1 E3102: A study guide and review, Version 1.2

Here is a list of subjects that *I* think we've covered in class (your mileage may vary). If you understand and can do the basic problems in this guide you should be in very good shape. This guide is probably over-thorough. The test itself will have about 6 questions covering the whole course but emphasizing the second half (which implicitly includes all of the first half). I'll try to avoid anything *overly* tricky.

1.1 Homogeneous Linear PDE's in 2 variables

Separation of variables a-go-go Be able to solve the following separable problems with homogeneous boundary conditions and no forcing terms. When in doubt use full-blown separation of variables. Alternatively, if you **know** the appropriate eigenfunctions you can solve these by eigenfunction expansion (the first three problems all have eigenfunctions that are some combinations of sines and cosines determined by the boundary conditions)

- *1-D time dependent heat flow equation*

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}$$

- *1-D vibrating string*

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

- *Laplace's Equation* $\nabla^2 u = 0$ in cartesian coordinates (rectangles)

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

  and polar coordinates (disks)

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial u}{\partial \theta} = 0$$

  Note: For the disk, the eigenfunctions are in θ. The separated equations for $f(r)$ is an *equidimensional equation* with trial solutions of form $f(r) = r^p$

- *Helmholtz Equation* $\nabla^2 u + \lambda u = 0$ in cartesian and polar coordinates. **Note:** Helmholtz equation will give Eigenfunctions in both directions for $\lambda > 0$. In cartesian coordinates you'll get combinations of sines and cosines in x and y. In polar coordinates you'll get sines and cosines in θ and the dreaded Bessel functions in r. If $\lambda < 0$ (modified helmholtz equation) you'll only get eigenfunctions in one direction.

1.2 Sturm-Liouville Boundary Value problems

Regular Sturm-Liouville Boundary Value problems are of the form

$$\frac{d}{dx}p(x)\frac{df}{dx} + q(x)f + \lambda\sigma(x)f = 0$$

with general homogeneous Boundary conditions at $x = a$ and $x = b$

$$\beta_1 f(a) + \beta_2 \frac{df}{dx}(a) = 0$$
$$\beta_3 f(b) + \beta_4 \frac{df}{dx}(b) = 0$$

and $p > 0$, $\sigma > 0$ for $a \leq x \leq b$. (This eigenvalue problem can also be written as $\mathcal{L}(f) = -\lambda\sigma f$.)
Know: the basic properties of these problems (page 157 in Haberman).

1. They have an infinite number of *Real* eigenvalues $\lambda_1 < \lambda_2 < \ldots < \lambda_n$ $n \to \infty$

2. For each eigenvalue λ_n there is a corresponding unique eigenfunction $\phi_n(x)$ (note: uniqueness is only for 1-D problems without periodic BC's).

3. The eigenfunctions are **orthogonal** under weight σ, i.e.

$$\int_a^b \phi_n \phi_m \sigma dx = 0 \qquad \text{for } m \neq n$$

4. The eigenfunctions are **complete** in the sense that *any* piecewise smooth function $g(x,t)$ can be written in terms of an infinite series of the eigenfunctions. i.e.

$$g(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x)$$

where the coefficients $a_n(t)$ are defined by the integrals

$$a_n(t) = \frac{\int_a^b g(x,t)\phi_n(x)\sigma(x)dx}{\int_a^b \phi_n^2(x)\sigma(x)dx}$$

Also know

Green's formula for SL problems (and where it can be useful)

$$\int_a^b u\mathcal{L}v - v\mathcal{L}u = p(x)\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Bigg|_a^b$$

for *any* functions $u(x)$ and $v(x)$. Also know for which boundary conditions, the right hand side vanishes (and therefore $\mathcal{L}$ is *self-adjoint*).

