Applications of the SVD

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Detail from Durer's Melancolia, dated 1514., 359x371 image

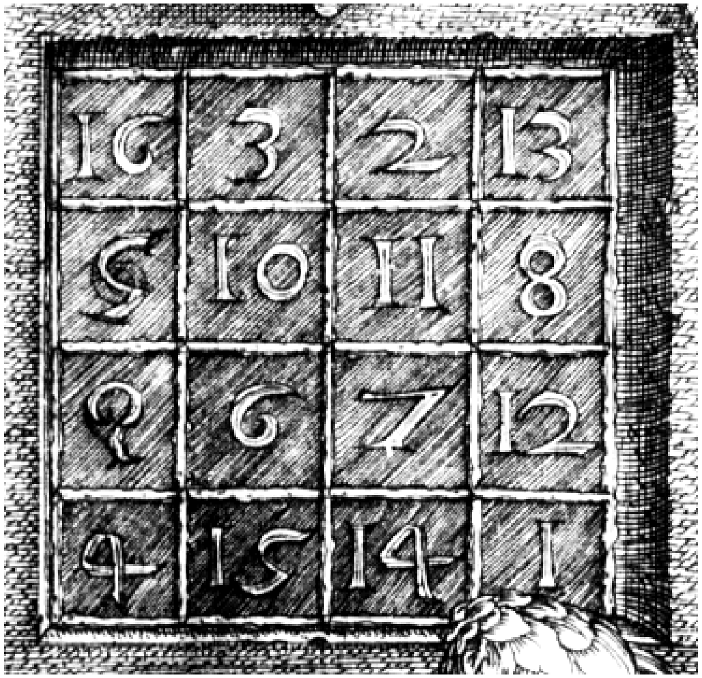
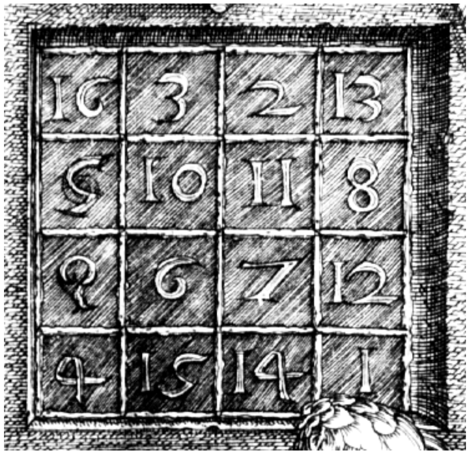


Image Compression

Given an original image (here 359×371 pixels)

Detail from Durer's Melancolia, dated 1514., 359x371 image



We can write it as a 359×371 matrix A which can then be decomposed via the singular value decomposition as

$$A = U \Sigma V^T$$

where U is 359×359, Σ is 359×371 and V is 371×371.

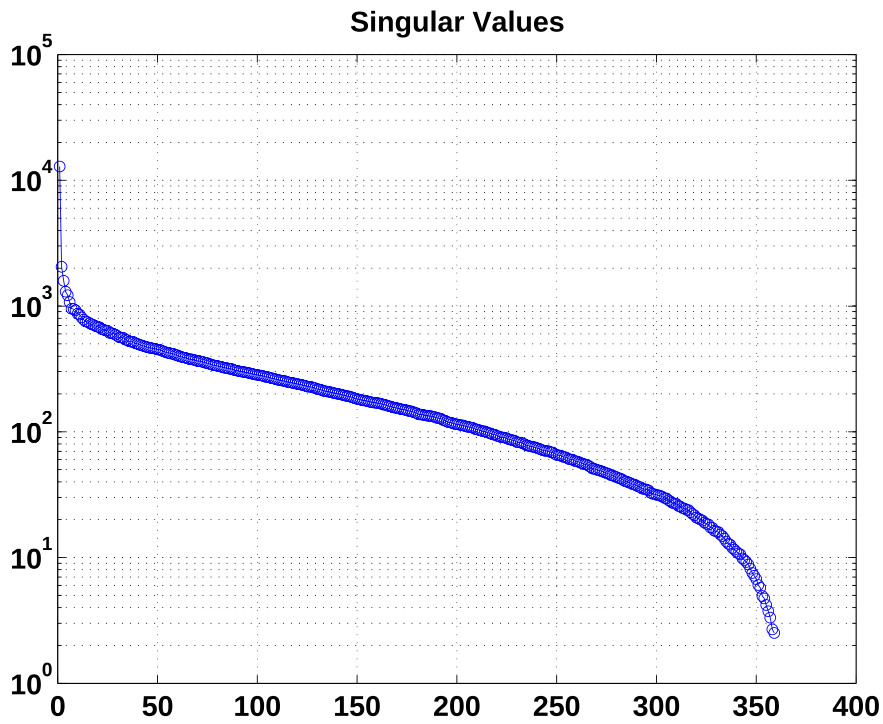
The matrix A however can also be written as a sum of rank 1 matrices

$$A = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T + \ldots + \sigma_n \mathbf{u}_n \mathbf{v}_n^T$$

where each rank 1 matrix $\mathbf{u}_i \mathbf{v}_i^T$ is the size of the original matrix. Each one of these matrices is a *mode*.

Because the singular values σ_i are ordered $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0$, however, significant compression of the image is possible if the *spectrum* of singular values has only a few very strong entries.

Spectrum of Singular values for A



Here the spectrum is contained principally in the first 100–200 modes (max).

We can therefore reconstruct the image from just a subset of modes. For example in matlabese we can write just the first mode as

```
[U,S,V]=svd(A)
B=U(:,1)*S(1,1)*V(:,1)'
```

