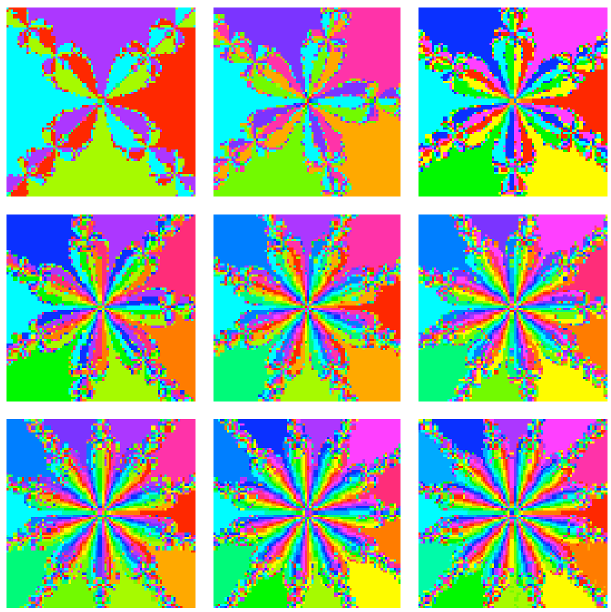
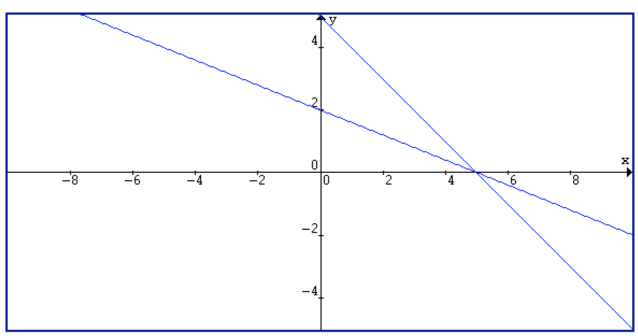
A Math Primer

A review of basic quantitative skills for the MPA Program in Environmental Science and Policy





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Algebraic Expressions

Variables

- Letters represent an unknown or generic real number
- Sometimes with restrictions, such as a member of a certain set, or the set of values that makes an equation true.
- Often a letter from the end of the alphabet: x, y, z
- Or a letter that stands for a physical quantity: d for distance, t for time, etc.

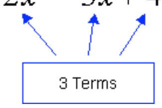
Constants

- Fixed values, like 2 or 7
- Can also be represented by letters: a, b, c, p, e, k

Terms

Terms are Separated by + or –

$$2x^2 - 3x + 4$$



Factors

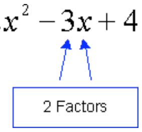
Factors are multiplied together.

Coefficients

Coefficients are constant factors that multiply a variable or powers of a variable

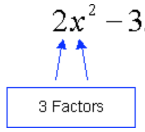
The middle term has 2 factors, –3 and x. We say that the coefficient of x is –3.

$$2x^2 - 3x + 4$$



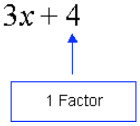
The first term has three factors, 2 and two factors of x. We say that 2 is the coefficient of x^2.

$$2x^2 - 3x + 4$$



The last term is a factor all by itself (although the number 4 could be factored into 2 x 2).

$$2x^2 - 3x + 4$$



Simplifying Algebraic Expressions

By "simplifying" an algebraic expression, we mean writing it in the most compact or efficient manner, without changing the value of the expression. This mainly involves *collecting like terms*, which means that we add together anything that can be added together. The rule here is that only *like* terms can be added together.

Like (or similar) terms

Like terms are those terms which contain the same powers of same variables. They can have different coefficients, but that is the only difference.

Examples:

$3x$, x, and $-2x$ are like terms.

$2x^2$, $-5x^2$, and are like terms.

xy^2, $3y^2x$, and $3xy^2$ are like terms.

xy^2 and x^2y are **NOT** like terms, because the same variable is not raised to the same power.

Combining Like terms

Combining like terms is permitted because of the distributive law. For example,

$$3x^2 + 5x^2 = (3 + 5)x^2 = 8x^2$$

What happened here is that the distributive law was used in reverse—we "undistributed" a common factor of x^2 from each term. The way to think about this operation is that if you have three x-squareds, and then you get five more x-squareds, you will then have eight x-squareds.

