Statistics and Quantitative Analysis U4320

Segment 10

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Key Points

- 1. Review Univariate Regression Model
- 2. Introduce Multivariate Regression Model
  - Assumptions
  - Estimation
  - Hypothesis Testing
- 3. Interpreting Multiple Regression Model
  - “Impact of X on Y controlling for ....”

I. Univariate Analysis

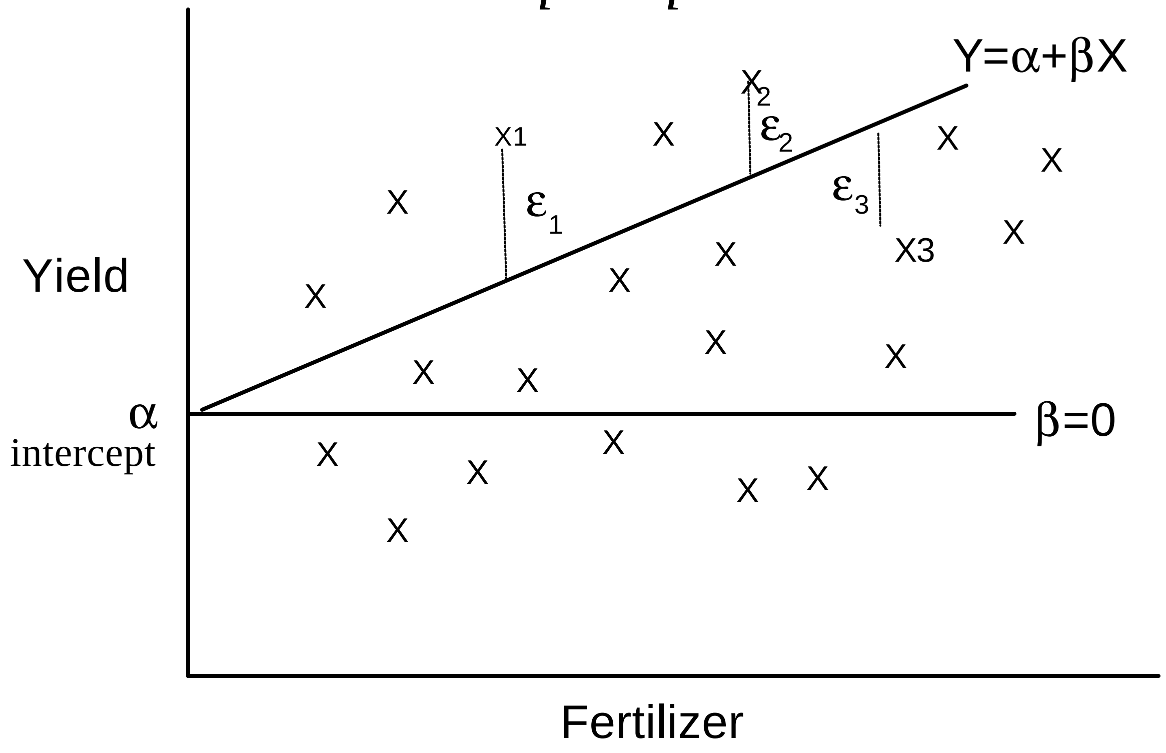
A. Assumptions of Regression Model

- 1. Regression Line
  - A. Population
    - The standard regression equation is
      - $Y_i = \alpha + \beta X_i + \varepsilon_i$
      - The only things that we observe is Y and X.
      - From these data we estimate α and β.
      - But our estimate will always contain some error.

Univariate Analysis (cont.)

- This error is represented by:

$$e_i = Y_i - Y$$



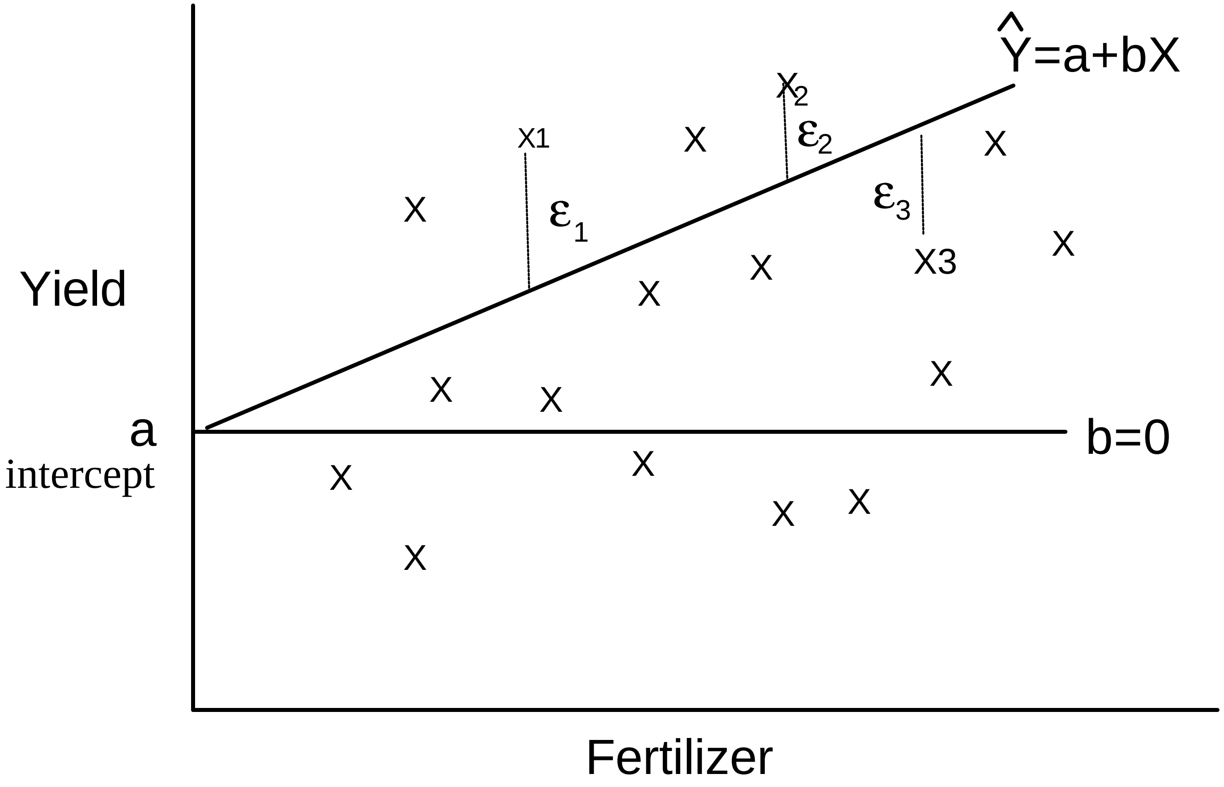
Univariate Analysis (cont.)

