 1 illustrates such networks, with circles representing individuals and lines connecting neighbors. The interaction radius is $r = 1$ in Fig. 1a and $r = 2$ in Fig. 1b.

We can rewrite Eqs. 1–3 for the ring lattice model as follows:[3]

$$\pi_i = -s_i + \frac{\beta}{2r}\left(\sum_{j=i-r}^{i-1} s_j + \sum_{j=i+1}^{i+r} s_j\right)$$

$$\bar{\pi}_i^e = \frac{\sum_{j=i-r}^{i+r}\left(1-s_j\right)\pi_j}{\sum_{j=i-r}^{i+r}\left(1-s_j\right)} \text{ and } \bar{\pi}_i^a = \frac{\sum_{j=i-r}^{i+r} s_j \pi_j}{\sum_{j=i-r}^{i+r} s_j}$$

[2] As a tie-breaking convention, assume that when both actions yield the same average payoff, the *egoistic* action is chosen. This makes it somewhat less likely that altruism will be sustained from any given initial state, but the bias is of little consequence since ties of this kind will not occur generically.

[3] For expositional reasons, we omit mod n where this is clearly understood and drop the dependence of the variables on time t.

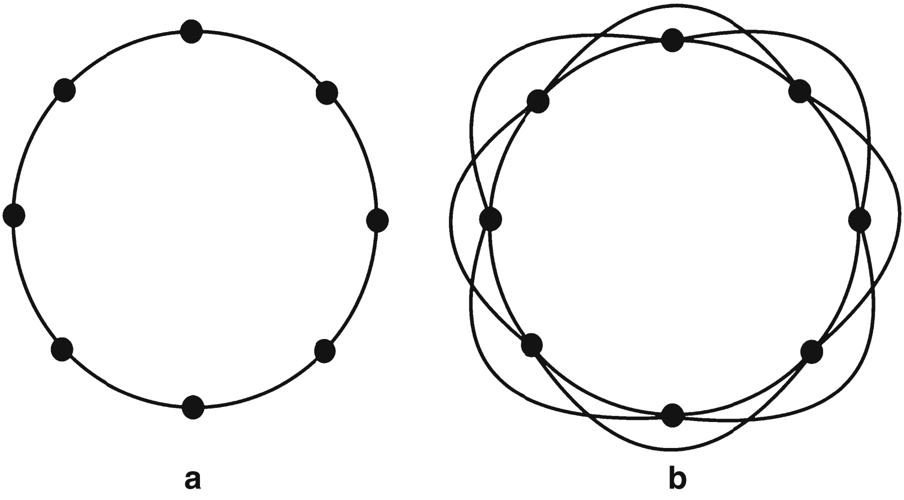

Fig. 1 Ring lattices with interaction radius **a** $r = 1$ and **b** $r = 2$

Given these payoffs, the dynamics (Eq. 4) govern the movement of the state over time.

4 Dynamics

For any given network structure, the dynamics of the distribution of actions over time will depend on two key elements of the model: the initial state $s(0)$ and the benefit–cost ratio β. Generally speaking, the survival of altruistic behavior in the long run will require both a favorable set of initial conditions and a sufficiently high value of β. Favorable initial conditions require that there be at least one altruistic cluster that is sufficiently large. Provided that β is large enough, altruistic clusters of sufficient size will expand when surrounded on both sides by large egoistic clusters. On the other hand, if altruistic clusters are too small, or the benefit–cost ratio is insufficiently high, altruism in the population will not be viable in the long run. For certain intermediate values of β the level of altruism in the population can increase in the short run before collapsing to zero in the long run.

In order to characterize the distribution of actions in the system, we introduce the following notation. Let $\mathcal{Q}$ be the set of all sequences $\{i, i+1, ..., i+l-1\}$ (mod n) where $i, l \in \{1, ..., n\}$. For any given state s, we say that $q = \{i, i+1, ..., i+l-1\} \in \mathcal{Q}$ is a *cluster of length* l if $s_i = s_j$ for all $i, j \in q$, and $s_{i-1} = s_{i+l} \neq s_i$. A cluster therefore corresponds to a set of adjacent players who take the same action, and which is of maximal length with respect to this property. For any state s, the set of players can be partitioned into a finite number $z \leq n$ of adjacent clusters $q_1, ..., q_z$. We shall say that a cluster q is altruist cluster if $s_i = 1$ for all $i \in q$. Egoist clusters are analogously defined.

The set of altruist clusters in period t is $\mathcal{A}(t)$; the set of egoist clusters is $\mathcal{E}(t)$. We say that an altruist cluster $q \in \mathcal{A}(t)$ *survives* in period $t+1$ if $q \in \mathcal{A}(t+1)$. It *expands* if $q \subset q' \in \mathcal{A}(t+1)$. It *contracts* if there exists $q' \subset q$ such that $q' \in \mathcal{A}(t+1)$, and all elements $i \in q$, $i \notin q'$ belong to egoist clusters in period $t+1$. It *vanishes* if $q \subseteq q' \in \mathcal{E}(t+1)$. Analogous definitions apply for egoist clusters. We say that a cluster which does not survive, expand, contract or vanish is *punctured*. Finally, let $L(a,b) \subset \mathcal{S}$ denote the set of states in which all altruist clusters are of length at least a and all egoist clusters are of length at least b, and let $U(a,b) \subset \mathcal{S}$ denote the set of states in which all altruist clusters are of length at most a and all egoist clusters are of length at most b.

To get a sense of the dynamic possibilities inherent in this system, consider first an initial state $s \in L(3r, 2r)$ so that all altruist clusters are of length at least $3r$ and all egoist clusters are of length at least $2r$. Without loss of generality, suppose that the set of players $\{1, ..., l\}$ constitute an altruist cluster, where $l \geq 3r$ by hypothesis. This cluster has (on either side) egoist clusters of length at least $2r$. Consider player m where $m \in \{1, ..., r\}$ from the border with the egoist cluster. The neighborhood of this player contains $r+m$ altruists (including herself). Of these, exactly m altruists are in homogeneous neighborhoods and hence obtain a payoff of $(\beta - 1)$. The payoffs of the remainder depend on their distance from the egoist cluster. From the perspective of player m, the mean altruist payoff in her interaction neighborhood is accordingly

$$\bar{\pi}_m^a = -1 + \frac{1}{r+m}\left(m\beta + \frac{\beta}{2r}\sum_{i=1}^{r}(r+i-1)\right) = \frac{1}{4}\beta\frac{4m+3r-1}{r+m} - 1. \tag{5}$$

Player m has $r+1-m$ egoists in her interaction neighborhood, and the mean egoist payoff in her neighborhood is given by

$$\bar{\pi}_m^e = \left(\frac{1}{r+1-m}\right)\frac{\beta}{2r}\sum_{i=1}^{r+1-m}(r+1-i) = \frac{1}{4}\beta\frac{r+m}{r}. \tag{6}$$

The payoff difference $\bar{\pi}_m^a - \bar{\pi}_m^e$, after some simplification, can be written as

$$\bar{\pi}_m^a - \bar{\pi}_m^e = \frac{\beta - \varphi(r,m)}{\varphi(r,m)} \tag{7}$$

where

$$\varphi(r,m) = \frac{4r\,(r+m)}{2rm + 2r^2 - r - m^2}.$$

Since $1 \leq m \leq r$, $\varphi(r,m)$ is strictly positive. $\varphi(r,m)$ can be interpreted as the minimum of altruist benefit for altruist m to remain as altruist in the next period. Hence altruist m will switch if and only if $\beta \leq \varphi(r,m)$. Note that

$$\frac{\partial\varphi(r,m)}{\partial m} = \frac{4r\left(2rm - r + m^2\right)}{\left(2rm + 2r^2 - r - m^2\right)^2} > 0. \tag{8}$$

So $\varphi(r, m)$ is greatest when $m = r$ and least when $m = 1$. This implies that the altruist who is furthest from the egoist cluster is most likely to be converted to egoist. Define $\beta_l(r) = \varphi(r, 1)$ and $\beta_h(r) = \varphi(r, r)$. Note that

$$\beta_l(r) = \frac{4r}{2r - 1},$$
$$\beta_h(r) = \frac{8r}{3r - 1}.$$

When $r = 1$, $\beta_l(r) = \beta_h(r) = 4$. Both functions are strictly decreasing in r and bounded below by 2 and 8/3 respectively. For all $r > 1$, $\beta_h(r) > \beta_l(r)$. Figure 2 plots β_h for various values of r.