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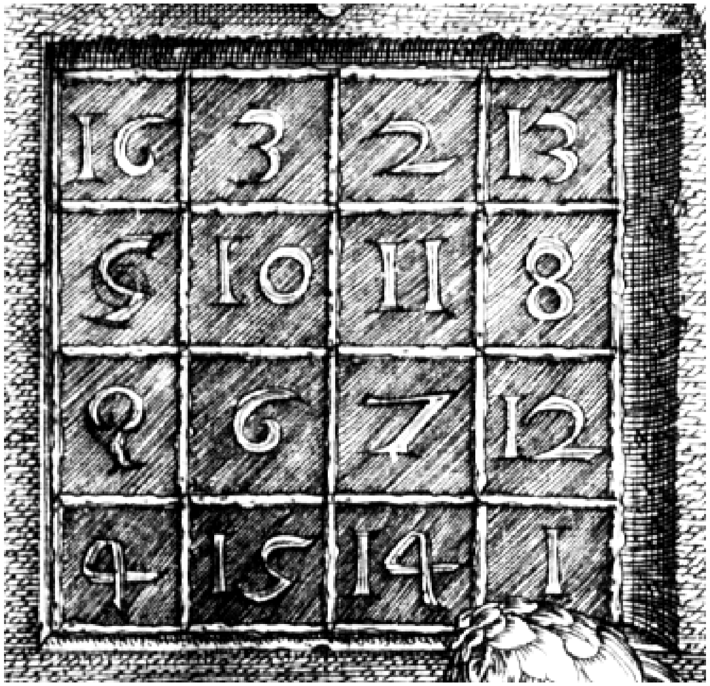
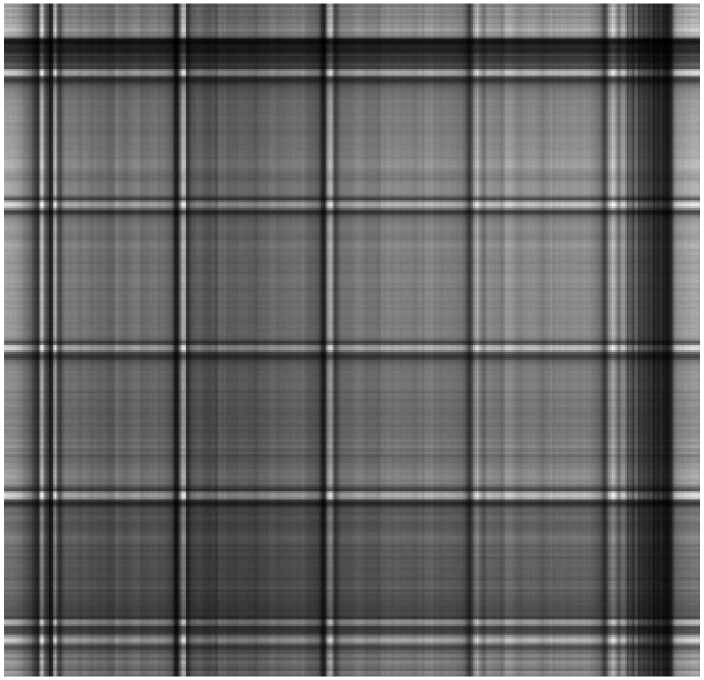


EOF reconstruction with 1 modes



Or as a sum of the first 10 modes as

```
B=U(:,1:10)*S(1:10,1:10)*V(:,1:10)'
```

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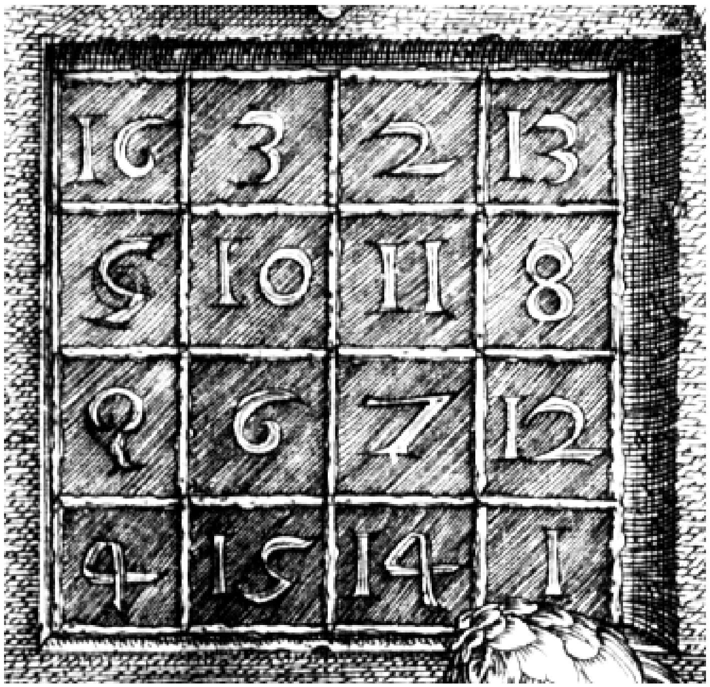
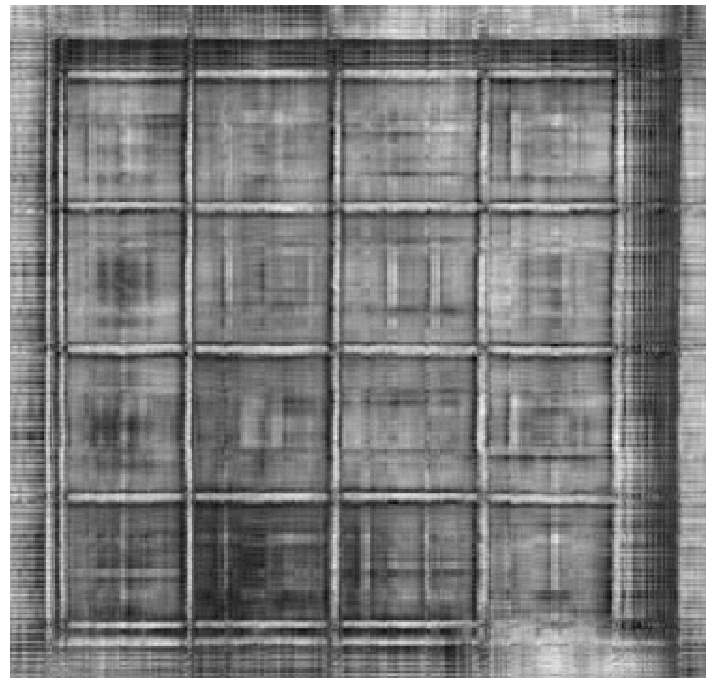


EOF reconstruction with 10 modes

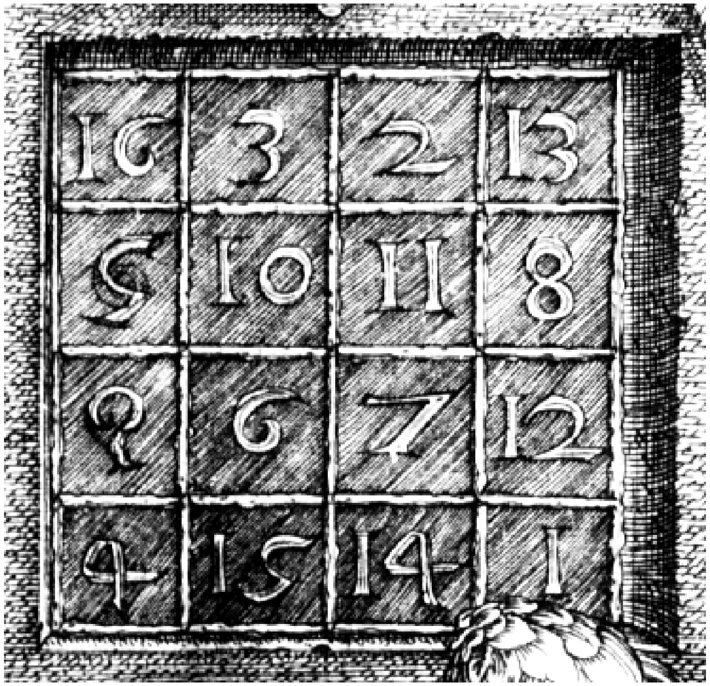


which only uses 5% of the storage ($10 \times 359 + 10 \times 371 + 10 = 7310$ pixels vs $359 \times 371 = 133189$ pixels.