Example: $x^2 + 2x + 3x^2 + 2 + 4x + 7$

Starting with the highest power of x, we see that there are four x-squareds in all $(1x^2 + 3x^2)$. Then we collect the first powers of x, and see that there are six of them $(2x + 4x)$. The only thing left is the constants $2 + 7 = 9$. Putting this all together we get

$$x^2 + 2x + 3x^2 + 2 + 4x + 7$$
$$=$$
$$4x^2 + 6x + 9$$

Parentheses

- Parentheses must be multiplied out before collecting like terms

You cannot combine things in parentheses (or other grouping symbols) with things outside the parentheses. Think of parentheses as opaque—the stuff inside the parentheses can't "see" the stuff outside the parentheses. If there is some factor multiplying the parentheses, then the only way to get rid of the parentheses is to multiply using the distributive law.

Example:

$$3x + 2(x - 4) = 3x + 2x - 8$$
$$= 5x - 8$$

Minus Signs: Subtraction and Negatives

Subtraction can be replaced by adding the opposite

$$3x - 2 = 3x + (-2)$$

Negative signs in front of parentheses

A special case is when a minus sign appears in front of parentheses. At first glance, it looks as though there is no factor multiplying the parentheses, and you may be tempted to just remove the parentheses. What you need to remember is that the minus sign indicating subtraction should always be thought of as adding the opposite. This means that you want to add the opposite of the entire thing inside the parentheses, and so you have to change the sign of each term in the parentheses. Another way of looking at it is to imagine an implied factor of one in front of the parentheses. Then the minus sign makes that factor into a negative one, which can be multiplied by the distributive law:

$$3x - (2 - x)$$
$$= 3x + (-1)[2 + (-x)]$$
$$= 3x + (-1)(2) + (-1)(-x)$$
$$= 3x - 2 + x$$
$$= 4x - 2$$

However, if there is only a plus sign in front of the parentheses, then you can simply erase the parentheses:

$$3x + (2 - x)$$
$$= 3x + 2 - x$$

A comment about subtraction and minus signs

Although you can always explicitly replace subtraction with adding the opposite, as in this previous example, it is often tedious and inconvenient to do so. Once you get used to *thinking* that way, it is no longer necessary to actually write it that way. It is helpful to always think of minus signs as being "stuck" to the term directly to their right. That way, as you rearrange terms, collect like terms, and clear parentheses, the "adding the opposite" business will be taken care of because the minus signs will go with whatever was to their right. If what is immediately to the right of a minus sign happens to be a parenthesis, and then the minus sign attacks every term inside the parentheses.

Solutions of Algebraic Equations

Up until now, we have just been talking about manipulating algebraic expressions. Now it is time to talk about *equations*. An expression is just a statement like

$$2x + 3$$

This expression might be equal to any number, depending on the choice of x. For example, if $x = 3$ then the value of this expression is 9. But if we are writing an equation, then we are making a statement about its value. We might say

$$2x + 3 = 7$$

A mathematical equation is either true or false. This equation, $2x + 3 = 7$, might be true or it might be false; it depends on the value chosen for x. We call such equations *conditional*, because their truth depends on choosing the correct value for x. If I choose $x = 3$, then the equation is clearly false because $2(3) + 3 = 9$, not 7. In fact, it is only true if I choose $x = 2$. Any other value for x produces a false equation. We say that $x = 2$ is the *solution* of this equation.

Solutions

- The solution of an equation is the value(s) of the variable(s) that make the equation a true statement.

An equation like $2x + 3 = 7$ is a simple type called a linear equation in one variable. These will always have one solution, no solutions, or an infinite number of solutions. There are other types of equations, however, that can have several solutions. For example, the equation

$$x^2 = 9$$

is satisfied by both 3 and –3, and so it has two solutions.

One Solution
This is the normal case, as in our example where the equation $2x + 3 = 7$ had exactly one solution, namely $x = 2$. The other two cases, no solution and an infinite number of solutions, are the oddball cases that you don't expect to run into very often. Nevertheless, it is important to know that they can happen in case you do encounter one of these situations.

Infinite Number of Solutions
Consider the equation

$$x = x$$

This equation is obviously true for every possible value of x. This is, of course, a ridiculously simple example, but it makes the point. Equations that have this property are called *identities*. Some examples of identities would be

$$2x = x + x$$

$$3 = 3$$

$$(x - 2)(x + 2) = x^2 - 4$$

All of these equations are true for any value of x. The second example, 3 = 3, is interesting because it does not even contain an x, so obviously its truthfulness cannot depend on the value of x! When you are attempting to solve an equation algebraically and you end up with an obvious identity (like 3 = 3), then you know that the original equation must also be an identity, and therefore it has an infinite number of solutions.