- B. Sample
  - Most times we don’t observe the underlying population parameters.
  - All we observe is a sample of X and Y values from which make estimates of α and β.

Univariate Analysis (cont.)

- So we introduce a new form of error in our analysis.

$$e_i = Y_i - \hat{Y}$$



Univariate Analysis (cont.)

- 2. Underlying Assumptions
  - Linearity
  - The true relation between Y and X is captured in the equation: $Y = a + bX$
  - Homoscedasticity (Homogeneous Variance)
    - Each of the e_i has the same variance.

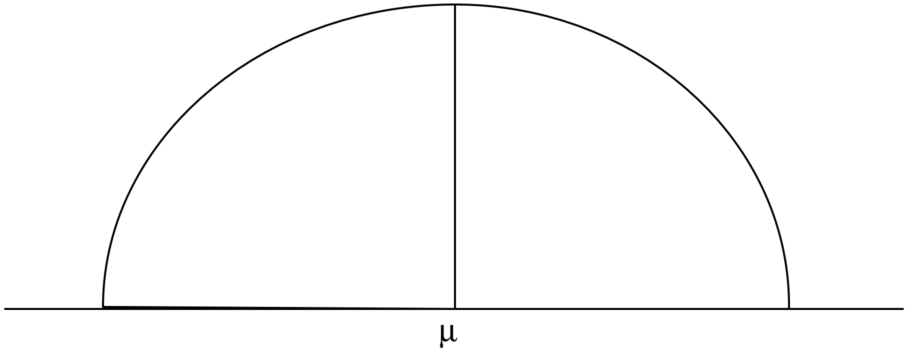
$$E(e_i2) = {}^2 \quad \text{for all } i$$

Univariate Analysis (cont.)

- Independence
  - Each of the e_i's is independent from each other. That is, the value of one does not effect the value of any other observation i's error.

$Cov(e_i,e_j) = 0$ for i j

- Normality
  - Each e_i is normally distributed.



Univariate Analysis (cont.)

- Combined with assumption two, this means that the error terms are normally distributed with mean = 0 and variance 2

We write this as $e_i \sim N(0, \sigma^2)$

Univariate Analysis (cont.)

B. Estimation: Make inferences about the population given a sample

- 1. Best Fit Line
  - We are estimating the population line by drawing the best fit line through our data,

$$\hat{Y} = a + bX$$

Univariate Analysis (cont.)

- That means we have to estimate both a slope and an intercept.

$$\mathrm{b}=\frac{\sum xy}{\sum x^2}$$

$$a = \overline{Y} - b\overline{X}$$

Univariate Analysis (cont.)

- Usually, we are interested in the slope.
- Why?
  - Testing to see if the slope is not equal to zero is testing to see if one variable has any influence on the other.

Univariate Analysis (cont.)

- 2. The Standard Error
  - To construct a statistical test of the slope of the regression line, we need to know its mean and standard error.
  - Mean
    - The mean of the slope of the regression line
      Expected value of b = β

Univariate Analysis (cont.)

- Standard Error
  - The standard error is exactly by how much our estimate of b is off.

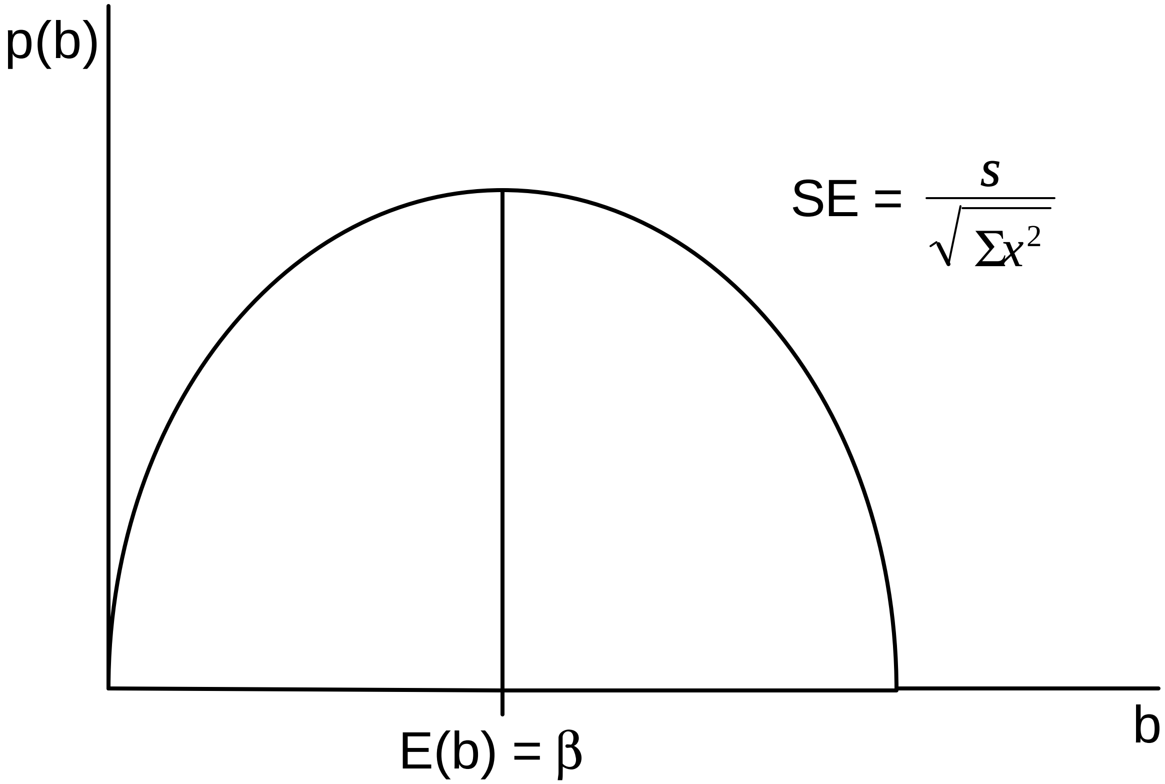
$$\text{Standard error of b} = \frac{s}{\sqrt{\Sigma x^2}}$$