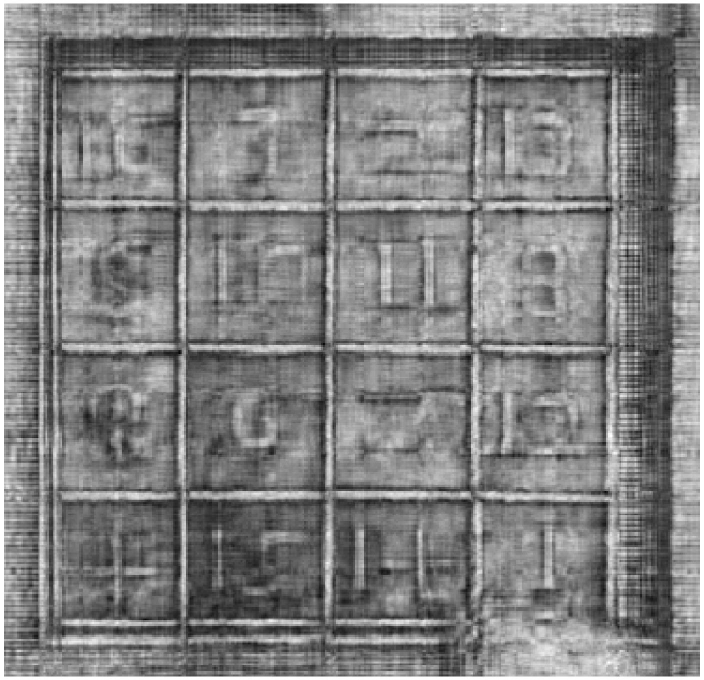
Adding modes, just adds resolution

```
B=U(:,1:20)*S(1:20,1:20)*V(:,1:20)'
```

Detail from Durer's Melancolia, dated 1514., 359x371 image



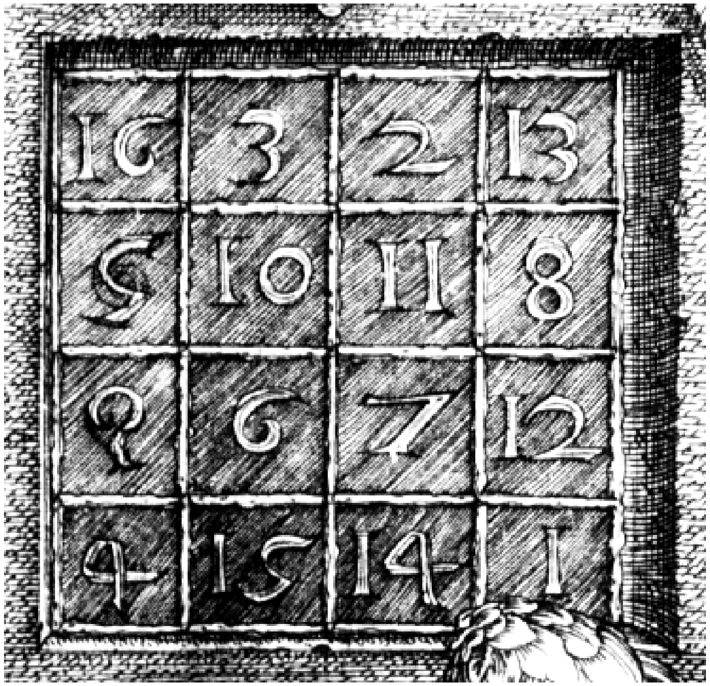
EOF reconstruction with 20 modes



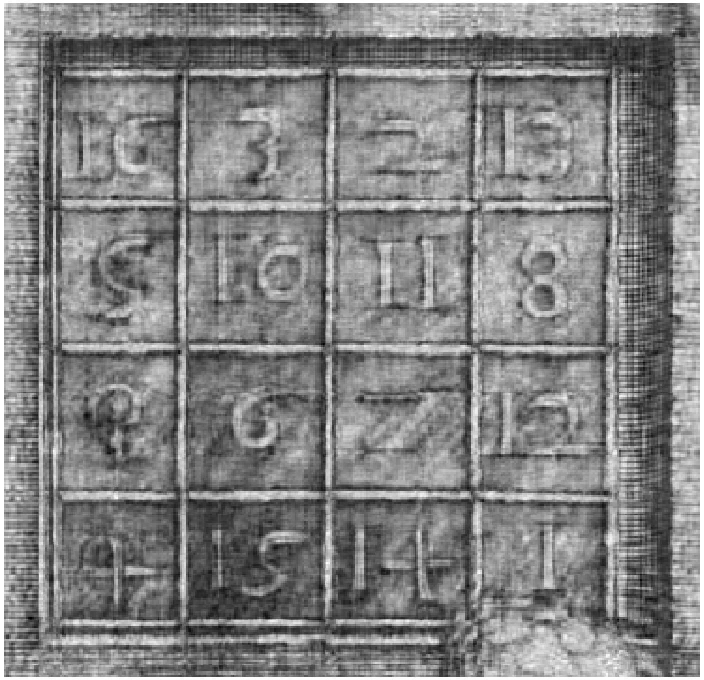
Adding modes, just adds resolution

```
B=U(:,1:30)*S(1:30,1:30)*V(:,1:30)'
```