No Solutions
Now consider the equation

$$x + 4 = x + 3$$

There is no possible value for x that could make this true. If you take a number and add 4 to it, it will never be the same as if you take the same number and add 3 to it. Such an equation is called a *contradiction*, because it cannot ever be true.

If you are attempting to solve such an equation, you will end up with an extremely obvious contradiction such as 1 = 2. This indicates that the original equation is a contradiction, and has no solution.

In summary,

- An *identity* is always true, no matter what x is
- A *contradiction* is never true for any value of x
- A *conditional equation* is true for some values of x

Addition Principle

Equivalent Equations
The basic approach to finding the solution to equations is to change the equation into simpler equations, but in such a way that the solution set of the new equation is the same as the solution set of the original equation. When two equations have the same solution set, we say that they are equivalent.

What we want to do when we solve an equation is to produce an equivalent equation that tells us the solution directly. Going back to our previous example,

$$2x + 3 = 7$$

we can say that the equation

$$x = 2$$

is an equivalent equation, because they both have the same solution, namely $x = 2$. We need to have some way to convert an equation like $2x + 3 = 7$ into an equivalent equation like $x = 2$ that tells us the solution. We solve equations by using methods that rearrange the equation in a manner that does not change the solution set, with a goal of getting the variable by itself on one side of the equal sign. Then the solution is just the number that appears on the other side of the equal sign.

The methods of changing an equation without changing its solution set are based on the idea that if you change both sides of an equation in the same way, then the equality is preserved. Think of an equation as a balance—whatever complicated expression might appear on either side of the equation, they are really just numbers. The equal sign is just saying that the value of the expression on the left side is the same number as the value on the right side. Therefore, no matter how horrible the equation may seem, it is really just saying something like 3 = 3.

The Addition Principle

Adding (or subtracting) the same number to both sides of an equation does not change its solution set.

Think of the balance analogy—if both sides of the equation are equal, then increasing both sides by the same amount will change the value of each side, but they will still be equal. For example, if

$$3 = 3,$$

then

$$3 + 2 = 3 + 2.$$

Consequently, if

$$6 + x = 8$$

for some value of x (which in this case is $x = 2$), then we can add any number to both sides of the equation and $x = 2$ will still be the solution. If we wanted to, we could add a 3 to both sides of the equation, producing the equation

$$9 + x = 11.$$

As you can see, $x = 2$ is still the solution. Of course, this new equation is no simpler than the one we started with, and this maneuver did not help us solve the equation.
If we want to solve the equation

$$6 + x = 8,$$

the idea is to get x by itself on one side, and so we want to get rid of the 6 that is on the left side. We can do this by subtracting a 6 from both sides of the equation (which of course can be thought of as adding a negative six):

$$6 - 6 + x = 8 - 6$$

or

$$x = 2$$

You can think of this operation as moving the 6 from one side of the equation to the other, which causes it to change sign

The addition principle is useful in solving equations because it allows us to move whole terms from one side of the equal sign to the other. While this is a convenient way to think of it, you should remember that you are not really "moving" the term from one side to the other—you are really adding (or subtracting) the term on both sides of the equation.

Notations

In the previous example, we wrote the –6 in-line with the rest of the equation. This is analogous to writing an arithmetic subtraction problem in one line, as in

$$234 - 56 = 178.$$

You probably also learned to write subtraction and addition problems in a column format, like

$$\begin{array}{r} 234 \\ -\ 56 \\ \hline 178 \end{array}$$

We can also use a similar notation for the addition method with algebraic equations.
Given the equation

$$x + 3 = 2,$$

we want to subtract a 3 from both sides in order to isolate the variable. In column format this would look like

$$\begin{array}{rcl} x + 3 &=& 2 \\ -3 &=& -3 \\ \hline x &=& -1 \end{array}$$

Here the numbers in the second row are negative 3's, so we are *adding* the two rows together to produce the bottom row.

The advantage of the column notation is that it makes the operation easier to see and reduces the chances for an error. The disadvantage is that it takes more space, but that is a relatively minor disadvantage. Which notation you prefer to use is not important, as long as you can follow what you are doing and it makes sense to you.