$$\text{Standard error of } \sigma = \Sigma\left(Y_i - \hat{Y}\right)^2$$

$$x^2 = (X_i - \overline{X})^2$$

Univariate Analysis (cont.)

- So we can draw this diagram



Univariate Analysis (cont.)

- This makes sense, b is the factor that relates the Xs to the Y, and the standard error depends on both which is the expected variations in the Ys and on the variation in the Xs.

Univariate Analysis (cont.)

- 3. Hypothesis Testing

  a) 95% Confidence Intervals (σ unknown)

  - Confidence interval for the true slope of β given our estimate b:

$$b = b \pm t_{.025}\ \mathrm{SE}$$

$$\beta = b \pm t_{.025}\ \mathrm{SE}\ \frac{s}{\sqrt{\Sigma x^2}}$$

Univariate Analysis (cont.)

b) P-values

- P-value is the probability of observing an event, given that the null hypothesis is true.
- We can calculate the p-value by:
  - Standardizing and calculating the t-statistic:

$$t = \frac{b - b_0}{SE}$$

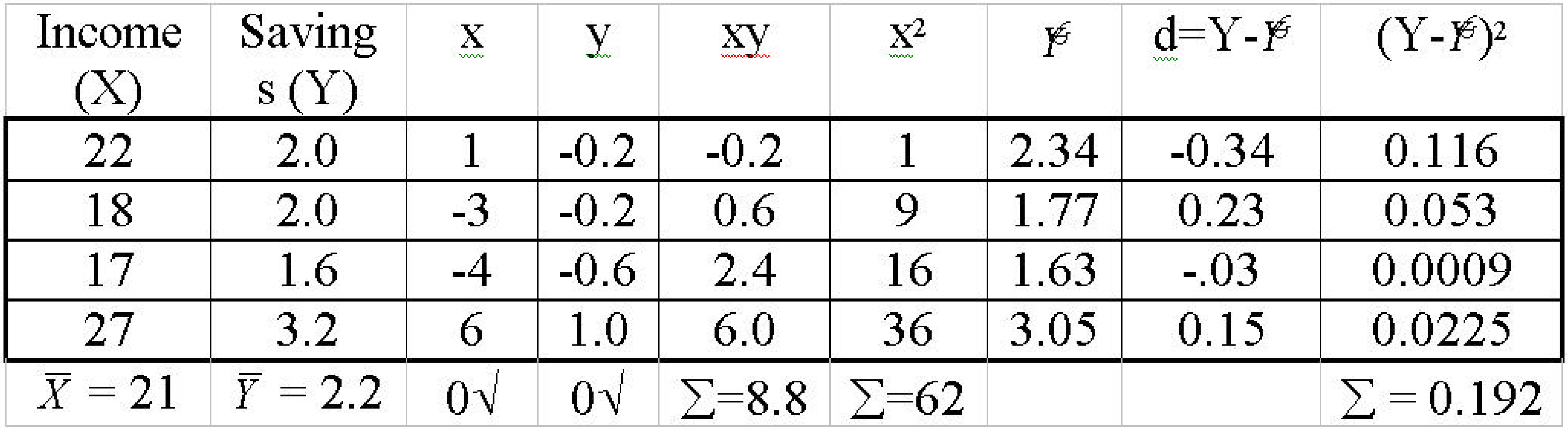
  - Determine the Degrees of Freedom:

    For univariate analysis = n-2
  - Find the probability associated with the t-statistics with n-2 degrees of freedom in the t-table.

Univariate Analysis (cont.)

C. Example

- Now we want to know do people save more money as their income increases?
- Suppose we observed 4 individual's income and saving rates?



Income (X)	Savings (Y)	x	y	xy	x^2	$\hat{Y}$	$d=Y-\hat{Y}$	$(Y-\hat{Y})^2$
22	2.0	1	-0.2	-0.2	1	2.34	-0.34	0.116
18	2.0	-3	-0.2	0.6	9	1.77	0.23	0.053
17	1.6	-4	-0.6	2.4	16	1.63	-.03	0.0009
27	3.2	6	1.0	6.0	36	3.05	0.15	0.0225
$\bar{X}=21$	$\bar{Y}=2.2$	0√	0√	∑=8.8	∑=62			∑ = 0.192

Univariate Analysis (cont.)