Rayleigh Quotient and how to use it to estimate eigenvalues (or show if positive)

$$\lambda = -\frac{\int_a^b \phi\mathcal{L}\phi dx}{\int_a^b \phi_n^2(x)\sigma(x)dx}$$

1.3 Fourier Series/Generalized Fourier Series

- Understand how Fourier series are a special case of Sturm-Liouville theory
- Be able to sketch full Fourier Series, Fourier Sine Series and Fourier Cosine series
- Realize that Fourier-Bessel series work the same way. I.e. for a disk $0 < r < a$ I can expand any function $g(r)$ (bounded at $r = 0$) in terms of bessel functions,e.g.

  $$g(r) = \sum_{n=1}^{\infty} a_n J_m(z_{mn}r/a)$$

  where

  $$a_n = \frac{\int_0^a g(r)J_m(z_{mn}r/a)rdr}{\int_0^a J_m^2(z_{mn}r/a)rdr}$$

  i.e. $\phi_n(r) = J_m(z_m nr/a)$ and $\sigma = r$.
- Understand when, and when not, to differentiate these infinite series term-by-term. (i.e. it is okay for continuous functions g with the same boundary conditions as the eigenfunctions).

1.4 Fourier Transforms

The fourier transform

$$\tilde{f}(k) = \mathcal{F}\left[f(x)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{ikx}dx$$

and inverse fourier transform

$$f(x) = \mathcal{F}^{-1}\left[\tilde{f}(k)\right] = \int_{-\infty}^{\infty} \tilde{f}(k)e^{ikx}dk$$

are the natural extension of the full fourier series for the infinite line $-\infty < x < \infty$. Note, in general $\tilde{f}$ is a complex number even if f is real. Some other useful properties of the fourier transform that we talked about in class are that they can be extended to functions of multiple variables and their derivatives. For example if $u(x,t)$ is well behaved such that $|\int_{-\infty}^{\infty} u(x,t)dx| < \infty$ then

$$\tilde{u}(k,t) = \mathcal{F}\left[u(x,t)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)e^{ikx}dx$$

$$\mathcal{F}\left[\frac{\partial u}{\partial t}\right] = \frac{\partial \tilde{u}}{\partial t} \quad (1)$$

$$\mathcal{F}\left[\frac{\partial u}{\partial x}\right] = -ik\tilde{u} \quad (2)$$

$$\mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right] = -k^2\tilde{u} \quad (3)$$

which can be used to transform PDE's and their boundary condions into ODES…

1.5 Homogeneous Linear PDE's in 3 or more variables

These problems are just more complicated versions of the first set of problems. In general they can be solved by either separation of variables or eigenfunction expansion. For eigenfunction expansion, however, it is most useful to use the *2-D* eigenfunctions of Helmholtz equation. Basic problems are

- *Time dependent heat-flow on a rectangle* $u(t,x,y)$ *or disk* $u(t,r,\theta)$

$$\frac{\partial u}{\partial t} = \kappa\nabla^2 u$$

- *Time dependent vibrations of a 2-d membrane*

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

- *3-D Laplace equation* $\nabla^2 u = 0$ *on rectangular solid or cylinder*

For the first two problems you can always separate out the time dependent parts using $u(t, \mathbf{x}) = h(t)w(\mathbf{x})$ where $\mathbf{x} = (x, y)$ for cartesian problems and $\mathbf{x} = (r, \theta)$ for polar problems. In both cases, w will satisfy *Helmholtz Equation*

$$\nabla^2 w + \lambda w = 0$$

Properties of Helmholtz equation For $\lambda > 0$ and w having homogeneous boundary conditions on some domain R (e.g. a rectangle or a disk), then many of the properties of the 1-D Sturm-Liouville theory are relevant to the 2-D (or 3-D) **Eigenfunction** problem defined by Helmholtz Equation **(See Sections 7.4-7.5, pages 280–290).** Important examples are

- There are an infinite number of *real* eigenvalues $\lambda_1 < \lambda_2 < \ldots < \lambda_n$ $n \to \infty$

- For each eigenvalue there *may* be multiple orthogonal eigenfunctions (this is different from the 1-D case).