Multiplication Principle

Multiplying (or dividing) the same non-zero number to both sides of an equation does not change its solution set.

Example:

$$6 \times 2 = 12$$
$$3 \times 6 \times 2 = 3 \times 12$$

so if $6x = 12$, then $18x = 36$ for the same value of x (which in this case is $x = 2$).

The way we use the multiplication principle to solve equations is that it allows us to isolate the variable by getting rid of a factor that is multiplying the variable.

Example: $2x = 6$

To get rid of the 2 that is multiplying the *x*, we can divide both sides of the equation by 2, or multiply by its reciprocal (one-half).
Either divide both sides by 2:

$$2x = 6$$
$$\frac{2x}{2} = \frac{6}{2}$$
$$x = 3$$

or multiply both sides by a half:

$$2x = 6$$
$$\left(\tfrac{1}{2}\right)2x = \left(\tfrac{1}{2}\right)6$$
$$x = 3$$

- Whether you prefer to think of it as dividing by the number or multiplying by its reciprocal is not important, although when the coefficient is a fraction it is easier to multiply by the reciprocal:

Example: $4/5x = 8$

Multiply both sides by the reciprocal of the coefficient, or 5/4

$$\frac{5}{4}\cdot\frac{4}{5}x = \frac{5}{4}\cdot 8$$
$$x = 10$$

Using the Principles Together

Suppose you were given an equation like

$$2x - 3 = 5.$$

You will need to use the addition principle to move the –3, and the multiplication principle to remove the coefficient 2. Which one should you use first? Strictly speaking, it does not matter—you will eventually get the right answer. In practice, however, it is usually simpler to use the addition principle first, and then the multiplication principle. The reason for this is that if we divide by 2 first we will turn everything into fractions:

Given: $2x - 3 = 5$

Suppose we first divide both sides by 2:

$$\frac{2x-3}{2} = \frac{5}{2}$$
$$\frac{2x}{2} - \frac{3}{2} = \frac{5}{2}$$
$$x - \frac{3}{2} = \frac{5}{2}$$

Now there is nothing wrong with doing arithmetic with fractions, but it is not as simple as working with whole numbers. In this example we would have to add 3/2 to both sides of the equation to isolate the *x*. It is usually more convenient, though, to use the addition principle first:

Given: $2x - 3 = 5$

Add 3 to both sides:

$$\begin{array}{r} 2x - 3 = 5 \\ 3 = 3 \\ \hline 2x = 8 \end{array}$$

At this point all we need to do is divide both sides by 2 to get $x = 4$.

Word Problems

Problem Solving Strategies

Understand

1. Read the problem carefully.
2. Make sure you understand the situation that is described.
3. Make sure you understand what information is provided, and what the question is asking.
4. For many problems, drawing a clearly labeled picture is very helpful.

Plan

1. First focus on the objective. What do you need to know in order to answer the question?
2. Then look at the given information. How can you use that information to get what you need to know to answer the question?
3. If you do not see a clear logical path leading from the given information to the solution, just try *something*. Look at the given information and think about what you can find from it, even if it is not what the question is asking for. Often you will find another piece of information that you can then use to answer the question.

Write equations

You need to express mathematically the logical connections between the given information and the answer you are seeking. This involves:

1. Assigning variable names to the unknown quantities. The letter x is always popular, but it is a good idea to use something that reminds you what it represents, such as d for distance or t for time. The trickiest part of assigning variables is that you want to use a minimum number of different variables (just one if possible). If you know how two quantities are related, then you can express them both with just one variable. For example, if Jim is two years older than John is, you might let x stand for John's age and $(x + 2)$ stand for Jim's age.
2. Translate English into Math. Mathematics is a language, one that is particularly well suited to describing logical relationships. English, on the other hand, is much less precise.

Solve

Now you just have to solve the equation(s) for the unknown(s). Remember to answer the question that the problem asks.

Check!

Think about your answer. Does your answer come out in the correct units? Is it reasonable? If you made a mistake somewhere, chances are your answer will not just be a little bit off, but will be completely ridiculous

General Word Problems

General Strategy

Recall the general strategy for setting up word problems. Refer to the Problem Solving Strategies page for more detail.

1. Read the problem carefully: Determine what is known, what is unknown, and what question is being asked.
2. Represent unknown quantities in terms of a variable.
3. Use diagrams where appropriate.
4. Find formulas or mathematical relationships between the knowns and the unknowns.
5. Solve the equations for the unknowns.
6. Check answers to see if they are reasonable.