Provided that the initial state $s \in L(3r, 2r)$, all altruists will remain altruists under the dynamics if and only if $\beta > \beta_h(r)$. Similarly, all altruists with at least one egoist neighbor will become egoists if and only if $\beta \leq \beta_l(r)$. For the intermediate range $\beta_l(r) < \beta \leq \beta_h(r)$, some but not all altruists will switch. That is, $\beta_h(r)$ is the threshold value of altruist benefit over which all altruists facing at least one egoist neighbor will remain altruists in the next period and $\beta_l(r)$ is the threshold value of altruist benefit below which all altruists facing at least one egoist neighbor will switch to egoists in the next period. The fact that $\beta_h(r)$ is declining implies that the range of parameter values for which altruist

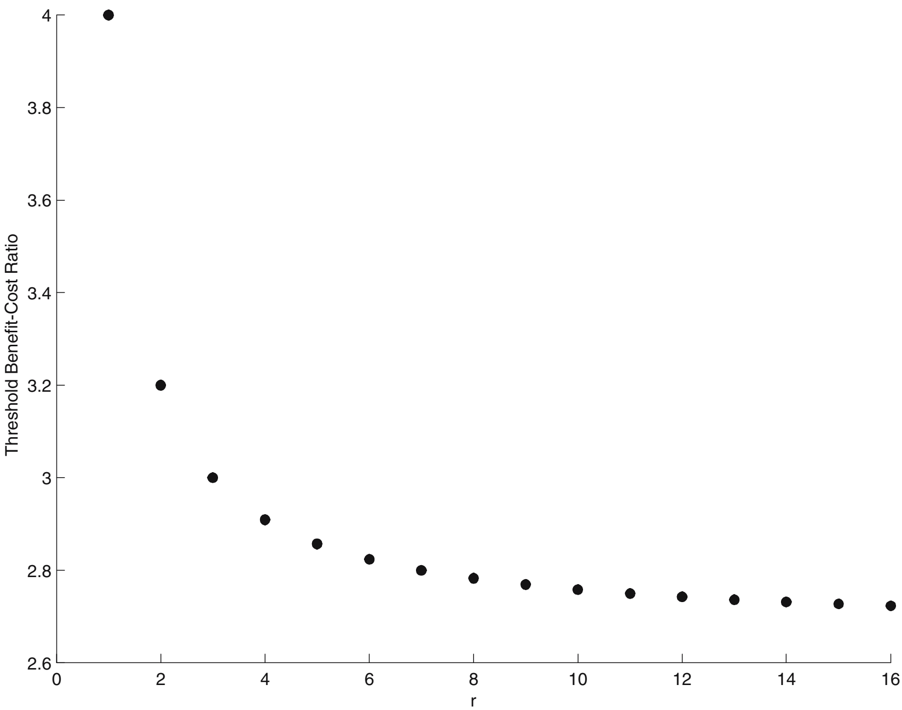

Fig. 2 Locus of $\beta_h(r)$

clusters remain intact (under the stated conditions) *rises* with the interaction radius r.

We next proceed with an initial state $s \in L(2r, 3r)$ and show that, in this case, if $\beta > \beta_h(r)$, then all egoists with one or more altruist neighbors become altruists. Without loss of generality, suppose that the set of players $\{1, ..., l\}$ constitute an egoist cluster, where $l \geq 3r$ by hypothesis. This cluster has (on either side) altruist clusters of length at least $2r$. Consider the player m, where $m \in \{1, ..., r\}$ from the border with altruist cluster and note that the neighborhood of this player contains $r + m$ egoists (including herself). Of these, exactly m egoists are in homogeneous neighborhoods and hence obtain a payoff of 0. From the perspective of player m, the mean egoist payoff in her interaction neighborhood is accordingly

$$\bar{\pi}_m^e = \frac{1}{r+m}\left(\frac{\beta}{2r}\sum_{i=1}^{r}(r+1-i)\right) = \frac{1}{4}\beta\frac{r+1}{r+m}. \tag{9}$$

Player m has $r + 1 - m$ altruists in her neighborhood, and the mean altruist payoff in her neighborhood is given by

$$\bar{\pi}_m^a = -1 + \frac{1}{r+1-m}\left(\frac{\beta}{2r}\sum_{i=1}^{r+1-m}(r+i-1)\right) = -1 + \frac{1}{4}(3r-m)\frac{\beta}{r}. \tag{10}$$

The payoff difference $\bar{\pi}_m^a(s^t) - \bar{\pi}_m^e(s^t)$, after some simplification, may be expressed as

$$\bar{\pi}_m^a - \bar{\pi}_m^e = \frac{\beta - \varphi(r,m)}{\varphi(r,m)}, \tag{11}$$

exactly as in Eq. 7. Following the same arguments made above, we conclude that provided the initial state $s \in L(2r, 3r)$, all egoists will remain egoists if and only if $\beta \leq \beta_l(r)$. Similarly, all egoists with at least one altruist neighbor will become altruists if and only if $\beta > \beta_h(r)$. For the intermediate range $\beta_l(r) < \beta \leq \beta_h(r)$, some but not all egoists will switch. Those who switch will be the ones closest to the altruist cluster, and hence the altruist cluster can spread. Hence, for parameter values in this intermediate range, altruist clusters can expand at the edges but are punctured from within. We illustrate this case with an example where $n = 50$, $k = 8$, and $\beta = 2.89 < \beta_h(r) = 2.9091$, simulated in Fig. 3.

Figure 3 shows that the population share of altruists grows in the short run but altruists are eventually eliminated from the population (the darker cells represent altruists and the lighter ones for egoists and time flows from top to bottom). In the initial period the population is composed of a single altruistic cluster with 33 individuals and a smaller egoistic cluster with 17 individuals. In the second period the total number of altruists rises to 37, although there are now three distinct altruistic clusters. This happens because the initial cluster of altruists is punctured from within, creating two egoistic clusters each composed of just one individual. In period 3 these egoistic clusters expand, resulting in a decline in the total number of altruists. On the three altruistic clusters

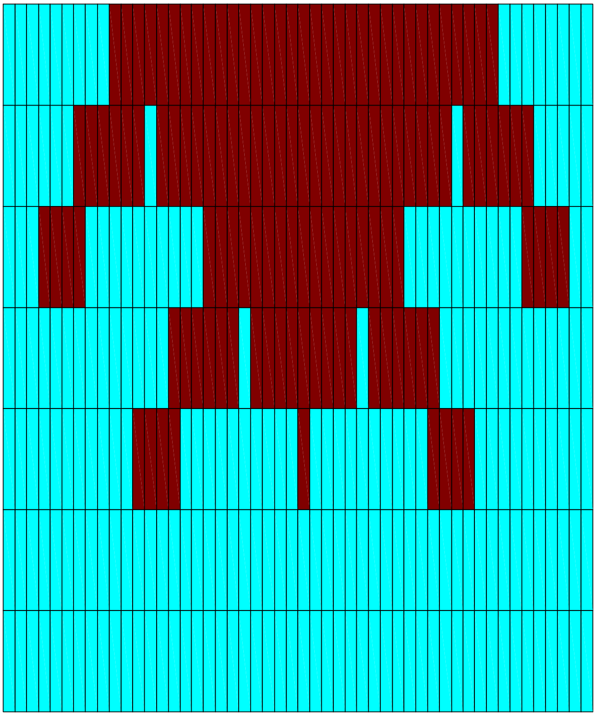

Fig. 3 Evolution of an altruist cluster when $\beta \in (\beta_l, \beta_h)$

remaining, the two smaller ones disappear in period 4. The larger altruistic cluster expands at the edges but is again punctured from within. By period 5, all altruistic clusters are too small to be viable and by period 6 the entire population is at a steady state composed only of egoists.