1) Calculate the fitted line

$$Y = a + bX$$

- Estimate b

  $b = \Sigma xy / \Sigma x^2 = 8.8 / 62 = 0.142$

- What does this mean?
  - On average, people save a little over 14% of every extra dollar they earn.

Univariate Analysis (cont.)

- Intercept a
  - a = $\overline{Y}$ - b$\overline{X}$ = 2.2 - 0.142 (21) = -0.782

- What does this mean?

  With no income, people borrow

- So the regression equation is:

  Y= -0.78 + 0.142X

Univariate Analysis (cont.)

2) Calculate a 95% confidence interval

- Now let's test the null hypothesis that $\beta = 0$. That is, the hypothesis that people do not tend to save any of the extra money they earn.

H_0: $\beta = 0$ H_a: $\beta \neq 0$;

at the 5% significance level

Univariate Analysis (cont.)

- What do we need to calculate the confidence interval?

$s2 = \Sigma d^2 / n\text{-}2 = .192 / 2 = 0.096$

$s = .096 = .309$

Univariate Analysis (cont.)

- What is the formula for the confidence interval?

$$\beta = \ b \ t_{.025} \ \frac{s}{\sqrt{\sum x^2}} .$$

$$\beta = \ .142 \pm 4.30 \cdot .309 / \sqrt{}62$$

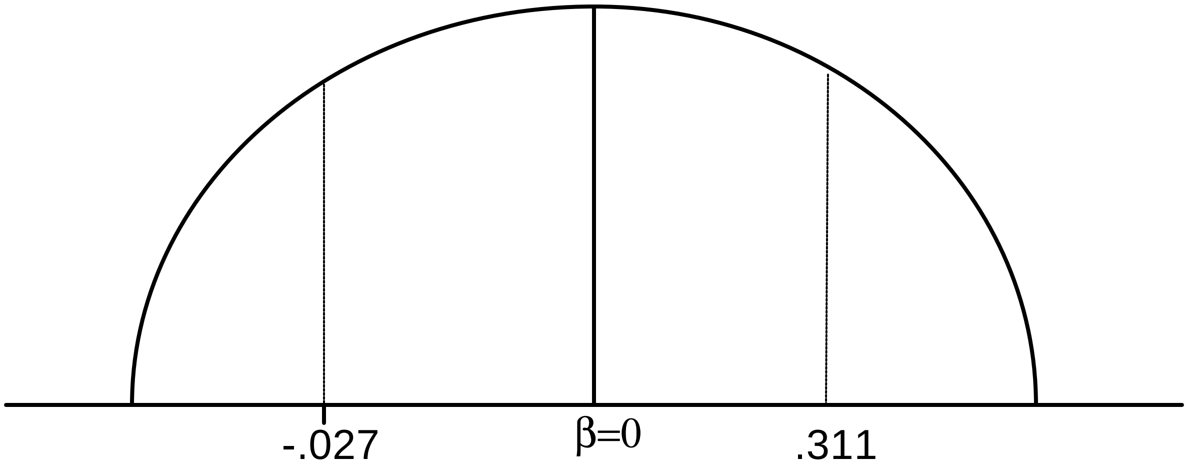
$$\beta = \ .142 \pm .169$$

$$-.027 \leq \beta \leq .311$$

Univariate Analysis (cont.)

3) Accept or reject the null hypothesis

- Since zero falls within this interval, we **cannot reject** the null hypothesis. This is probably due to the small sample size.



Univariate Analysis (cont.)

D. Additional Examples

- 1. How about the hypothesis that $\beta = .50$, so that people save half their extra income?
  - It is outside the confidence interval, so we can **reject** this hypothesis

Univariate Analysis (cont.)

- 2. Let's say that it is well known that Japanese consumers save 20% of their income on average. Can we use these data (presumably from American families) to test the hypothesis that Japanese save at a higher rate than Americans?

  - Since 20% also falls within the confidence interval, we **cannot reject** the null hypothesis that Americans save at the same rate as Japanese.

II. Multiple Regression

A. Casual Model

- 1. Univariate
  - Last time we saw that fertilizer apparently has an effect on crop yield
  - We observed a positive and significant coefficient, so more fertilizer is associated with more crops.
  - That is, we can draw a causal model that looks like this:

```
                          +
FERTILIZER ----------------------------->  YIELD
```

Multiple Regression (cont.)

- 2. Multivariate
  - Let's say that instead of randomly assigning amounts of fertilizer to plots of land, we collected data from various farms around the state.
  - Varying amounts of rainfall could also affect yield.
  - The causal model would then look like this:

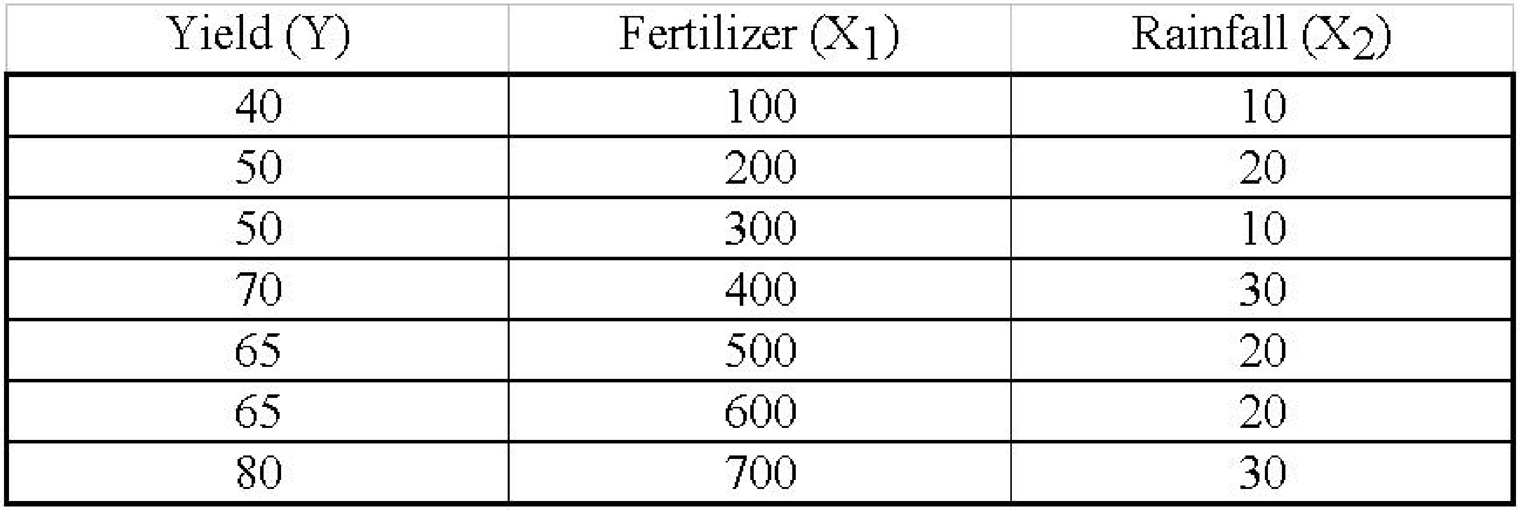
FERTILIZER -----------------------------> YIELD

RAIN

Multiple Regression (cont.)

B. Sample Data

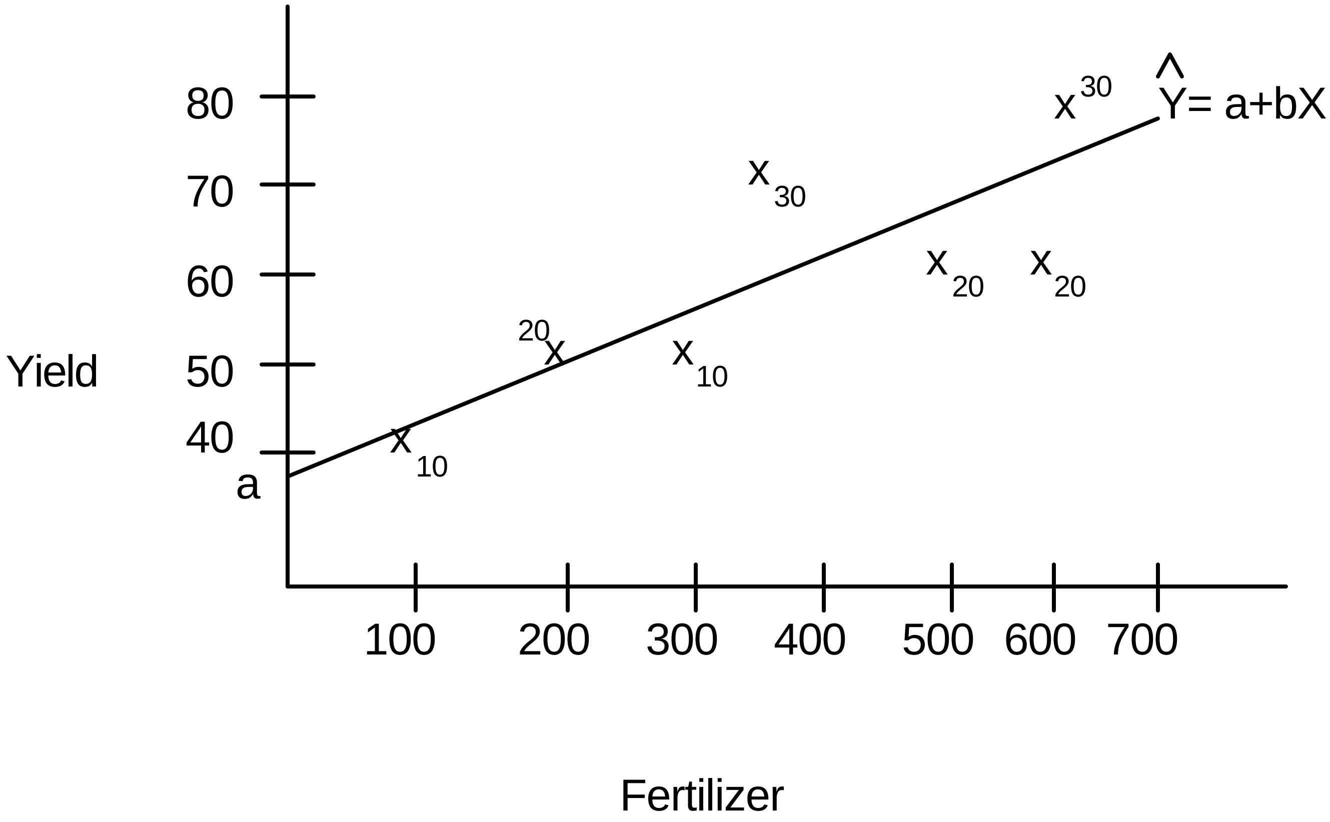
- 1. Data
  - Let's add a new category to our data table for rainfall.



Yield (Y)	Fertilizer (X_1)	Rainfall (X_2)
40	100	10
50	200	20
50	300	10
70	400	30
65	500	20
65	600	20
80	700	30

Multiple Regression (cont.)

- 2. Graph



Multiple Regression (cont.)

C. Analysis

- 1. Calculate the predicated line
  - Remember the last time

    $$\hat{Y} = 36.4 + .059X$$

  - How do we calculate the slopes when we have two variables?
    - For instance, there are two cases for which rainfall = 10.
    - For these two cases,

      $\bar{X}$ = 200 and $\bar{Y}$ = 45.

Multiple Regression (cont.)