- Eigenfunctions with different eigenvalues are orthogonal with regard to the **area** integral over the domain R

$$\iint_R \phi_i \phi_j dx dy = 0 \qquad \text{for } i \neq j$$

  this can also be made generally true for any two eigenfunctions with the same eigenvalue. (see below)

- The eigenfunctions are **complete** in the sense that any piecewise-smooth 2-D function can be written as an infinite sum of appropriately weighted eigenfunctions

$$g(x, y) = \sum_i a_i \phi_i(x, y)$$

  where

$$a_i = \frac{\iint_R g(x, y)\phi_i(x, y) dA}{\iint_R \phi_i^2(x, y) dA}$$

Some example solutions of Helmholtz Eq.

rectangular region $0 \le x \le L$, $0 \le y \le H$ with $w = 0$ on the boundary

$$\lambda_{mn} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2 \qquad \phi_{mn}(x,y) = \sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}$$

Note 1: if L=H (square region), then ϕ_{mn} and ϕ_{nm} are orthogonal but have the same eigenvalue $\lambda_{mn} = \lambda_{nm}$

Note 2: if the boundaries in x are homogeneous but instead were $\partial w/\partial x(0) = \partial w/\partial x(L) = 0$, the eigenvalues would be the same (and also include $\lambda = 0$), but the eigenfunctions would be $\phi_{mn} = \cos\frac{n\pi x}{L}\sin\frac{m\pi y}{H}$

circular disk $0 \le r \le a$, $-\pi \le \theta \le \pi$ with $w(a,\theta) = 0$ on the boundary (and $w(0,\theta)$ is bounded).

$$\lambda_{mn} = \left(\frac{z_{mn}}{a}\right)^2$$

with **two** orthogonal eigenfunctions for each λ_{mn}.

$$\phi^1_{mn}(r,\theta) = J_m\left(z_{mn}\frac{r}{a}\right)\cos m\theta \qquad \phi^2_{mn}(r,\theta) = J_m\left(z_{mn}\frac{r}{a}\right)\sin m\theta$$

where $J_m(r)$ is the Bessel function of the first kind of order m and z_{mn} is the nth zero of the mth Bessel function. Note for the disk $dA = r dr d\theta$.

1.6 Non-Homogeneous PDE's and method of eigenfunction expansion

Here we extended the homogeneous problems to problems with both non-homogeneous source terms and non-homogeneous boundary conditions. For the latter problems however it was always possible to set $u(t,\mathbf{x}) = v(t,\mathbf{x}) + r(t,\mathbf{x})$ where r is any function that satisfies the non-homogeneous boundary conditions. Substituting this into the original PDE, will produce a new equation for v where v has *homogeneous* boundary conditions. Given these reduced problems, there is a general recipe for solving the non-homogeneous source terms using the **method of eigenfunction expansion** which I will illustrate with the simplified time-dependent problem

$$\frac{\partial v}{\partial t} = \frac{1}{\sigma(\mathbf{x})}\mathcal{L}v + Q(t,\mathbf{x})$$

with

$$v(\mathbf{x}, 0) = f(\mathbf{x})$$

and v has homogeneous boundary conditions and $\mathcal{L}$ is a 2nd order differential operator that only includes spatial derivatives. Some examples in 1-D would be $\mathcal{L}v = k\partial^2 v/\partial x^2$, $\sigma = 1$, or $\mathcal{L}v = k\frac{\partial}{\partial r}r\frac{\partial f}{\partial r}$, $\sigma = r$ for polar coordinates, or in 2-D and 3-D $\mathcal{L}v = k\nabla^2 v$ and $\sigma = 1$.

1. Use separation of variables on the associated homogeneous problem (assume $Q = 0$) to find the eigenvalues and eigenfunctions of the spatial boundary value problem $\mathcal{L}\phi_n = -\lambda_n\sigma\phi_n$ and ϕ_n has the **same** homogeneous boundary conditions as v.