Detail from Durer's Melancolia, dated 1514., 359x371 image



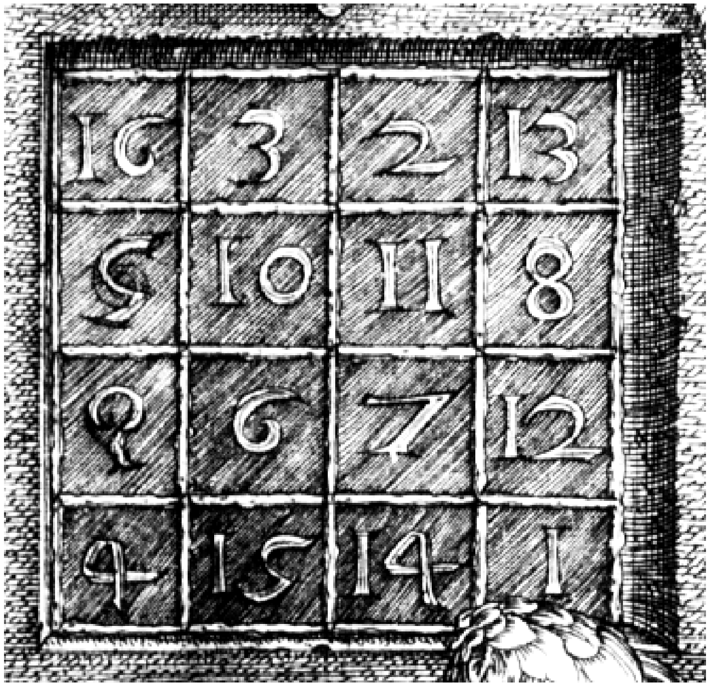
EOF reconstruction with 30 modes



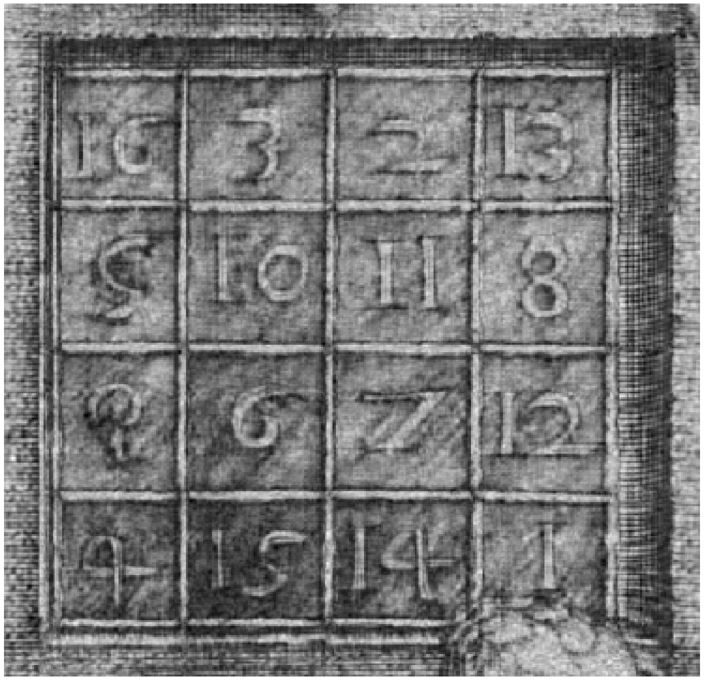
Adding modes, just adds resolution

```
B=U(:,1:40)*S(1:40,1:40)*V(:,1:40)'
```

Detail from Durer's Melancolia, dated 1514., 359x371 image



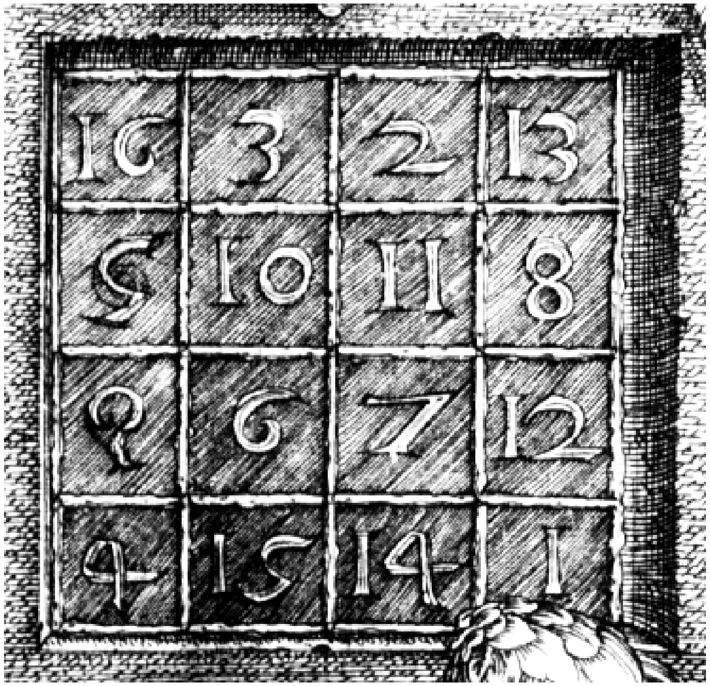
EOF reconstruction with 40 modes



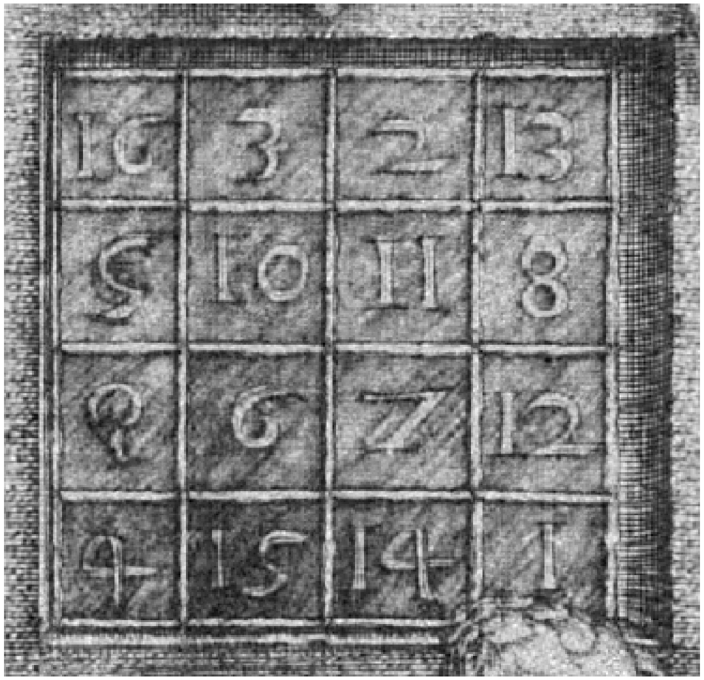
Adding modes, just adds resolution

```
B=U(:,1:50)*S(1:50,1:50)*V(:,1:50)'
```