- So we can calculate the slope and intercept of the line between these points:

$$b = \Sigma xy \,/\, \Sigma x^2$$

where $x = (X_i - \overline{X})$ and $y = (Y_i - \overline{Y})$

$$b = \frac{(-100 * -5) + (100 * 5)}{(100^2 + 100^2)}$$

$$b = .05$$

$$a = \overline{Y} - b\overline{X}$$

$$a = 45 - .05(200)$$

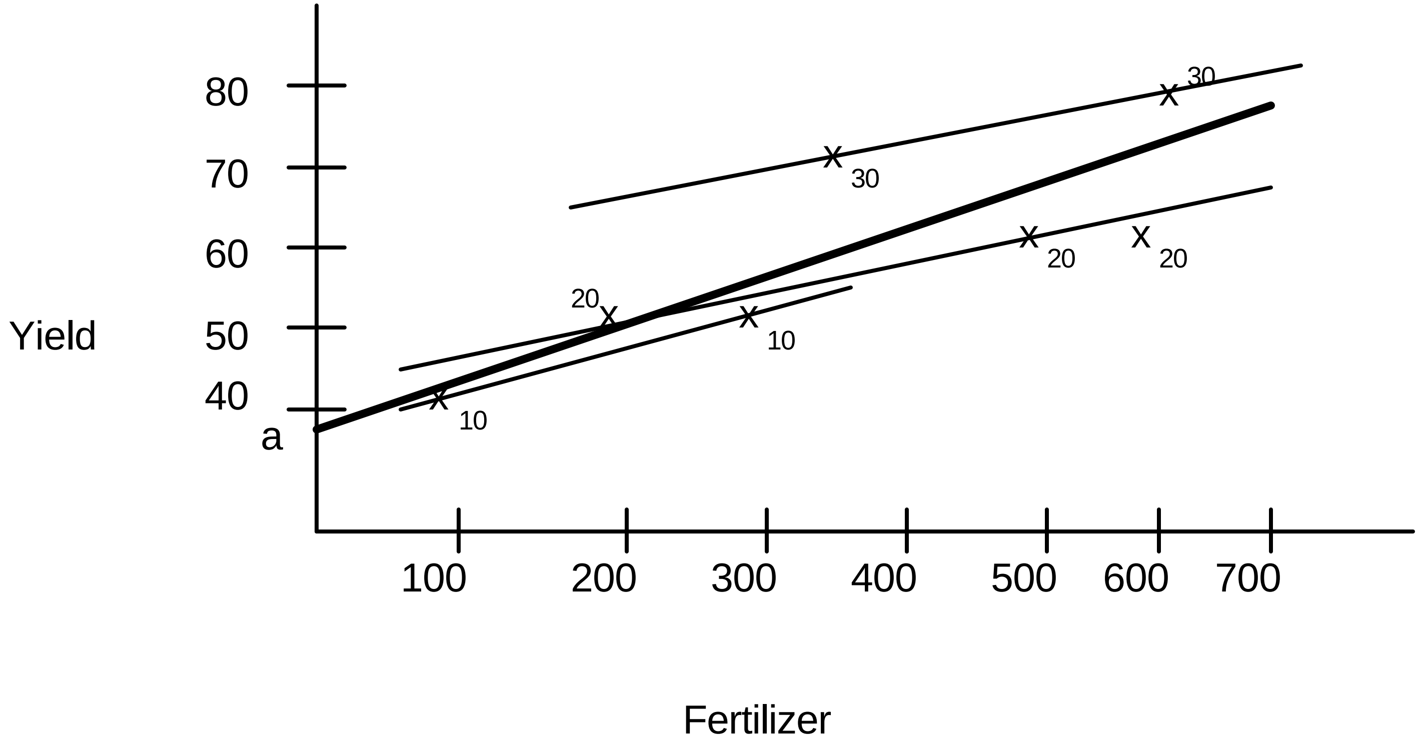
$$a = 35$$

So the regression line is:

$$Y = 35 + .05X$$

Multiple Regression (cont.)

- 2. Graph
  - We can do the same thing for the other two lines, and the results look like this:



Multiple Regression (cont.)

- You can see that these lines all have about the same slope, and that this slope is **less** than the one we calculated without taking rainfall into account.

- We say that in calculating the new slope, we are **controlling** for the effects of rainfall.

Multiple Regression (cont.)

- 3. Interpretation
  - When rainfall is taken into account, fertilizer is **not** as significant a factor as it appeared before.
  - One way to look at these results is that we can gain more accuracy by incorporating extra variables into our analysis.

III. Multiple Regression Model and OLS Fit

A. General Linear Model

- 1. Linear Expression
  - We saw that fertilizer apparently has an We write the equation for a regression line with two independent variables like this:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2.$$

Multiple Regression Model and OLS Fit (cont.)

- Intercept
  - Here, the y-intercept (or constant term) is represented by b_0.
  - How would you interpret β_0?
  - β_0 is the level of the dependent variable when both independent variables are set to zero.

Multiple Regression Model and OLS Fit (cont.)

- Slopes
  - Now we also have two slope terms, β_1 and β_2.
    - β_1 is the change in Y due to X_1 when X_2 is **held constant**. It's the change in the dependent variable due to changes in X_1 alone.
    - β_2 is the change in Y due to X_2 when X_1 is **held constant**.

Multiple Regression Model and OLS Fit (cont.)

- 2. Assumptions
  - We can write the basic equation as follows:

    $$Y= b_0 + b_1X_1 + b_2X_2 + e.$$

  - The four assumptions that we made for the one-variable model still hold.
  - we assume:
    - Linearity
    - Normality
    - Homoskedasticity, and
    - Independence

Multiple Regression Model and OLS Fit (cont.)