5 Main results

Putting together the above arguments in the previous section and applying them iteratively, we find that if $\beta > \beta_h(r)$ and the initial state $s(0)$ belongs to $L(3r, 3r)$, then altruist clusters will expand and egoist clusters contract as long as the condition $s \in L(3r, 3r)$ continues to be satisfied. Eventually a state will be reached in which at least one egoist cluster is shorter than $3r$, while all altruist clusters are at least $5r$ in length. The dynamics from this point onwards are much more difficult to characterize, but it can be shown that the following holds.

Proposition 1 *Suppose* $\beta > \beta_h(r)$ *and the initial state* $s(0) \in L(3r, 3r)$. *Then there exists an integer* τ *such that* $\rho(t) \geq 0.5$ *for all* $t \geq \tau$.

Proof See the Appendix. □

The intuition for this result is as follows. Since $\beta > \beta_h(r)$, initially the altruist clusters expand and the egoist clusters contract. The question is then whether these contracted egoist clusters can revive back to its original or greater size. Equation 8 implies that the altruist who neighbors with one single egoist is most likely to be converted to egoist. If she is converted, this altruist cluster rots from the core out and collapses. However, since $\beta > \beta_h(r)$, altruist cluster of sufficiently large size will not be punctured. This implies that once egoist cluster contract to less than $3r$, it will either remain less than $3r$ or vanish.

The set of initial conditions for which the above result applies is very restrictive in a probabilistic sense, particularly when k is relatively large. The same logic that underlies the proof of this proposition, however, suggests that if $\beta > \beta_h(r)$ and the initial state $s(0)$ contains at least one sufficiently large altruist cluster, a significant population share of altruists persists in the long run. Numerical results supportive of this conjecture are presented in Section 6.

What about the case in which $\beta \leq \beta_h(r)$? As the example of Fig. 3 shows, movements in population shares can be quite complex and nonmonotonic. One feature of dynamics that is observed when $\beta \leq \beta_h(r)$ but not when $\beta > \beta_h(r)$ is that altruist clusters may expand at the edges but are punctured from within (see Fig. 3 as an example). The reason can be discerned as follows. Equations 7 and 11 suggest that there exists asymmetry in the likelihood of being converted to the alternative action for members of altruist and egoist clusters in a sense that egoist who is *nearer* to the altruist cluster is more likely to be converted, while altruist who is further *away* from the egoist cluster is more likely to be converted. This implies that altruist clusters can be punctured but egoist clusters will never be. This process of collapsing within the altruist cluster is attributable to the break-down of large altruist clusters, which would otherwise survive, into several smaller clusters, which eventually vanish. Based on this observation, in the next section we show by simulations that if $\beta \leq \beta_h(r)$ and the initial state $s(0)$ contains at least one egoist, then there exists τ such that $\rho(t) = 0$ for all $t \geq \tau$.

Before presenting numerical results supportive of the above claims, we provide a partial characterization of the dynamics in the special case where the interaction radius is $r = 2$.

Proposition 2 *Suppose $r = 2$, and $\beta < \beta_h(r)$. Then, if the initial state contains at least one egoist, an absorbing state is reached in which all players are egoists.*

Proof See the Appendix. □

The underlying mechanism behind Proposition 2 is that the puncturing of altruist clusters can lead to the collapse of altruism in the steady state, even if the altruist population grows in the short-run (as in Fig. 3). Taken together, Propositions 1 and 2 suggest that $\beta_h(r)$ can serve as a bifurcation value below which a large number of states in which the majority of individuals are altruists lose stability: when $\beta > \beta_h(r)$ the long run viability of altruism is practically ensured, while when $\beta \leq \beta_h(r)$, altruism is not viable under any initial conditions. Although these results have been proved only for special cases (with respect to initial conditions and the interaction radius), the numerical results in the next section suggest that they apply much more generally.

6 Simulations

In this section we argue on the basis of simulation results that the analytical results of the previous section hold under a more general set of conditions. Define $\beta_\epsilon^+ = \beta_h(r) + \varepsilon$ and $\beta_\epsilon^- = \beta_h(r) - \varepsilon$ for some small ε. If $\beta_h(r)$ is indeed a bifurcation value as claimed, the limiting properties of the system should be dramatically different when $\beta = \beta_\epsilon^+$ relative to the case of $\beta = \beta_\epsilon^-$.

For some given population size n and a range of values for the interaction radius r, the dynamics are simulated for a large number of randomly selected initial states, using parameters β_ϵ^+ and β_ϵ^-. Given the population size and interaction radius, we compute the limiting share of altruists from the given initial state and benefit–cost ratio $\beta \in \{\beta_\epsilon^+, \beta_\epsilon^-\}$. Different initial conditions s will, in general, result in different values of the limiting share. To test the claims made above, we partition the set of initial conditions according to the size of the largest altruist cluster l. To do so, we let $F(a, b)$ denote the set of initial conditions in which the largest altruist clusters are of length at least a but strictly less than b.

Each cell in Table 1 reports the mean of this limiting share of altruists over the set of initial states according to $F(a, b)$ for the specified parameter values. These results are based on $n = 200$ and r ranging from 1 to 5. We choose ε to be 0.0001. For each parameter configuration, 10,000 randomly selected initial configurations of states are explored. Each of such configurations is run until convergence to some absorbing state or cycle. Then each $\rho(t)$ in absorbing state or cycle is computed and averaged over in all initial configurations, which is presented in Table 1.

In all simulations, regardless of interaction radius r, all altruists are eliminated in the long run whenever $\beta = \beta_\epsilon^- < \beta_h(r)$. This can be seen by observing

Table 1 Steady state mean share of altruists for values of β and r

β	r	$F(1,r)$	$F(r,2r)$	$F(2r,3r)$	$F(3r,4r)$	$F(4r,5r)$	$F(5r,n)$
β_ε^+	1	0	0	0	0.9306	0.9157	0.9094
	2	0	0	0	0.2238	0.4388	0.9749
	3	0	0	0	0.0439	0.1498	0.9709
	4	0	0	0	0.0096	0.0524	0.9604
	5	0	0	0	0.0081	0.0274	0.9498
β_ε^-	1	0	0	0	0	0	0
	2	0	0	0	0	0	0
	3	0	0	0	0	0	0
	4	0	0	0	0	0	0
	5	0	0	0	0	0	0

that the mean share of altruists in steady state or cycle is zero for all classes when $\beta = \beta_\epsilon^-$. When $\beta = \beta_\epsilon^+ > \beta_h(r)$, on the other hand, the long-run population share of altruists depends on whether the initial state contains a sufficiently large cluster of altruists. Table 1 shows that whenever the size of the largest altruists cluster in the initial state is at least $3r$, altruists can survive and spread under the dynamics. With this set of initial conditions, altruism is potentially viable but not ensured. For any given interaction radius, the larger the size of the largest altruist cluster in the initial state, the greater the likelihood that altruists will survive in the long run.