Detail from Durer's Melancolia, dated 1514., 359x371 image



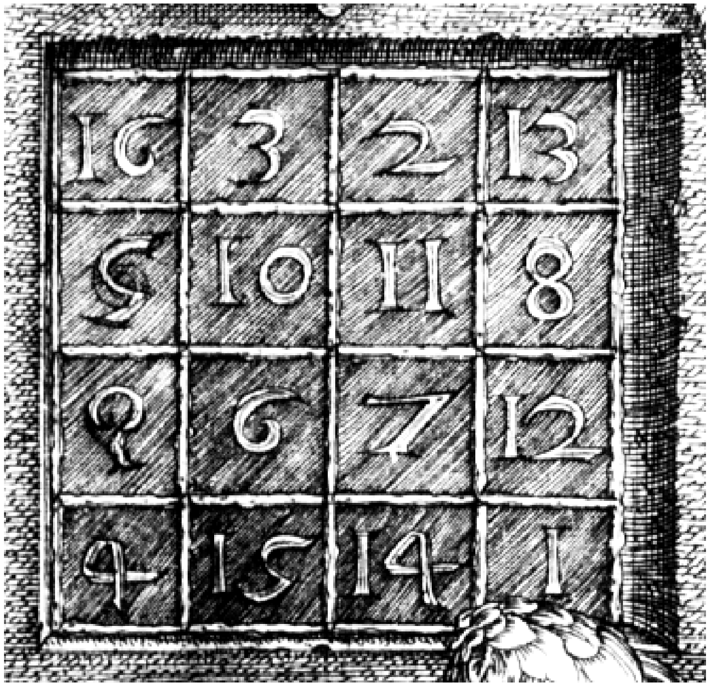
EOF reconstruction with 50 modes



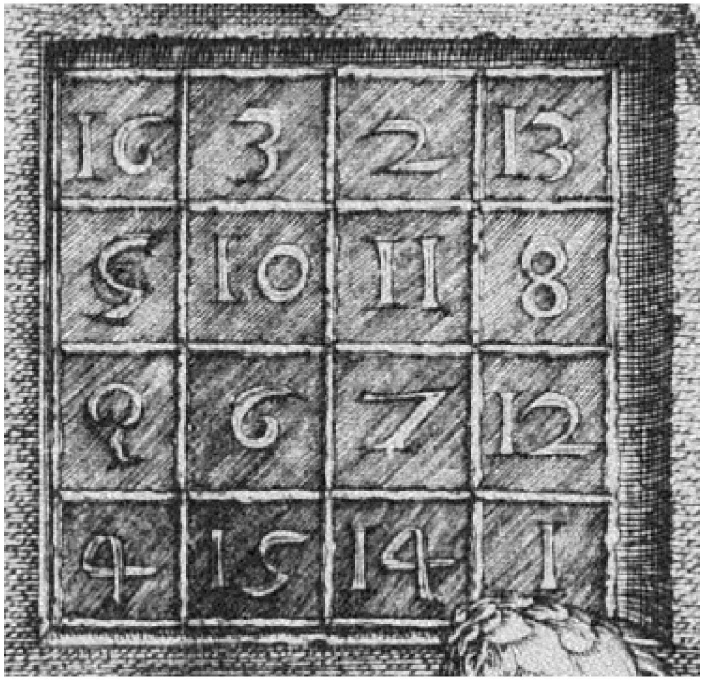
Adding modes, just adds resolution

```
B=U(:,1:100)*S(1:100,1:100)*V(:,1:100)'
```

Detail from Durer's Melancolia, dated 1514., 359x371 image



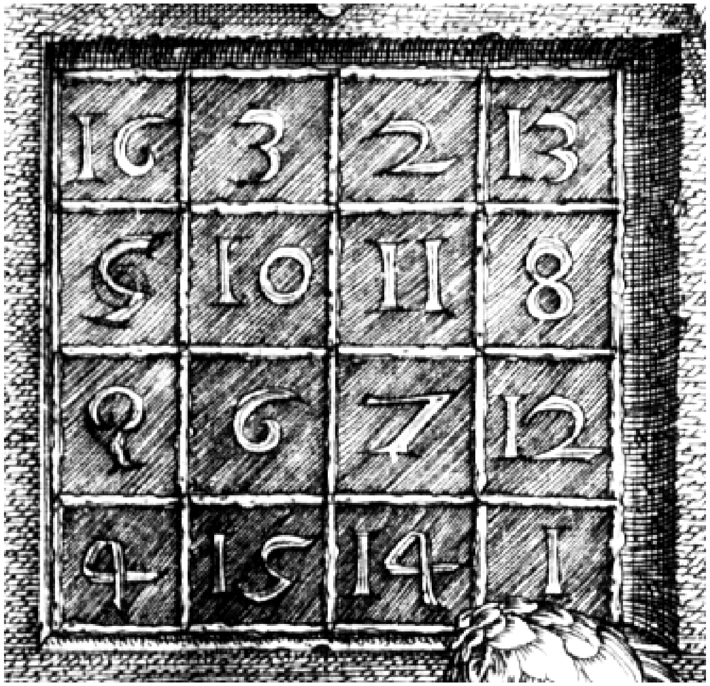
EOF reconstruction with 100 modes



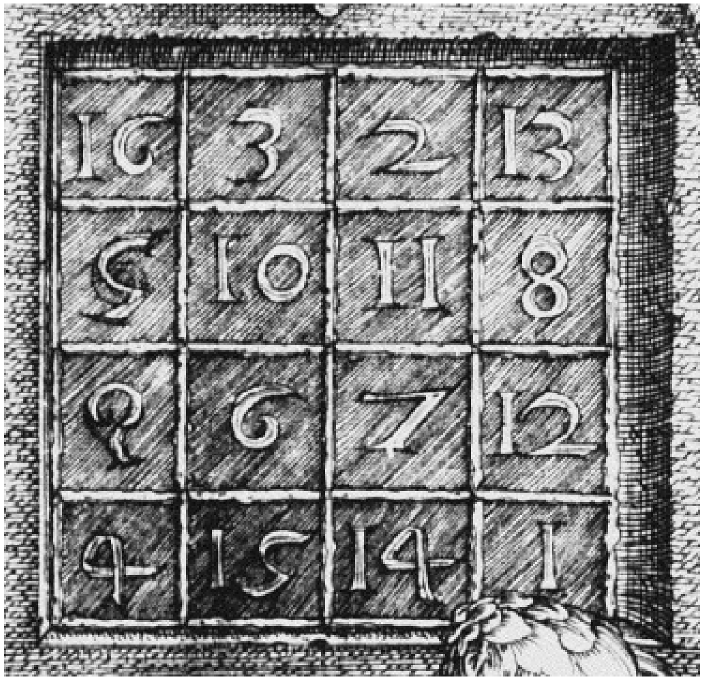
Adding modes, just adds resolution

```
B=U(:,1:200)*S(1:200,1:200)*V(:,1:200)'
```