Number Problems

Example: Find a number such that 5 more than one-half the number is three times the number.
Let x be the unknown number.

Translating into math: $5 + x/2 = 3x$

Solving:
(First multiply by 2 to clear the fraction)

$$5 + x/2 = 3x$$

$$10 + x = 6x$$

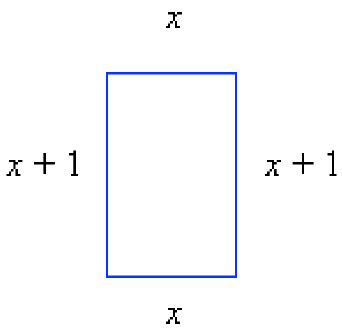
$$10 = 5x$$

$$x = 2$$

Geometry Problems

Example: If the perimeter of a rectangle is 10 inches, and one side is one inch longer than the other, how long are the sides?

Let one side be x and the other side be $x + 1$.



Then the given condition may be expressed as

$$x + x + (x + 1) + (x + 1) = 10$$

Solving:

$$4x + 2 = 10$$
$$4x = 8$$
$$x = 2$$

so the sides have length **2 and 3**.

Rate-Time Problems

Rate = Quantity/Time

or

Quantity = Rate x Time

Example 1: A fast employee can assemble 7 radios in an hour, and another slower employee can only assemble 5 radios per hour. If both employees work together, how long will it take to assemble 26 radios?

The two together will build 7 + 5 = 12 radios in an hour, so their combined rate is 12 radios/hr.

Using Time = Quantity/Rate, Time = 26/12 = 2 1/6 h

or

2 hours 10 minutes

Example 2: you are driving along at 55 mph when you are passed by a car doing 85 mph. How long will it take for the car that passed you to be one mile ahead of you?

We know the two rates, and we know that the difference between the two distances traveled will be one mile, but we don't know the actual distances. Let D be the distance that you travel in time t, and $D + 1$ be the distance that the other car traveled in time t.

Using the rate equation in the form *distance = speed • time* for each car we can write

$$D = 55\,t, \text{ and } D + 1 = 85\,t$$

Substituting the first equation into the second,

$$55t + 1 = 85t$$

$$-30t = -1$$

$t = 1/30$ hr(or **2 minutes**)

Mixture Problems

Example: How much of a 10% vinegar solution should be added to 2 cups of a 30% vinegar solution to make a 20% solution?

Let x be the unknown volume of 10% solution. Write an equation for the volume of vinegar in each mixture:

(amount of vinegar in first solution) + (amount of vinegar in second solution) = (amount of vinegar in total solution)

$$0.1x + 0.3(2) = 0.2(x + 2)$$

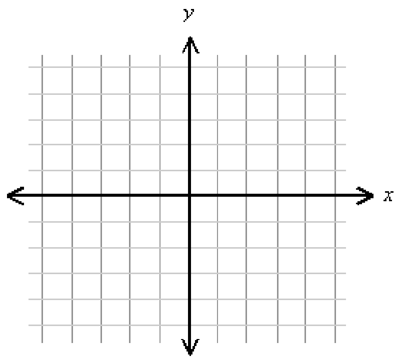
$$0.1x + 0.6 = 0.2x + 0.4$$

$$-0.1x = -0.2$$

x = **2 cups**

Graphing and straight Lines

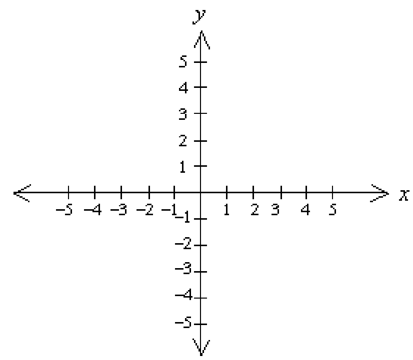
A. Rectangular Coordinates



The rectangular coordinate system is also known as the *Cartesian* coordinate system after Rene Descartes, who popularized its use in analytic geometry. The rectangular coordinate system is based on a grid, and every point on the plane can be identified by unique *x* and *y* coordinates, just as any point on the Earth can be identified by giving its latitude and longitude.