These results suggest that the condition $\beta > \beta_h(r)$ is necessary (and sufficient, subject to appropriate initial conditions) for the persistence of cooperative behavior under the specified imitation dynamics. Given that $\beta_h(r)$ is a strictly decreasing function of r (see Fig. 2), this in turn implies that the range of benefit–cost ratios for which cooperation is possible becomes larger with increasing interaction radius. It is in this sense that more *dense* networks can be said to be more conductive to the propagation of cooperation.

7 Conclusions

Cooperative behavior is widespread in many societies, and is commonly exhibited in experimental environments (Fehr and Fischbacher 2003). There is now a large literature on the evolution of cooperation, most of which is based on random matching (see, for instance, Axelrod 1984 and Bowles and Gintis 2004). Such models ignore the structure of the social network within which interactions occur, which is a serious limitation since "all economic action is embedded in networks" (Granovetter 1985) and network structure can have important effects on behavior (Coleman 1988). We have investigated the effects of changes in interaction radius on the survival and stability of cooperative behavior in a local interaction environment. The viability of cooperative behavior depends in a systematic way on interaction radius and the benefit–cost ratio. We identified for each interaction radius a critical value of the benefit–cost ratio which serves as the threshold below which cooperation cannot be sustained. When the benefit–cost ratio falls below this threshold, the incidence of cooperation can increase for some time, but eventually collapses as large altruist clusters are punctured and erode from within. Since the threshold itself declines as the interaction radius rises, there is a precise sense in which dense networks are more conducive to the evolution of cooperation.

An interesting direction for future research is the exploration of *irregular* networks. Regular networks studied in the literature to date satisfy two important properties. First, each individual has the same number of neighbors and, provided that this number is not too low, the proportion of one's neighbors who are also connected to each other is very high. In other words, such networks exhibit a high degree of clustering or cliquishness: two persons who are connected to each other have a large number of social neighbors in common. Second, the average distance or "degrees of separation" between two individuals (defined as the length of the shortest sequence of connected individuals which contains both of them) can be very large. Such networks are said to have high *characteristic path length*. However, there is considerable evidence that real world social networks satisfy the first property (high clustering) but have remarkably *low* characteristic path length. This combination of attributes is typical of “small world” networks (Watts and Strogatz 1998; Watts 1999). Whether or not cooperation is viable such small world networks is an important question to consider, but one beyond the scope of the present paper.

Appendix

Proof of Proposition 1 We shall prove that if $s(0) \in L(3r, 3r)$ and $\beta > \beta_h(r)$ then there exists τ such that for all $t \geq \tau$, $s(t) \in L(3r, 0) \cap U(n, 3r)$, from which the result follows. We begin with the following results (the first two of which follow from the discussion in the text): □

Lemma 1 *Suppose* $s(t) \in L(3r, 2r)$. *Then* $s_i(t) = 1 \Rightarrow s_i(t+1) = 1$ (*all altruists remain altruists*) *if and only if* $\beta > \beta_h(r)$.

Lemma 2 *Suppose* $s(t) \in L(2r, 3r)$. *Then* $s_i(t) = 0$ *and* $\sum_{j=i-r}^{i+r} s_j(t) \neq 0 \Rightarrow s_i(t+1) = 1$ *(all egoists with one or more altruist neighbors become altruists) if and only if* $\beta > \beta_h(r)$.

Lemma 3 *Suppose that* $\beta > \beta_h(r)$ *and* $s(t) \in L(3r, 2r+1)$. *Then all altruist clusters expand and all egoist clusters contract.*

Proof of Lemma 3 From Lemma 1, all altruist clusters expand or survive. From Lemma 2, all egoist clusters of length at least $3r$ contract. If we can show that all egoist clusters of length l with $2r + 1 \leq l \leq 3r - 1$ also contract then it follows from Lemma 1 that all altruist clusters expand. Accordingly, let the players $\{1, ..., l\}$ constitute such a cluster. By hypothesis, this cluster must be adjacent (on either side) to an altruist cluster of length at least $3r$. Exactly $l - 2r > 0$ players in the egoist cluster $\{1, ..., l\}$ are in homogeneous egoist neighborhoods and thus will not switch. To see which, if any, of the remaining egoists switch, consider the egoist players m, where $m \in \{1, ..., r\}$. From the perspective of player m, the average altruist payoff $\bar{\pi}_m^a$ is exactly as given in Eq. 10 in the text:

$$\bar{\pi}_m^a = -1 + \frac{1}{4}(3r - m)\frac{\beta}{r}$$

The average egoist payoff $\bar{\pi}_m^e$ observed by player m depends on whether or not $m > l - 2r$. For all players $m \in \{1, ..., l - 2r\}$, $\bar{\pi}_m^e$ is exactly as given in Eq. 9 in the text, and hence (using the same argument as in the text), since $\beta > \beta_h(r)$

all these egoists will switch to altruists. For $m \in \{l - 2r + 1, ..., r\}$, on the other hand, the average egoist payoff from the perspective of player m is

$$\bar{\pi}_m^e = \left(\frac{1}{r+m}\right)\frac{\beta}{2r}\left(\sum_{i=1}^{r}(r+1-i) + \sum_{i=1}^{l-2r} 0 + \sum_{i=1}^{m+2r-l} i\right)$$
$$= \frac{1}{4}\beta\frac{5r^2 + 3r + m^2 + 4mr - 2ml + m - 4rl + l^2 - l}{(r+m)\,r} \qquad (12)$$

The average payoff difference is

$$\bar{\pi}_m^a - \bar{\pi}_m^e$$
$$= -1 + \frac{1}{4}(3r - m)\frac{\beta}{r} - \frac{1}{4}\beta\frac{5r^2 + 3r + m^2 + 4mr - 2ml + m - 4rl + l^2 - l}{(r+m)\,r}$$
$$= -\frac{1}{4}\beta\frac{2r^2 + 2mr + 2m^2 + 3r - 2ml + m - 4rl + l^2 - l}{(r+m)\,r}$$

Solution: Differentiating with respect to m and simplifying yields

$$\frac{\partial\left(\bar{\pi}_m^a - \bar{\pi}_m^e\right)}{\partial m} = -\frac{1}{4}\beta\frac{4mr + 2m^2 + 2rl - 2r - l^2 + l}{(r+m)^2\,r}$$

Since we are considering the case $m \in \{l - 2r + 1, ..., r\}$, we have $l \le m+2r-1$. Hence, provided that $2r \le l$, the numerator of the above expression is

$$4mr + 2m^2 + 2rl - 2r - l^2 + l$$
$$\ge 4mr + 2m^2 + 2r\,(2r) - 2r - (m + 2r - 1)^2 + 2r$$
$$= m^2 + 2m + 4r - 1 > 0.$$

Therefore $\bar{\pi}_m^a - \bar{\pi}_m^e$ is strictly decreasing in m and if egoist m switches, then so does egoist $m - 1$ for all $m \in \{l - 2r + 2, ..., r\}$. Together with the fact that players $\{1, ..., l - 2r\}$ switch to altruists, this proves that the egoist cluster contracts. Hence from Lemma 1 both adjacent altruist clusters expand. Note that since $l > 2r$, the egoist cluster shrinks to a length strictly less than $2r$ in state $s(t + 1)$. □

Lemma 4 *Suppose that* $\beta > \beta_h(r)$ *and* $s^t \in L(3r, 2r)$ *. Then all altruist clusters survive or expand and all egoist clusters survive, contract, or vanish.*

Proof of Lemma 4 From Lemma 3, all egoist clusters of length greater than $2r$ contract so we need only consider clusters of length $2r$. Let the players $\{1, ..., 2r\}$ constitute such a cluster. By hypothesis, this cluster must be adjacent (on either side) to an altruist cluster of length at least $3r$. Consider the egoist player m, where $m \in \{1, ..., r\}$. From the perspective of player m, the average altruist payoff $\bar{\pi}_m^a$ is exactly as given in Eq. 10 above and the average egoist payoff is exactly as given in Eq. 12 above. As in the proof of Lemma 3 therefore, if any egoist switches then so do all egoists which are closer to an altruist cluster. Hence the egoist cluster cannot be punctured. Since all altruist clusters survive or expand from Lemma 1, the egoist cluster cannot expand. It must therefore survive, contract or vanish. □

Lemma 5 *Suppose that $s^t \in L(3r, r+1)$. Then no new cluster appears, and all egoist cluster of length less than 3r continue to remain of length less than 3r.*

Proof of Lemma 5 To show that no new cluster appears, we need to show that no cluster is punctured, and that two heterogeneous adjacent players cannot *both* switch. Given the above results, we need only consider egoist clusters of length l where $r+1 \le l \le 2r-1$, and their adjacent altruist clusters. Without loss of generality, let the players $\{1, ..., l\}$ constitute an egoist cluster with $r+1 \le l \le 2r-1$. By hypothesis, this cluster must be adjacent (on either side) to an altruist cluster of length at least $3r$. □

Claim 1 For any $m \in \{l-r, ..., r\}$, $s_{m+1}^{t+1} = 0$ if $s_m^{t+1} = 0$.