Detail from Durer's Melancolia, dated 1514., 359x371 image



EOF reconstruction with 200 modes



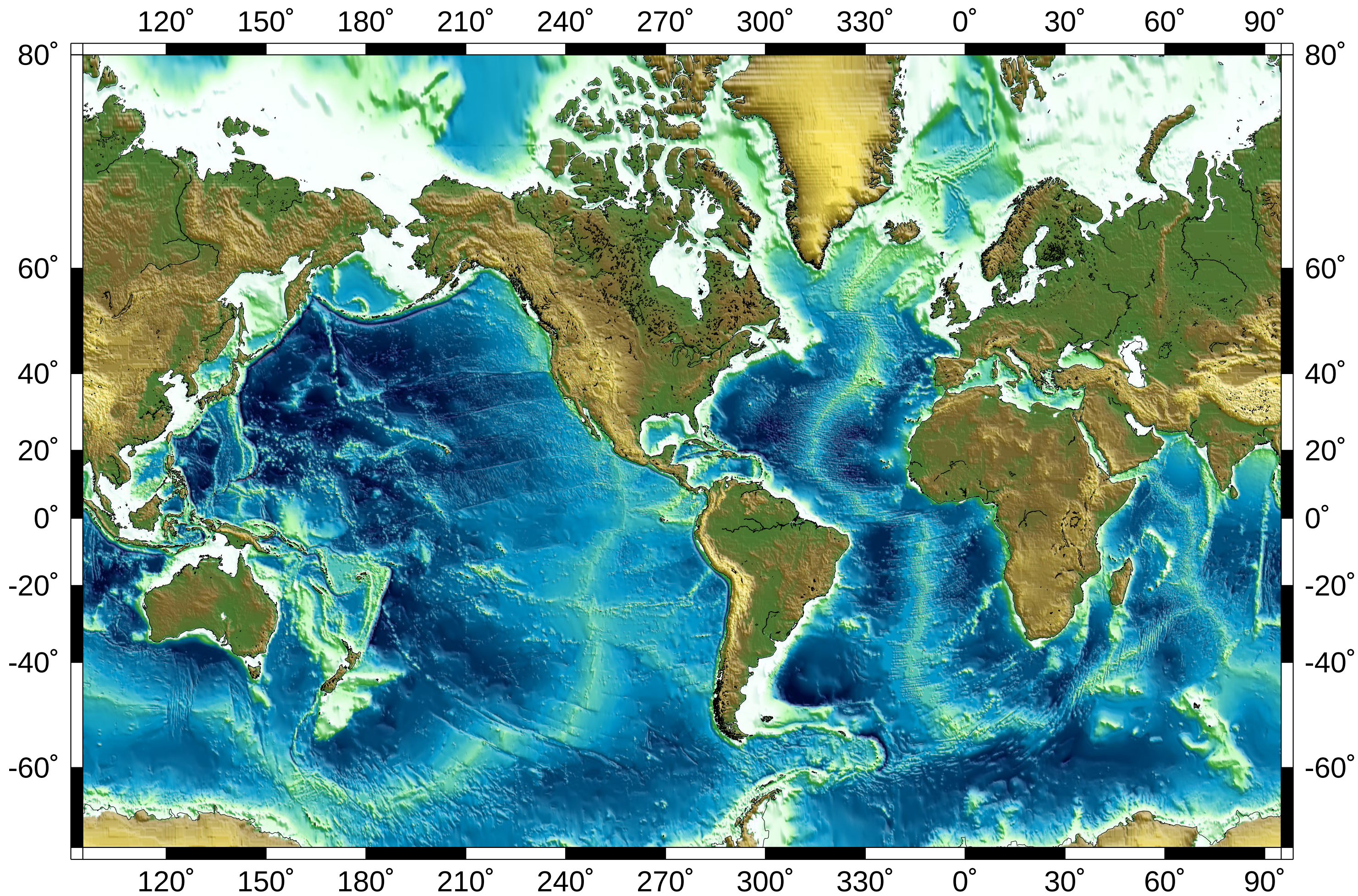
Application 2: EOF analysis

Pattern extraction—Mid-ocean ridge topography

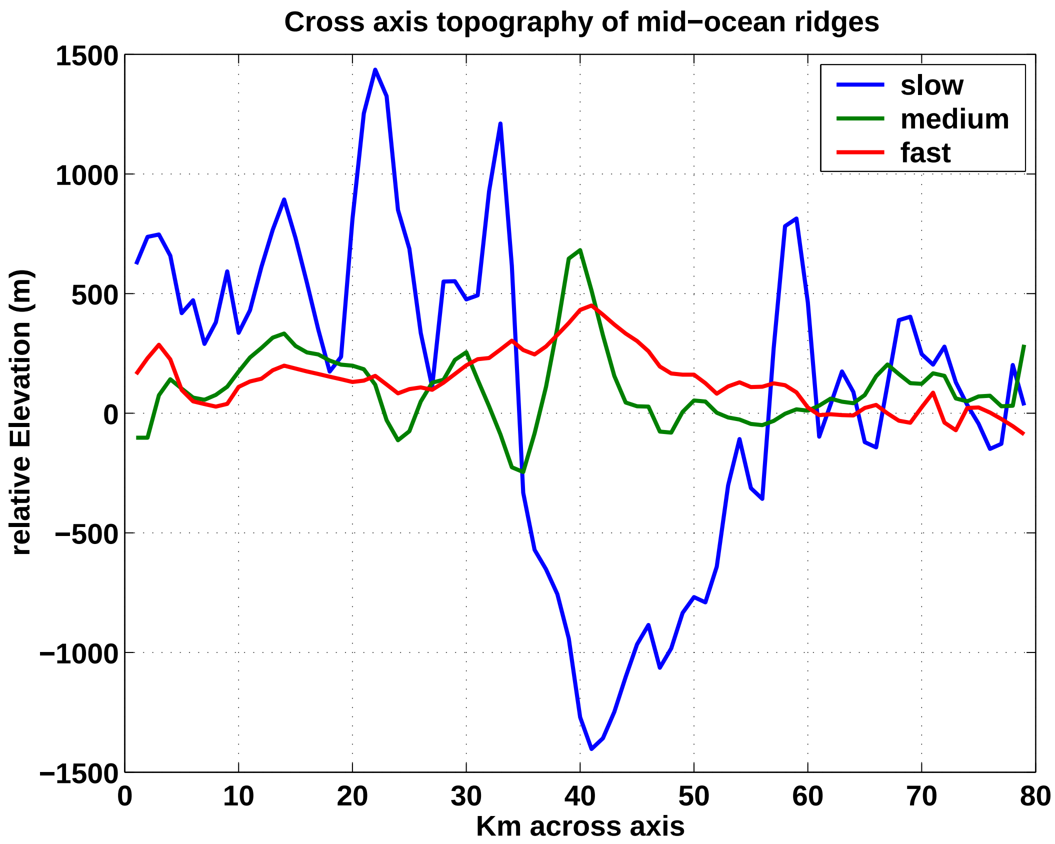
Here we consider a real research use of the SVD by Chris Small (LDEO)