- You can see that we can extend this type of equation as far as we'd like. We can just write:

$$Y = b_0 + b_1X_1 + b_2X_2 + b_3X_3 + \ldots + e.$$

Multiple Regression Model and OLS Fit (cont.)

- 3. Interpretation
  - The interpretation of the constant here is the value of Y when all the X variables are set to zero.
    - a. Simple regression slope (Slope)

Y = a+ bX

coefficient b = slope

ΔY/ ΔX= b => ΔY = b Δ X

The change in Y = b*(change in X)

b = the change in Y that accompanies a unit change in X.

Multiple Regression Model and OLS Fit (cont.)

- b. Multiple Regression (slope)
  - The slopes are the effect of one independent variable on Y when all other independent variables are held constant
  - That is, for instance, b_3 represents the effect of X_3 on Y after controlling for X_1, X_2, X_4, X_5, etc.

Multiple Regression Model and OLS Fit (cont.)

B. Least Square Fit

- 1. The Fitted Line

  $Y = b_0 + b_1X_1 + b_2X_2 + e.$

- 2. OLS Criteria
  - Again, the criterion for finding the best line is least squares.
  - That is, the line that minimizes the sum of the squared distances of the data points from the line.

$$\Sigma\left(Y_i - \hat{Y}\right)^2$$

Multiple Regression Model and OLS Fit (cont.)

- 3. Benefits of Multiple Regression

  - Reduce the sum of the squared residuals.

  - Adding more variables **always** improves the fit of your model.

Multiple Regression Model and OLS Fit (cont.)

C. Example

- For example, if we plug the fertilizer numbers into a computer, it will tell us that the OLS equation is:

  Yield = 28 + .038(Fertilizer) + .83(Rainfall)

- That is, when we take rainfall into account, the effect of fertilizer on output is only .038, as compared with .059 before.

IV. Confidence Intervals and Statistical Tests

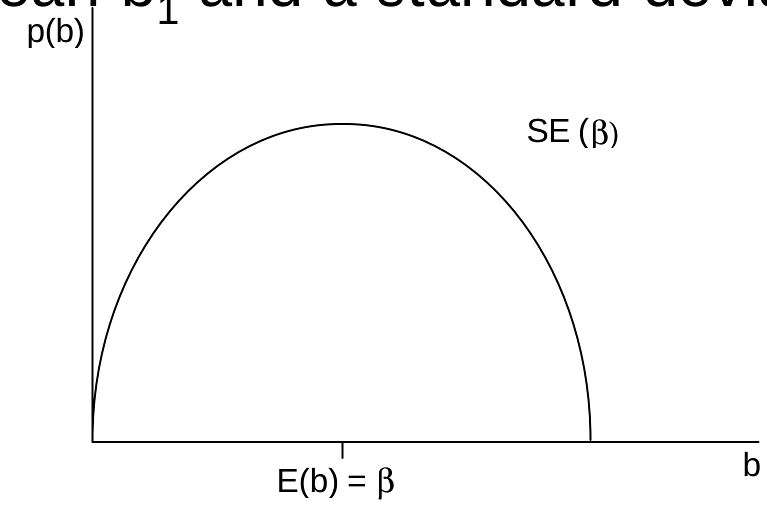
- Question:

  Does fertilizer still have a significant effect on yield, after controlling for rainfall?

Confidence Intervals and Statistical Tests (cont.)

A. Standard Error

- We want to know something about the distribution of our test statistic b_1 around β_1, the true value.
- Just as before, it's normally distributed, with mean b_1 and a standard deviation:



Confidence Intervals and Statistical Tests (cont.)

B. Confidence Intervals and P-Values

- Now that we have a standard deviation for b_1, what can we calculate?
- That's right, we can calculate a **confidence interval** for b_1.

Confidence Intervals and Statistical Tests (cont.)

1. Formulas

- Confidence Interval

  $CI\ (\beta_1) = b_1\ t_{.025} * SE_{b_1}$

Confidence Intervals and Statistical Tests (cont.)

- Degrees of Freedom
  - First, though, we'll need to know the degrees of freedom.
    - Remember that with only one independent variable, we had n-2 degrees of freedom.
    - If there are two independent variables, then degrees of freedom equals n-3.
  - In general, with k independent variables.

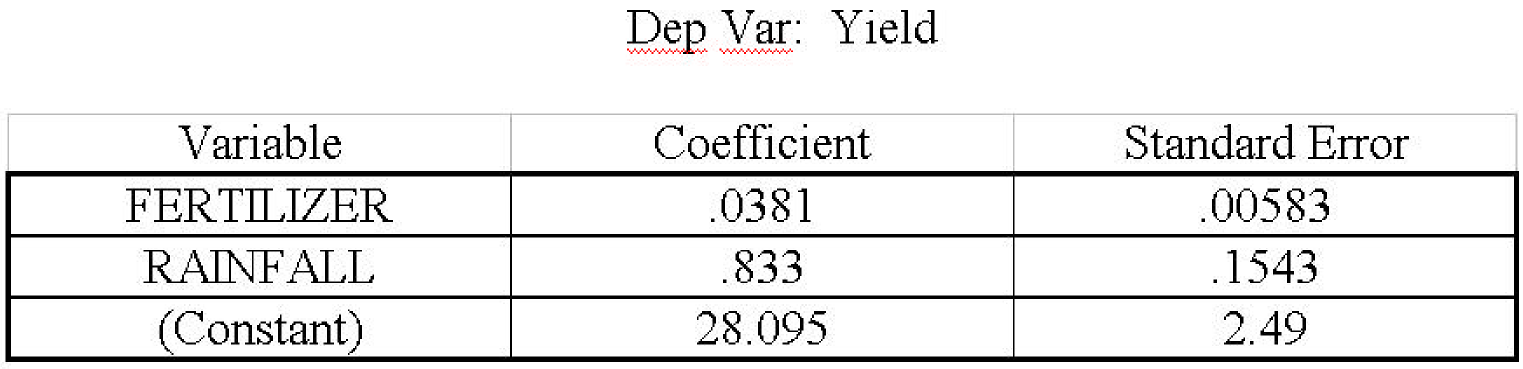
    d.f. = (n - k - 1)

    - This makes sense: one degree of freedom used up for each independent variable and one for the y-intercept.
    - So for the fertilizer data with the rainfall added in, d.f. = 4.