Proof of Claim 1 All (egoist)players m with $m \in \{l+1-r, ..., r\}$ have the same payoffs since all have exactly l egoist neighbors. They also observe the same mean egoist payoff since they observe all egoists in the cluster. The mean altruist payoff they observe is greater for values of m closer to the boundary of the cluster. Hence if player r does not switch, neither do any of the players $\{l+1-r, ..., r\}$. Player $l-r$ observes a higher mean altruist payoff than player $l+1-r$, and the same mean egoist payoff. Hence if $l-r$ switches, $l+1-r$ must switch. □

Claim 2 For any $m \in \{1, ..., l-r-1\}$, $s_{m+1}^{t+1} = 0$ if $s_m^{t+1} = 0$.

Proof of Claim 2 Consider player m with $m \in \{1, ..., l-r\}$.

$$\bar{\pi}_m^e(s^t) = \frac{1}{r+m}\left(\sum_{i=1}^{l-r} \frac{\beta\,(r+1-i)}{2r} + \frac{(2r-l)\,\beta\,(2r+1-l)}{2r} + \sum_{i=1}^{m} \frac{\beta\,(2r-l+i)}{2r}\right)$$

$$\bar{\pi}_{m+1}^e(s^t) = \frac{1}{r+m+1}\left(\sum_{i=1}^{l-r} \frac{\beta\,(r+1-i)}{2r} + \frac{(2r-l)\,\beta\,(2r+1-l)}{2r} + \sum_{i=1}^{m+1} \frac{\beta\,(2r-l+i)}{2r}\right)$$

and

$$\bar{\pi}_m^a(s^t) = \frac{1}{r+1-m}\sum_{i=1}^{r+1-m} \frac{\beta\,(r+i-1)}{2r} - 1$$

$$\bar{\pi}_{m+1}^a(s^t) = \frac{1}{r-m}\sum_{i=1}^{r-m} \frac{\beta\,(r+i-1)}{2r} - 1$$

Now suppose that $\bar{\pi}^e_m(s^t) - \bar{\pi}^a_m(s^t) \geq 0$ (Player m remains an egoist). Then

$$\begin{aligned}
&\bar{\pi}^e_{m+1}(s^t) - \bar{\pi}^a_{m+1}(s^t)\\
&\geq \left(\bar{\pi}^e_{m+1}(s^t) - \bar{\pi}^e_m(s^t)\right) - \left(\bar{\pi}^a_{m+1}(s^t) - \bar{\pi}^a_m(s^t)\right)\\
&= \left(-\frac{1}{4}\beta\frac{r-l-m+l^2-2rm-m^2-2rl+r^2}{(r+m+1)\,r\,(r+m)}\right) - \left(-\frac{1}{4}\frac{\beta}{r}\right)\\
&= \frac{1}{4}\beta\frac{l+2m-l^2+4rm+2m^2+2rl}{(r+m+1)\,r\,(r+m)} > 0.
\end{aligned}$$

Hence $s^{t+1}_m = 0$ implies that $s^{t+1}_{m+1} = 0$. □

Claim 3 An altruist cluster cannot be punctured.

Proof of Claim 3 Altruist clusters adjacent to egoist clusters of length at least $2r$ cannot be punctured from Lemma 1. Accordingly, let the players $\{1, ..., l\}$ constitute an egoist cluster with $r+1 \leq l \leq 2r-1$. Consider (altruist) players $l+m$ with $m \in \{1, ..., r\}$. For $2r-l < m \leq r$, The average payoff of altruists is given in Eq. 5. The average payoff of egoists is given by Eq. 6. By Lemma 1, that player $l+m$ remains an altruist. For $1 \leq m \leq 2r-l$, the average payoff of altruists is given in Eq. 5 and, for $1 \leq m \leq 2r-l$,

$$\bar{\pi}^e_m = \frac{1}{r-m+1}\left(\sum_{i=1}^{l-r}\frac{\beta\,(r+1-i)}{2r} + \frac{\beta\,(2r-l+1)\,(2r-l+1-m)}{2r}\right)$$

Differentiating $\bar{\pi}^a_m - \bar{\pi}^e_m$ with respect to m,

$$\frac{\partial\left(\bar{\pi}^a_m - \bar{\pi}^e_m\right)}{\partial m} = \frac{1}{4}\beta\frac{r+1}{(r+m)^2} - \frac{1}{4}\beta\frac{-2rl+r^2+r-l+l^2}{(r-m+1)^2\,r} \tag{13}$$

where both $\frac{\partial\bar{\pi}^a_m}{\partial m}$ and $\frac{\partial\bar{\pi}^e_m}{\partial m}$ are positive. Since Eq. 13 is decreasing in m,

$$\frac{\partial\left(\bar{\pi}^a_m - \bar{\pi}^e_m\right)}{\partial m} > \frac{1}{4}\beta\frac{r+1}{(r+(2r-l))^2} - \frac{1}{4}\beta\frac{-2rl+r^2+r-l+l^2}{(r-(2r-l)+1)^2\,r} \tag{14}$$

Note that the second term in Eq. 14 decreases in l. Substituting $r+1$ for l in the second term,

$$\begin{aligned}
\frac{\partial\left(\bar{\pi}^a_m - \bar{\pi}^e_m\right)}{\partial m} &> \frac{1}{4}\beta\frac{r+1}{(r+(2r-l))^2} - \frac{1}{4}\beta\frac{-2r\,(r+1)+r^2+r-(r+1)+(r+1)^2}{(r-(2r-(r+1))+1)^2\,r}\\
&= \frac{1}{4}\beta\frac{r+1}{(3r-l)^2} > 0
\end{aligned}$$

Noting this, suppose now that player $l+m$ switches. For player $l+m-1$, $\bar{\pi}^a$ decreases more than $\bar{\pi}^e$. Player $l+m-1$ also switches. Hence the altruist cluster cannot be punctured. □

Claim 4 Two heterogeneous adjacent players cannot *both* switch.