A Global Analysis Of Midocean Ridge Axial Topography

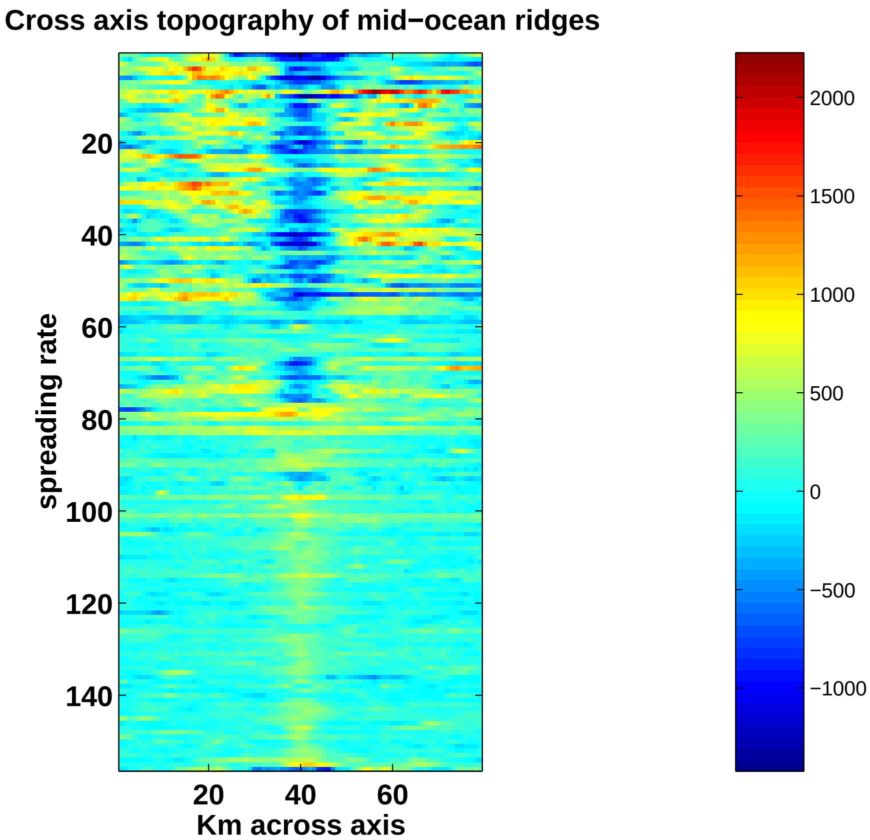
GEOPHYSICAL JOURNAL INTERNATIONAL 116 (1): 64-84 JAN 1994


120° 150° 180° 210° 240° 270° 300° 330° 0° 30° 60° 90°
80° 60° 40° 20° 0° -20° -40° -60°

The data: cross axis topography profiles from different spreading rates



Form a matrix A (179×80) of elevation vs. distance across the ridge



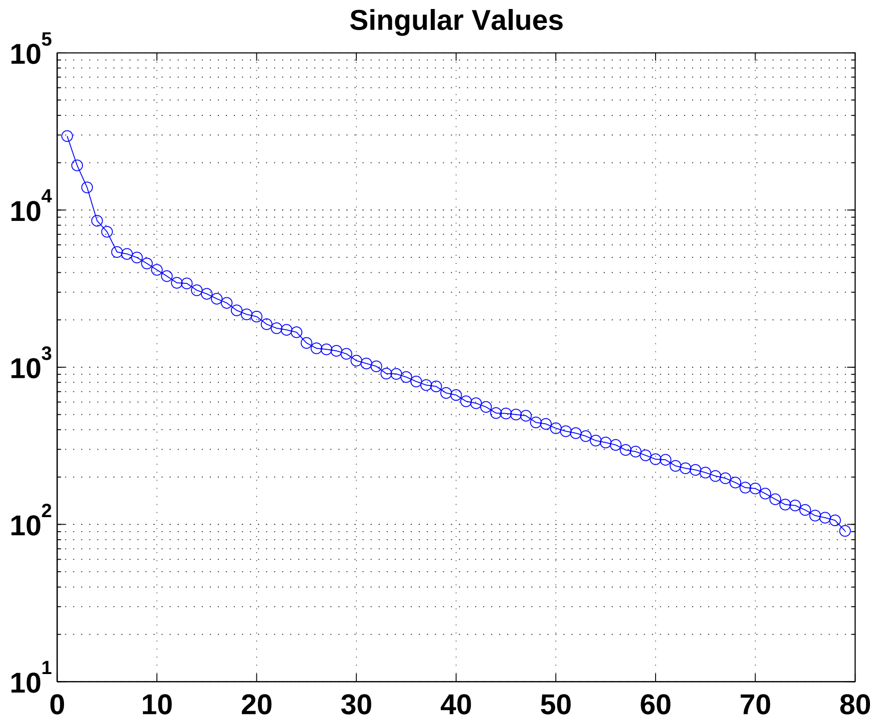
and again take the SVD $A = U\Sigma V^T$. Here U is the same size as A and Σ and V are both square 80×80 matrices.

Now the rows of V^T form an orthonormal basis for the row space of A, i.e. each profile (row of A) can be written as a linear combination of the rows of V^T or

$$A = CV^T$$

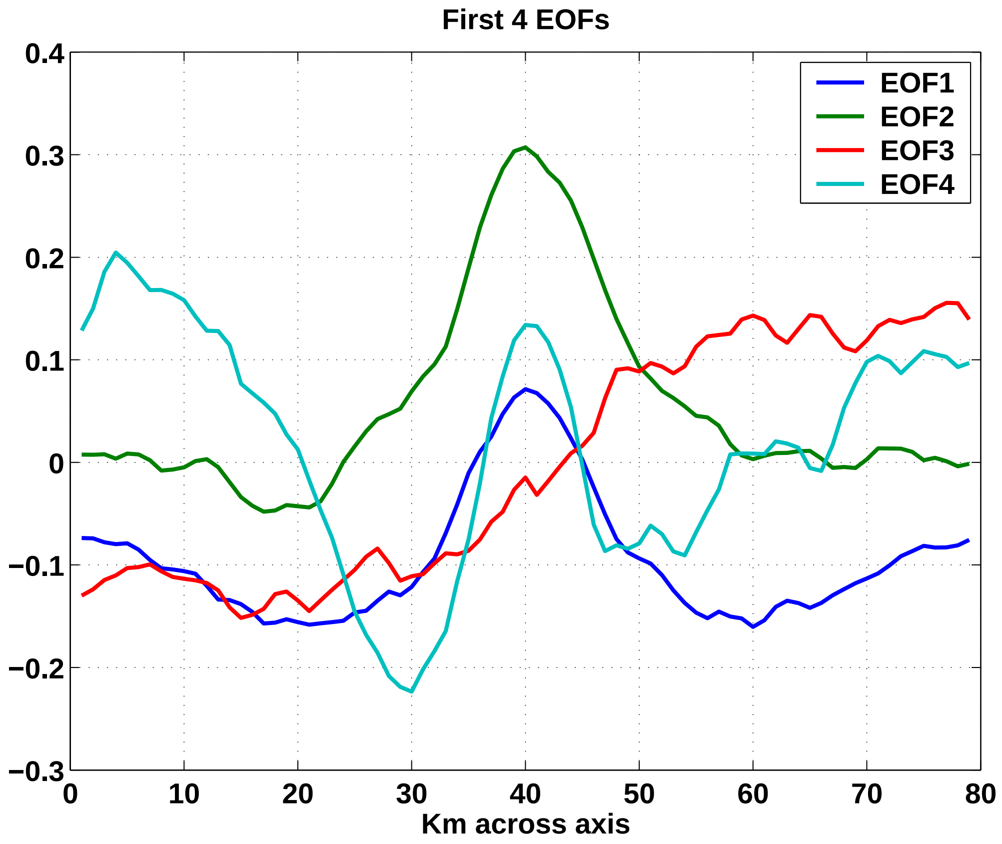
which by inspection of the SVD shows that $C = U\Sigma$. Here, the rows of V^T are known as *Empirical Orthogonal Functions* or EOFs.