Confidence Intervals and Statistical Tests (cont.)

2. Example

- Let's say the computer gives us the following information:



Dep Var: Yield

Variable	Coefficient	Standard Error
FERTILIZER	.0381	.00583
RAINFALL	.833	.1543
(Constant)	28.095	2.49

Confidence Intervals and Statistical Tests (cont.)

- Then we can calculate a 95% confidence interval for b1:

$$\beta_1 = b_1 \; t_{.025} \; *$$

$$\beta_1 = .0381 \; 2.78 * .00583$$

$$\beta_1 = .0381 \; .016$$

$$\beta_1 = .022 \text{ to } .054$$

Confidence Intervals and Statistical Tests (cont.)

- So we can still **reject** the hypothesis that $\beta_1 = 0$ at the 5% level, since 0 does not fall within the confidence interval.
- With p-values, we do the same thing as before:

$$H_o:\beta_1 = 0$$

$$H_a: \beta_1 \neq 0$$

$$t = b - b_0 / SE.$$

- When we're testing the null hypothesis that b = 0, this becomes:

$$t = b / SE.$$

Confidence Intervals and Statistical Tests (cont.)

3. Results

- The t value for fertilizer is:

$$t = \frac{0.0381}{0.00583} = 6.53.$$

  - We go to the t-table under four degrees of freedom and see that this corresponds to a probability p<.0025.

  - So again we'd reject the null at the 5%, or even the 1% level.

Confidence Intervals and Statistical Tests (cont.)

- What about rainfall?

$$t = \frac{0.833}{0.154} = 5.41.$$

- This is significant at the .005 level, so we'd reject the null that rainfall has no effect.

$$\begin{array}{rcl} \hat{Y} = 28.095 & +\,0.0381X_1 & +\,0.833X_2 \\ & (0.0058) & (0.1543) \\ & 6.53 & 5.41 \end{array}$$

Confidence Intervals and Statistical Tests (cont.)

C. Regression Results in Practice

- 1. Campaign Spending
  - The first analyzes the percentage of votes that incumbent congressmen received in 1984 (Dep. Var). The independent variables include:
    1. the percentage of people registered in the same party in the district,
    2. Voter approval of Reagan,
    3. their expectations about their economic future,
    4. challenger spending, and
    5. incumbent spending.
  - The estimated coefficients are shown, with the standard errors in parentheses underneath.

Confidence Intervals and Statistical Tests (cont.)

- 2. Obscenity Cases
  - The Dependent Variable is the probability that an appeals court decided "liberally" in an obscenity case.
  - The independent variables include:
    1. Whether the case came from the South (this is Region)
    2. who appointed the justice,
    3. whether the case was heard before or after the landmark 1973 *Miller* case,
    4. who the accused person was,
    5. what type of defense the defendant offered, and
    6. what type of materials were involved in the case.

V. Homework

A. Introduction

- In your homework, you are asked to add another variable to the regression that you ran for today's assignment. Then you are to find which coefficients are significant and interpret your results.

Homework (cont.)

- 1. Model

MONEY--------------------> PARTYID

GENDER

Homework (cont.)

```
* * * *  M U L T I P L E   R E G R E S S I O N   * * * *
Equation Number 1    Dependent Variable..   MYPARTY
Block Number  1.  Method:  Enter      MONEY

Variable(s) Entered on Step Number
   1..    MONEY
Multiple R              .13303
R Square                .01770
Adjusted R Square        .01697
Standard Error          2.04682
Analysis of Variance
                  DF      Sum of Squares      Mean Square
Regression          1         101.96573        101.96573
Residual         1351         5659.96036           4.18946

F =      24.33863      Signif F =  .0000
```

Homework (cont.)

```
* * * *  MULTIPLE  REGRESSION  * * * *

Equation Number 1    Dependent Variable..   MYPARTY


------------------ Variables in the Equation ------------------

Variable           B            SE B        Beta        T        Sig T

MONEY         .052492      .010640     .133028     4.933
.0000
(Constant)      2.191874      .154267                  14.208
.0000


End Block Number   1   All requested variables entered.
```

Homework (cont.)

```
* * * *  MULTIPLE REGRESSION  * * * *

Equation Number 2    Dependent Variable..   MYPARTY

Block Number  1.  Method:  Enter      MONEY    GENDER
```

Homework (cont.)

```
* * * *  M U L T I P L E   R E G R E S S I O N   * * * *

Equation Number 2    Dependent Variable..   MYPARTY
Variable(s) Entered on Step Number
   1..    GENDER
   2..    MONEY
Multiple R              .16199
R Square               .02624
Adjusted R Square   .02480
Standard Error       2.03865
Analysis of Variance
                         DF      Sum of Squares      Mean Square
Regression          2          151.18995         75.59497
Residual         1350        5610.73614           4.15610

F =      18.18892      Signif F =  .0000
```

Homework (cont.)

```
 * * * *  MULTIPLE REGRESSION  * * * *

Equation Number 2    Dependent Variable..   MYPARTY

------------------ Variables in the Equation ------------------

Variable              B          SE B         Beta          T        Sig T
GENDER          -.391620      .113794    -.093874      -3.441      .0006
MONEY            .046016      .010763     .116615       4.275      .0000
(Constant)      2.895390      .255729                  11.322      .0000
```