Proof of Claim 4 This follows from the above results when the egoist cluster is at least $2r$. Accordingly, let the players $\{1, ..., l\}$ constitute an egoist cluster with $r + 1 \leq l \leq 2r - 1$. We need to show that if player l switches then player $l + 1$ does not, and if player $l + 1$ switches then player l does not. Note that

$$\bar{\pi}_l^e(s^t) = \frac{1}{r+1}\left(\sum_{i=1}^{l-r}\frac{\beta\,(r+1-i)}{2r} + \frac{(2r-l)\,\beta\,(2r+1-l)}{2r} + \frac{\beta\,(2r-l+1)}{2r}\right),$$

$$\bar{\pi}_l^a(s^t) = \frac{1}{r}\sum_{i=1}^{r}\left(\frac{\beta\,(r+i-1)}{2r}\right) - 1.$$

Player l remains an egoist if and only if $\bar{\pi}_l^a(s^t) - \bar{\pi}_l^e(s^t) \leq 0$, or

$$\beta \leq \frac{4r\,(r+1)}{4rl - 2r^2 - 5r + 3l - 3 - l^2} = \beta''\,(r, l)\,.$$

Now consider (altruist) player $l + 1$.

$$\bar{\pi}_{l+1}^a(s^t) = \frac{1}{r+1}\sum_{i=1}^{r+1}\left(\frac{\beta\,(r+i-1)}{2r}\right) - 1$$

$$\bar{\pi}_{l+1}^e(s^t) = \frac{1}{r}\left(\sum_{i=1}^{l-r}\frac{\beta\,(r+1-i)}{2r} + \frac{(2r-l)\,\beta\,(2r+1-l)}{2r}\right)$$

So this player switches if and only if $\bar{\pi}_{l+1}^a(s^t) - \bar{\pi}_{l+1}^e(s^t) \leq 0$ or

$$\beta \leq \frac{4r\,(r+1)}{4rl - 2r^2 - 5r + 3l - 3 - l^2} = \beta''\,(r, l)\,.$$

It is easily verified that $\beta'\,(r, l) < \beta''\,(r, l)$ for all r and l in the admissible range. If any altruists switch ($\beta \leq \beta'$) then all egoists remain egoists ($\beta < \beta''$). If any egoists switch ($\beta > \beta''$) then all altruists remain altruists ($\beta > \beta'$).

Claims 1–4 establish that no new clusters appear. Claim 3 establishes that, for $m \geq 2r + 1 - l$, player $l + m$ remains an altruist. Hence the egoist cluster cannot expand by more than $2r - l$ on either side, or $4r - 2l$ in all. Hence it cannot expand to exceed $4r - l \leq 3r - 1$. ‖ □

Lemma 6 *Suppose that* $s^t \in L(3r, 1)$. *Then no new cluster appears.*

Proof of Lemma 6 To show that no new cluster appears, we need to show that no cluster is punctured, and that two heterogeneous adjacent players cannot *both* switch. Given the above results, we only need to consider egoist clusters of length l where $1 \leq l \leq r$, and their adjacent altruist clusters. Without loss of generality, let the players $\{1, ..., l\}$ constitute an egoist cluster $1 \leq l \leq r$. By hypothesis, this cluster must be adjacent (on either side) to an altruist cluster of length at least $3r$. Since $l \leq r$ all (egoist) players $m \in \{1, ..., l\}$ observe the same mean egoist payoff, given by

$$\bar{\pi}^e_m(s^t) = \frac{\beta\,(2r+1-l)}{2r}.$$

The mean altruist payoff observed by player m is

$$\bar{\pi}^a_m(s^t) = \frac{1}{2r+1-l}\left(\frac{2\,(r-l)\,(2r-l)\,\beta}{2r} + \sum_{i=1}^{m} \frac{\beta\,(2r-l+i-1)}{2r} + \sum_{i=1}^{l-m+1} \frac{\beta\,(2r-l+i-1)}{2r}\right) - 1.$$

This payoff $\bar{\pi}^a_m(s^t)$ is greater for values of m closer to the boundaries of the cluster $\{1, ..., l\}$. This is because players closer to the boundary are in contact with more altruists deeper within altruist clusters and hence with altruists who are earn higher payoffs. Hence player m switches to altruism only if all egoists closer to the boundary of the cluster $\{1, ..., l\}$ also switch. This implies that the egoist cluster cannot be punctured.

Consider whether player l remains an egoist.

$$\bar{\pi}^e_l(s^t) - \bar{\pi}^a_l(s^t) = 1 - \frac{1}{4}\beta\frac{-4r-2+l+l^2}{(2r+1-l)\,r}$$

Player l (and hence all egoists) remain egoists if and only if

$$\beta \leq 4\frac{(2r+1-l)\,r}{-4r-2+l+l^2} = \beta''(r, l)$$

We next show that if egoist l switches, then no altruists switch. As before, the altruist most likely to switch is player $l+1$. This follows from the facts that (a) all egoists have the same payoffs and hence the mean egoist payoffs are the same for all altruists who observe an egoist, (b) all altruists in N_{l+1} are also in N_{l+m} where $2 \leq m \leq r$, and (c) any altruists in N_{l+m} who is not in N_{l+1} is in a homogeneous neighborhood and hence obtains $\beta - 1$, the highest payoff possible for an altruist is

$$\bar{\pi}^a_{l+1}(s^t) = \frac{1}{2r+1-l}\left(\frac{2\,(r-l)\,\beta\,(2r-l)}{2r} + \sum_{i=1}^{l+1} \frac{\beta\,(2r-l+i-1)}{2r}\right) - 1.$$

Hence

$$\begin{aligned}
&\bar{\pi}^e_{l+1}(s^t) - \bar{\pi}^a_{l+1}(s^t) \\
&\quad= \frac{\beta\,(2r+1-l)}{2r} - \frac{\frac{2(r-l)\beta(2r-l)}{2r} + \sum_{i=1}^{l+1} \frac{\beta(2r-l+i-1)}{2r}}{2r+1-l} + 1 \\
&\quad= 1 - \frac{1}{4}\beta\frac{-4r-2+3l+l^2}{(2r+1-l)\,r}
\end{aligned}$$

Player $l+1$ switches if and only if

$$\beta \leq \frac{4(2r+1-l)r}{-4r-2+3l+l^2} = \beta'(r,l).$$

Note that $\beta'(r,l) < \beta''(r,l)$. Hence if $\beta > \beta''(r,l)$, which makes one or more egoists switch, then $\beta > \beta'(r,l)$, so all altruists remain altruists. In this case no new cluster appears. On the other hand if $\beta \leq \beta'(r,l)$, so that one or more altruists switch, then $\beta < \beta''(r,l)$, so all egoists remain egoists. Again no new cluster appears. Finally, if $\beta'(r,l) < \beta \leq \beta''(r,l)$, then no player in the egoist cluster and no adjacent altruist player switches. In this case too, no new cluster appears. ‖ □

The proof of Proposition 1 may now be completed as follows. If $s^0 \in L(3r, 3r)$ then from Lemmas 1 and 2, all altruist clusters expand by $2r$ and all egoist clusters contract by $2r$ until some period t_1 in which one or more egoist clusters are shorter than $3r$. In this period, all altruist clusters are of length at least $5r$, and egoist clusters fall into one of five categories; (I) length $l \geq 3r$, (II) $2r+1 \leq l \leq 3r-1$, (III) $2r = l$, (IV) $r+1 \leq l \leq 2r-1$, or (V) $1 \leq l \leq r$. From Lemmas 2–6 Clusters in categories (I) and (II) contract, those in (III) survive, contract or vanish, and those in (IV) and (V) expand, contract, survive or vanish but cannot expand to exceed length $3r$. Hence no egoist cluster can return to category (I) after leaving it, and no altruist cluster can contract to a length less than $3r$. Since all egoist clusters in category (I) contract in each period from Lemma 2, they must eventually fall into another category. Hence there exists τ such that for all $t \geq \tau$, $s^t \in L(3r, 0) \cap U(n, 3r)$. Since egoist and altruist clusters alternate by definition, ρ^t must be at least 0.5 at all states s^t such that $t \geq \tau$.