Again, if the spectrum of Singular values contains a few large values and a long tail of very small values, it may be possible to reconstruct the rows of A with only a small number of EOFs. The spectrum for this data looks like



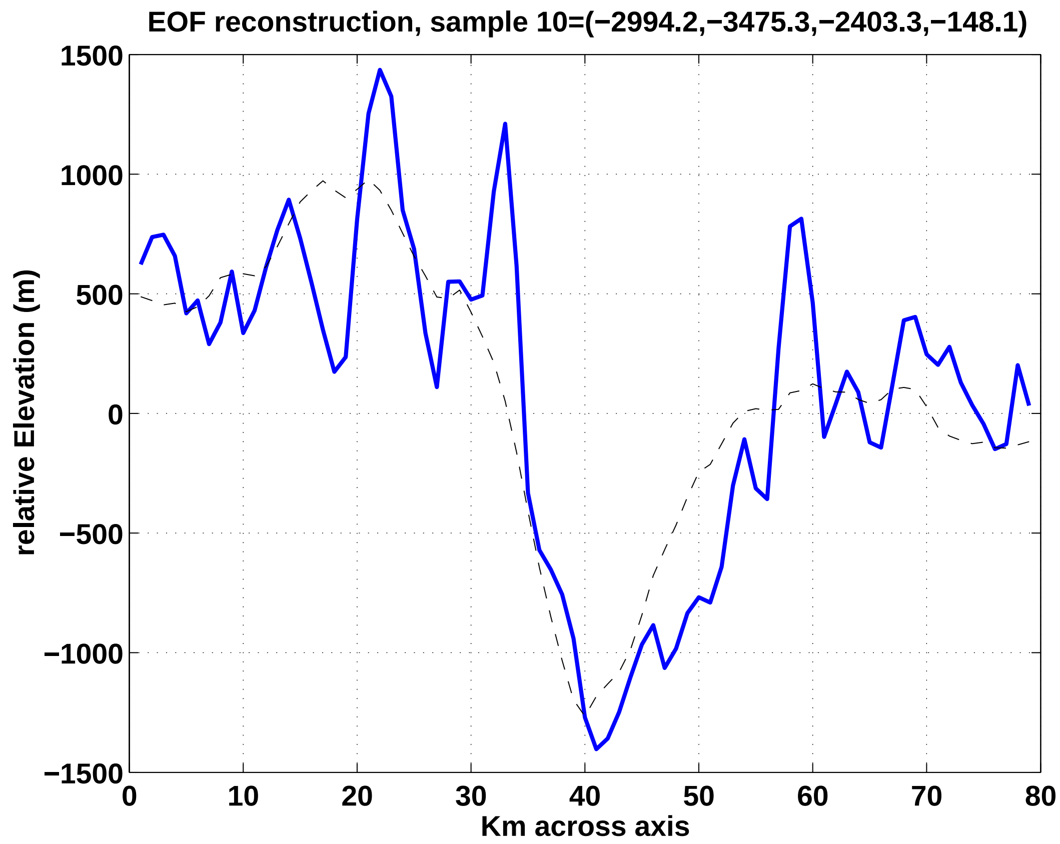
which suggests that you only need about 4 EOF's to explain most of the data.

The first 4 EOFs

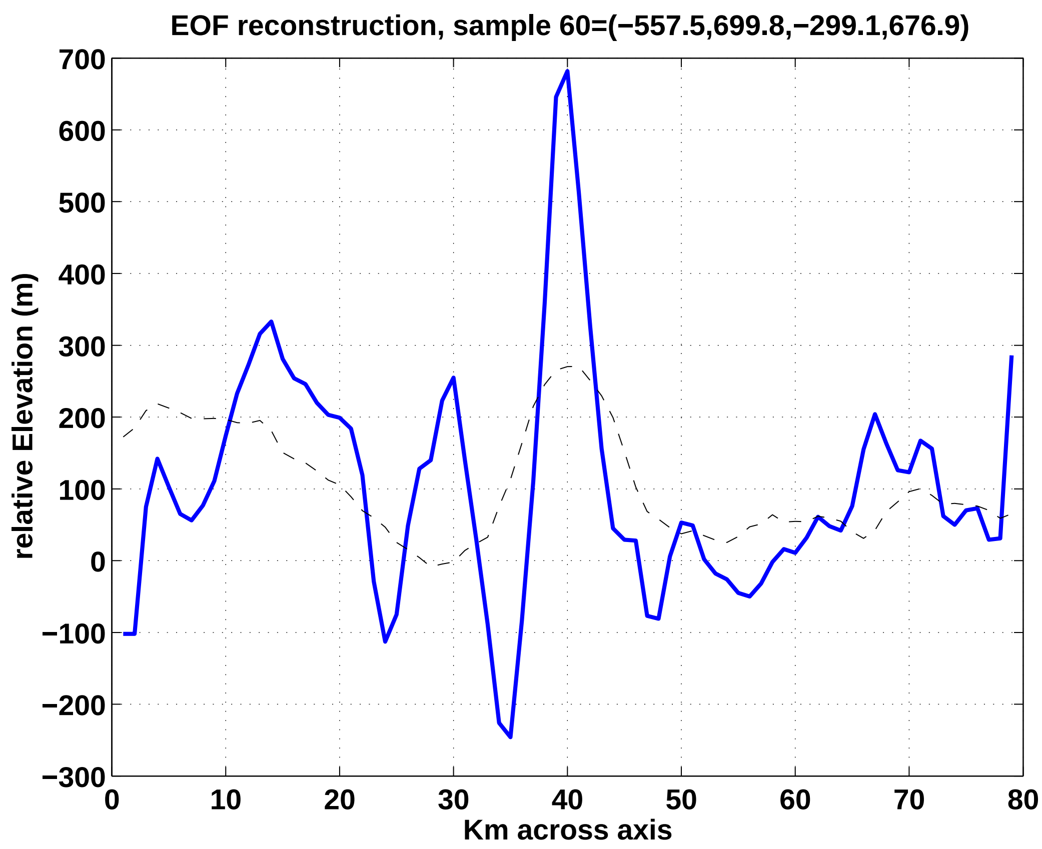


And we can reconstruct individual profiles as combinations of the first 4 EOF’s.

For example here is one for a *slow* spreading rate



Intermediate spreading rate



Fast spreading rate