1. Toss a coin three times. Let $X$ be the number of heads in the first toss and let $Y$ be the number of heads in the first three tosses.
   
   (a) Find $p(x, y) = P(X = x, Y = y)$ the joint p.m.f. of $(X, Y)$
   
   (b) Find the marginal probability distributions $p_X(x) = P(X = x)$ and $p_Y(y) = P(Y = y)$.
   
   (c) Find the conditional probability that $Y = 2$ given that $X = 0$.
   
   (d) Repeat (c) when $X = 1$.
   
   (e) Find $E[Y - X]$.
   
   (f) Find $\text{Cov}(X, Y)$.

2. Let $\Omega = \{ (\omega_1, \omega_2) : \omega_1^2 + \omega_2^2 \leq 1 \}$. Geometrically $\Omega$ is the unit circle (including its interior) in 2-dimensions. Let $A \subset \Omega$ let $a(A)$ denote its area. Clearly $a(\Omega) = \pi$. It is natural to define a probability measure that assigns
   
   $$P(A) = \frac{a(A)}{a(\Omega)} = \frac{a(A)}{\pi}$$
   
   to subsets $A \subset \Omega$ for which $a(A)$ is well defined. This measure conveys the idea of a uniform distribution over the unit circle. Given $\omega = (\omega_1, \omega_2) \in \Omega$, let $X(\omega) = \omega_1$ and $Y(\omega) = \omega_2$. It is not difficult to see that $(X, Y)$ has joint density function

   $$f(x, y) = \frac{1}{\pi} \{ (x, y) : x^2 + y^2 \leq 1 \}.$$ 

   (a) Find the marginal density functions $f_X(x)$ and $f_Y(y)$.
   
   (c) Find the conditional density function of $X$ given $Y$ for values of $Y \in (-1, 1)$.
   
   (d) Find $E[3X + 2Y]$.
   
   (e) Find $\text{Cov}(X, Y)$.

3. Let $X_1, X_2, \ldots, X_n$ be independent random variables with mean $\mu$ and variance $\sigma^2$ and let $S_n = X_1 + X_2 + \ldots + X_n$.
   
   (a) Find $E[S_n]$.
   
   (b) Find $\text{Var}[S_n]$.
   
   (c) Let $\bar{X}_n = \frac{S_n}{n}$. Find $E[\bar{X}_n]$ and $\text{Var}[\bar{X}_n]$.
   
   (d) Use Chebyshev’s inequality to bound $P(|\bar{X}_n - \mu| \geq c)$. Hint: Write $c = k\sqrt{\text{Var}[X_n]}$.
   
   What happens as $n$ increases?
   
   (e) Let $Z_n = \frac{S_n - n\mu}{\sqrt{n}\sigma}$. Find $E[Z_n]$ and $\text{Var}[Z_n]$. 