Proof of Proposition 2 We start with two lemmata for any r. □

Lemma 7 *Suppose that $\beta \leq \beta_h(r)$. Then all altruist clusters of length at least $2r$ contract or are punctured.*

Proof of Lemma 7 Suppose $\beta \leq \beta_h(r)$ and let the players $\{1, ..., l\}$ constitute an altruist cluster, where $l \geq 2r$. Consider player $l+1-r$. Note that this player has only one egoist neighbor, player $l+1$. Hence $\bar{\pi}^a_{l+1-r} - \bar{\pi}^e_{l+1-r} = \bar{\pi}^a_{l+1-r} - \pi_{l+1}$. We claim that this difference is greatest when players $\{l+2, ..., l+r+1\}$ are all egoists. To see this, observe that if any number m of these players switches from E to A then π_{l+1} increases by $\beta m/2r$ while $\bar{\pi}^a_{l+1-r}$ increases by strictly less than $\beta m/2r$. This is because at most $r-1$ of the altruists in the set $\{l+1-2r, ..., l\}$ have their payoffs raised, and these payoffs are raised by at most $\beta m/2r$.The remaining altruists experience no change in payoff. This proves that $\bar{\pi}^a_{l+1-r} - \bar{\pi}^e_{l+1-r}$ is maximized when players $\{l+2, ..., l+r+1\}$ are all egoists. Hence if altruist $l+1-r$ switches when players $\{l+2, ..., l+r+1\}$ are all egoists, then altruist $l+1-r$ will also switch regardless of the composition of $\{l+2, ..., l+r+1\}$. Accordingly, suppose that players $\{l+2, ..., l+r+1\}$ are all egoists. First consider the case $l \geq 3r$. Then

$$\bar{\pi}^{a}_{l+1-r} - \bar{\pi}^{e}_{l+1-r} = \frac{1}{2r}\left(\sum_{i=1}^{r} \frac{\beta\,(r+i-1)}{2r} + r\beta\right) - 1 - \frac{\beta r}{2r} = \frac{1}{8r}\,(3r\beta - \beta - 8r)$$

Hence if $\beta \leq \beta_h(r)$, altruist $l+1-r$ switches, so the cluster $\{1, ..., l\}$ either contracts or is punctured. Next consider the case $2r \leq l < 3r$. In this case $\bar{\pi}^{e}_{l+1-r}$ is the same as it would be when $l \geq 3r$, while $\bar{\pi}^{a}_{l+1-r}$ is strictly less than it would be when $l \geq 3r$. Since altruist $l+1-r$ switches when $l \geq 3r$, this player must also switch when $2r \leq l < 3r$. ‖ □

Lemma 8 *Suppose that $\beta \leq \beta_h(r)$. Then any egoist who have only one altruist in his neighborhood remain as egoist.*

Proof of Lemma 8 Suppose $\beta \leq \beta_h(r)$. The egoist we consider must belong to the egoist cluster of length at least $2r$. Let the players $\{1, ..., l\}$ constitute an egoist cluster, where $l \geq 2r$. Consider player $l+1-r$. Note that this player has only one altruist neighbor, player $l+1$. Hence $\bar{\pi}^{a}_{l+1-r} - \bar{\pi}^{e}_{l+1-r} = \pi_{l+1} - \bar{\pi}^{e}_{l+1-r}$. We claim that this difference is greatest when players $\{l+2, ..., l+r+1\}$ are all altruists. To see this, observe that if any number m of these players switches from A to E then π_{l+1} decreases by $\beta m/2r$ while $\bar{\pi}^{a}_{l+1-r}$ decreases by strictly less than $\beta m/2r$. This is because at most $r-1$ of the egoists in the set $\{l+1-2r, ..., l\}$ have their payoffs decreased, and these payoffs are decreased by at most $\beta m/2r$. The remaining egoists experience no change in payoff. This proves that $\pi_{l+1} - \bar{\pi}^{e}_{l+1-r}$ is maximized when players $\{l+2, ..., l+r+1\}$ are all altruists. Hence if egoist $l+1-r$ remain an egoist when players $\{l+2, ..., l+r+1\}$ are all altruists, then egoist $l+1-r$ will also remain an egoist regardless of the composition of $\{l+2, ..., l+r+1\}$. Accordingly, suppose that players $\{l+2, ..., l+r+1\}$ are all altruists. First consider the case $l \geq 3r$. Then

$$\bar{\pi}^{a}_{l+1-r} - \bar{\pi}^{e}_{l+1-r} = \frac{\beta r}{2r} - 1 - \frac{1}{2r}\left(\sum_{i=1}^{r} \frac{\beta i}{2r}\right) = \frac{1}{8r}\,(3r\beta - \beta - 8r)$$

Hence if $\beta \leq \beta_h(r)$, egoist $l+1-r$ remains an egoist. Next consider the case $2r \leq l < 3r$. In this case $\bar{\pi}^{e}_{l+1-r}$ is strictly greater than it would be when $l \geq 3r$, while $\bar{\pi}^{a}_{l+1-r}$ is the same as it would be when $l \geq 3r$. Since egoist $l+1-r$ remains when $l \geq 3r$, this player must also remain an egoist when $2r \leq l < 3r$. ‖

Now assume that $r = 2$ and thus $\beta_h\,(2) = \frac{16}{5}$ Let $A\,(m)$ denote the altruist cluster of length m and $E\,(m)$ for the egoist cluster of length m. Let a and e with superscript $*$ is the player in $A\,(m)$ and $E\,(m)$ respectively, while $a\,(e)$ be any other altruist(egoist). Let x denote the player with the unspecified type. Lemma 8 implies that with $r = 2$, the egoist cluster is never punctured. This and Lemma 7 implies that any newly formed altruist cluster will be of length at most 2. □

Claim 1 All the altruist in $A(m)$ for $m \leq 4$ switch to egoist.

Proof of Claim 1 Note that we can possibly consider two type of altruists in $A(m)$: either (*Case I*) all the egoist in his neighborhood has only one altruist neighbor, namely the altruist in $A(m)$, or (*Case II*) at least one egoist in the neighborhood whose payoff is at least $\frac{2}{4}\beta$. *Case I* is feasible iff $e, e, e, e, a^*, e, e, e, e$. where it is easy to see that this altruist will switch to egoist next period(this is true for any β). *Case I* implies that if not *Case I* , *all* the altruists in $A(m)$ have at least one egoist neighbor whose payoff is at least $\frac{2}{4}\beta$.

Now consider *Case II*. There are two possible cases: either (*Case II-1*) at least one altruist in his neighborhood has the payoff of less than $-1+\frac{3}{4}\beta$ or (*Case II-2)* all altruist neighbors including himself have the payoff $-1+\frac{3}{4}\beta$. In *Case II-2*, all the egoist the altruists face have the payoff $\frac{3}{4}\beta$. Thus these altruists will convert to egoist. Consider *Case II-1*. There are two possible cases for this: either there is only one egoist neighbor for $A(m)$ who has at least two altruist neighbors or there are more than one such egoist neighbor. The first is only feasible with $x_1, x_2, a, e, a^*, e, e, e, e$ (and its mirror $e, e, e, e, a^*, e, a, x_2, x_1$). The best case for the altruist a^* to remain an altruist is when $x_1 = x_2 = a$. In this case

$$\bar{\pi}^a - \bar{\pi}^e = \frac{1}{12}\beta - 1 \leq -\frac{11}{15} < 0$$

Hence this altruist player will switch to egoist. For the second case, the maximum of $\bar{\pi}^a$ among $A(m)$ is

$$\bar{\pi}^a = \frac{\frac{2}{4}\beta + (n-1)\frac{3}{4}\beta}{n} - 1 \quad \text{for } n = 1, 2, 3$$

while the minimum of the average payoff of egoist around the altruist in $A(m)$ is

$$\bar{\pi}^e = \frac{2\frac{2}{4}\beta + (5-n-2)\frac{1}{4}\beta}{5-n} \quad \text{for } n = 1, 2, 3$$

The difference is

$$\bar{\pi}^a - \bar{\pi}^e = \frac{1}{4}\frac{5\beta - 9\beta n + 2\beta n^2 + 20n - 4n^2}{n(-5+n)}$$

which is increasing in β for all n. Substituting $\frac{16}{5}$ for β and then we have

$$\bar{\pi}^a - \bar{\pi}^e = -\frac{1}{5}\frac{20 - 11n + 3n^2}{n(5-n)} < 0$$ □

Claim 2 All the egoist in $E(m)$ for $m \leq 4$ will remain an egoist.