2. Expand both the solution and Q in terms of these eigenfunctions, e.g.

$$v(\mathbf{x}, t) = \sum_n a_n(t)\phi_n(\mathbf{x}) \qquad Q(\mathbf{x}, t) = \sum_n q_n(t)\phi_n(\mathbf{x})$$

3. Substitute these sums into the PDE for v (and you can take all the derivatives term by term because v and ϕ_n all have the same boundary conditions) to get

$$\sum_i \left[\frac{da_n}{dt} + \lambda_n a_n(t) - q_n(t)\right]\phi_n(\mathbf{x}) = 0$$

   where we have used the relationship

$$\mathcal{L}v = \sum_i a_n(t)\mathcal{L}\phi_n(\mathbf{x}) = -\sum_i a_n(t)\lambda_n\sigma\phi_n(\mathbf{x}) =$$

   using the definition of the eigenfunctions of $\mathcal{L}$

4. Use orthogonality of the ϕ_n's to get the set of 1st-order *Non-homogeneous Ordinary differential equations*

$$\frac{da_n}{dt} + \lambda_n a_n = q_n(t)$$

5. solve this using variation of parameters (and initial conditions) to find $a_n(t)$ (and therefore v). Here you will need to use the initial conditions

$$a_n(0) = \frac{\int_R f(x)\phi_n\sigma dA}{\int_R \phi_n^2\sigma dA}$$

6. reconstruct the full solution $u(x, t) = v(x, t) + r(x, t)$*. . . the end*

simple problems with equilibrium solutions In addition to the full eigenfunction expansion technique, sometimes it is easier to solve problems with steady forcing terms $Q(x)$ by looking for a a steady state solution $u_e(\mathbf{x})$ that satisfies

$$\mathcal{L}u_e = -Q(x)$$

and non-homogeneous boundary conditions, then look for a transient solution $v(x,t)$ of the remaining homogeneous problem with homogeneous BC's. and reconstruct the full solution $u(x,t) = v(x,t) + u_e(x)$. For example you can use this to solve the heat flow equation with steady forcing and fixed temperature boundary conditions.

1.7 Green's Functions

Given a boundary value problem of form

$$\mathcal{L}u = f(x)$$

with homogeneous boundary conditions, find the **Green's Functions** with the same boundary conditions defined by

$$\mathcal{L}G(x,x_0) = \delta(x - x_0)$$

where $\delta(x - x_0)$ is a Dirac delta function at point x_0. Given $G(x,x_0)$ the general solution for u is

$$u(x,t) = \int_R f(x_0)G(x,x_0)dx_0$$

Basic problems

1. Understand basic behaviour of the delta function i.e.

   $$f(x_0) = \int_{-\infty}^{\infty} f(x)\delta(x - x_0)dx$$

   therefore

   $$1 = \int_{-\infty}^{\infty} \delta(x - x_0)dx$$

   and thus the Heaviside step function is defined by

   $$H(x - x_0) = \int_{-\infty}^{x} \delta(x - x_0)dx$$

2. Know how to find the 1-D green's functions for Sturm Louiville operaters (in particular $\mathcal{L}u = d^2u/dx^2$ or $\mathcal{L}u = d^2u/dx^2+u$) and appropriate boundary conditions. The basic recipe is

   (a) find the general solution of the *homogeneous* problem away from the delta spike

   (b) Find a specific solution for either side of the delta spike that satisfies the homogeneous equation and *one* boundary condition.

   (c) Make the functions continuous at the delta function ($x = x_0$)

   (d) integrate the ODE for a small region around x_0 and find the constraints on the derivatives dG/dx on either side of x_0. And that should do it.

3. Use the same approach to find the infinite space Green's functions for $\nabla^2 u = f(x)$ in 2 and 3-D.

1.8 Wave Equations and the method of characteristics

Understand

- How to find solutions of simple 1-D wave equation

$$\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0$$

  with initial conditions $w(x, 0) = f(x)$ on $-\infty < x < \infty$, using the method of characteristics.

- Know how to extend it to more general linear problems like

$$\frac{\partial w}{\partial t} + c(x,t)\frac{\partial w}{\partial x} = -w$$

- and to non-linear shock problems like

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial x} = 0$$

  with $w(x, 0) = f(x)$

for each of these problems know how to qualitatively sketch what is going on in space and time. A graphical answer will go a long way.

1.9 P.S.

That's it for now...watch this space for anything new and/or corrections. if you have any questions come and see me in office hours or send me e-mail at mspieg@ldeo.columbia.edu to set up an appointment. Good luck and relax.