Proof of Claim 2 All the egoist neighbors of the egoist in $E(m)$ including himself is of payoff at least $\frac{1}{4}\beta$. There are possibly two types of egoists: either (*Case I*) all the egoist in the neighborhood including himself face only one altruist. or (*Case II*) at least one egoist in the neighborhood including himself has the payoff of at least $\frac{2}{4}\beta$. *Case I* is feasible iff the altruist neighbor is surrounded by egoists and thus the average payoff of this altruist is $-1 + \frac{1}{4}\beta$. Thus $\bar{\pi}^a < \bar{\pi}^e$ for the egoists in $E(m)$. Now consider *Case II*. Notice that amongst the altruist neighbors that the egoist in $E(m)$ face, there exist at least one altruist neighbor whose payoff is less than $-1 + \frac{3}{4}\beta$, except the case of

$$a, a, a, a, e^*, a, a, a, a$$

in which case the egoist will survive as egoist. Other than this case, we can possibly consider two separate cases: either (*Case II-1)* there is only one altruist whose payoff is at most $-1 + \frac{2}{4}\beta$ in the neighborhood of $E(m)$ or (*Case II-2)* there are more than one such altruist. *Case II-1* is feasible only with $x_1, x_2, e, a, e^*, a, a, a, a$ (and its mirror image) and $e, a, a, a, e^*, a, a, a, a$ (and its mirror image). It is easy to see that e^* will remain in the latter case. For the first case, the best situation for the egoist e^* to switch to altruist is $x_1 = x_2 = e$. In this case, $\bar{\pi}^e = \frac{1}{2}\left(\frac{3}{4}\beta + \frac{1}{4}\beta\right) = \frac{1}{2}\beta$ and $\bar{\pi}^a = \frac{1}{3}\left(\frac{6}{4}\beta + \frac{1}{4}\beta\right) - 1 = \frac{7}{12}\beta - 1$. Thus $\bar{\pi}^a - \bar{\pi}^e = \frac{1}{12}\beta - 1 < -\frac{11}{15} < 0$. Hence the egoist will remain as egoist. Now consider *Case II-2*. The minimum average payoff of egoists in $E(m)$ is

$$\bar{\pi}^e = \frac{\frac{2}{4}\beta + (5 - n - 1)\frac{1}{4}\beta}{5 - n} \quad \text{for } n = 2, 3, 4$$

while the maximum average payoff of altruists in $E(m)$ is

$$\bar{\pi}^a = \frac{2\frac{2}{4}\beta + (n - 2)\frac{3}{4}\beta}{n} - 1 \quad \text{for } n = 2, 3, 4$$

Hence

$$\bar{\pi}^a - \bar{\pi}^e = -\frac{1}{4}\frac{10\beta - 11\beta n + 2\beta n^2 + 20n - 4n^2}{n(5 - n)}$$

which is increasing in β. Substituting $\frac{16}{5}$ for β, $\bar{\pi}^a - \bar{\pi}^e = -\frac{1}{5}\frac{40-19n+3n^2}{n(5-n)} < 0$.

What remains to be shown is that the newly created altruist cluster after punctured by egoists will eventually disappear without expanding indefinitely into the egoist cluster. We claim that this newly created altruist cluster will disappear in the very next period without seeding altruist in the egoist cluster. □

Claim 3 The altruist in $A(m)$ for $m = 1, 2$ disappear and do not expand.

Proof of Claim 3 For $m = 1$,we have the following situation; ...e_2, a^*, e_1, ...By Claim 1, a^* will switch to egoist. We need to show that the surrounding egoists e_1 (e_2) will remain as egoist in the next period. This is sufficient since the egoist cluster can not be punctured by Lemma 8 for $r = 2$. We will consider e_1 only since the result also applies to e_2 by symmetry. By Claim 2, it is sufficient to consider the case of $x_1, x_2, e_2, a^*, e_1, E(4)$. It is easy to see that the worst situation for e_1 to remain an egoist is $x_1 = e$ and $x_2 = a$. In this case, $\bar{\pi}^a_{e_1} - \bar{\pi}^e_{e_1} = -1 < 0$ Thus $e_1$will remain an egoist.

Suppose now $m = 2$. Following argument for $m = 1$, it is sufficient to consider the case of x, e_2, a^*, a^*, e_1, E (4). The worst situation for e_1 to remain an egoist is $x = a$. Then

$$\bar{\pi}^e_{e_1} = \frac{\frac{2}{4}\beta + \frac{1}{4}\beta}{3} = \frac{1}{4}\beta$$

$$\bar{\pi}^a_{e_1} = \frac{\frac{2}{4}\beta + \frac{1}{4}\beta}{2} - 1 = \frac{3}{8}\beta - 1$$

Hence $\bar{\pi}^a_{e_1} - \bar{\pi}^e_{e_1} = \frac{1}{8}\beta - 1$. Substituting $\frac{16}{5}$ for β, $\bar{\pi}^a_{e_1} - \bar{\pi}^e_{e_1} < -\frac{3}{5} < 0$. Thus e_1 will remain as egoist.

The proof of Proposition 2 can be completed as follows. By Lemma 7, the altruist cluster of length at least $2r$ either contracts until its length is less than or equal to $2r$ and then it disappears by Claim 1 and 3 or it is punctured and the newly created altruist cluster, whose length is at most $2r$ by Lemma 8, disappears without expansion by Claim 1 and 3. Eventually the length of the cluster reaches at most $2r$, at the point in which it disappears. □

References

Albin PS, Foley DK (2001) The co-evolution of cooperation and complexity in a multi-player, local-interaction prisoners' dilemma. Complexity 6:54–65

Axelrod RM (1984) The evolution of cooperation. New York, Basic Books

Bergstrom T (2002) Evolution of social behavior: individual and group selection. J Econ Perspect 16:67–88

Bergstrom T, Stark O (1993) How altruism can prevail in an evolutionary environment. Am Econ Rev 83:149–155

Bowles S, Gintis H (2004) The evolution of strong reciprocity: cooperation in heterogeneous populations. Theor Popul Biol 65:17–28

Coleman JS (1988) Free riders and zealots: the role of social networks. Sociol Theory 6:52–57

Eshel I (1972) On the neighborhood effect and the evolution of altruistic traits. Theor Popul Biol 3:258–277

Eshel I, Herreiner D, Samuelson L, Sansone E, Shaked A (2000) Cooperation, mimesis, and local interaction. Sociol Methods Res 28:341–364

Eshel I, Samuelson L, Shaked A (1998) Altruists, egoists, and hooligans in a local interaction model. Am Econ Rev 88:157–179

Eshel I, Sansone E, Shaked A (1999) The emergence of kinship behavior in structured populations of unrelated individuals. Int J Game Theory 28:447–463

Fehr E, Fischbacher U (2003) The nature of human altruism. Nature 425:785–791

Granovetter M (1985) Economic action and social structure: the problem of embeddedness. Am J Sociol 91:481–510

Nowak MA, May RM (1992) Evolutionary games and spacial chaos. Nature 359:826–829

Nowak MA, May RM (1993) The spatial dilemmas of evolution. Int J Bifurc Chaos Appl Sci Eng 3:35–78

Watts D (1999) Small worlds. Princeton University Press, Princeton, New Jersey

Watts D, Strogatz SH (1998) Collective dynamics of 'small-world' networks. Nature 393:440–442