Axes

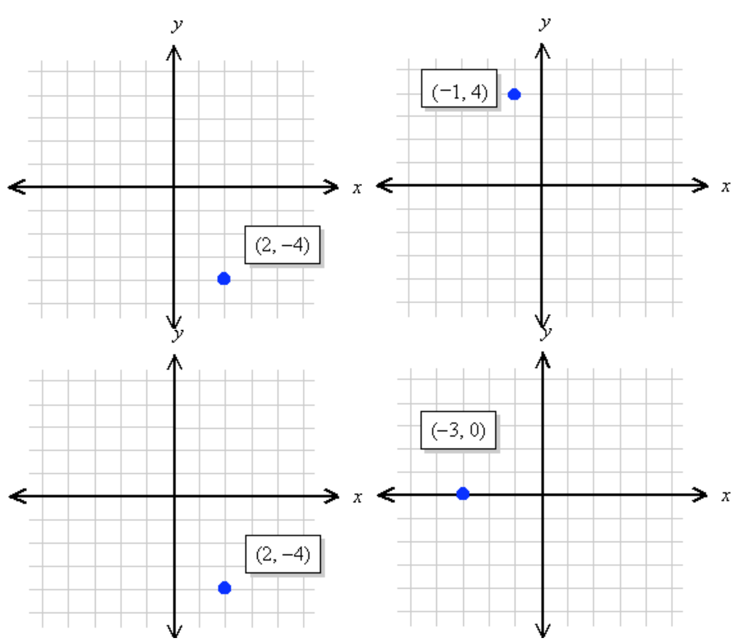
Locations on the grid are measured relative to a fixed point, called the *origin*, and are measured according to the distance along a pair of axes. The *x* and *y* axes are just like the number line, with positive distances to the right and negative to the left in the case of the *x* axis, and positive distances measured upwards and negative down for the *y* axis. Any displacement away from the origin can be constructed by moving a specified distance in the *x* direction and then another distance in the *y* direction. Think of it as if you were giving directions to someone by saying something like "go three blocks East and then 2 blocks North."



Coordinates, Graphing Points

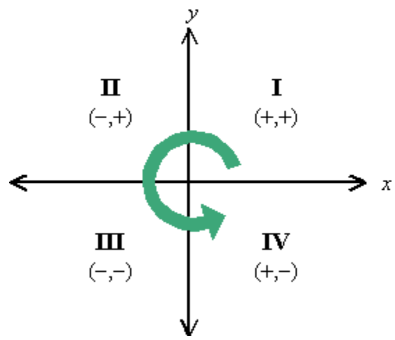
We specify the location of a point by first giving its *x* coordinate (the left or right displacement from the origin), and then the *y* coordinate (the up or down displacement from the origin). Thus, every point on the plane can be identified by a pair of numbers (x, y), called its *coordinates*.

Examples:



Quadrants

Sometimes we just want to know what general part of the graph we are talking about. The axes naturally divide the plane up into quarters. We call these *quadrants*, and number them from one to four. Notice that the numbering begins in the upper right quadrant and continues around in the counter-clockwise direction. Notice also that each quadrant can be identified by the unique combination of positive and negative signs for the coordinates of a point in that quadrant.



B. Graphing Functions

Consider an equation such as

$$y = 2x - 1$$

We say that y is a *function* of x because if you choose any value for x, this formula will give you a unique value of y. For example, if we choose $x = 3$ then the formula gives us

$$y = 2(3) - 1$$

or

$$y = 5$$

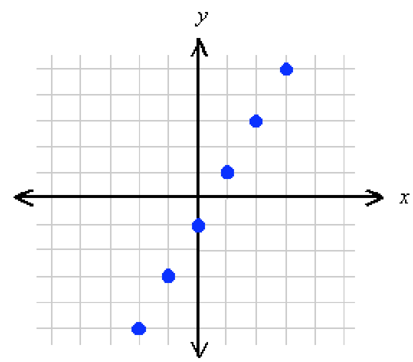
Thus we can say that the value $y = 5$ is generated by the choice of $x = 3$. Had we chosen a different value for x, we would have gotten a different value for y. In fact, we can choose a whole bunch of different values for x and get a y value for each one. This is best shown in a table:

x (input)	*x – Formula - y*	*y (output)*
–2	2(–2) – 1 = –5	–5
–1	2(–1) – 1 = –3	–3
0	2(0) – 1 = –1	–1
1	2(1) – 1 = 1	1
2	2(2) – 1 = 3	3
3	2(3) – 1 = 5	5

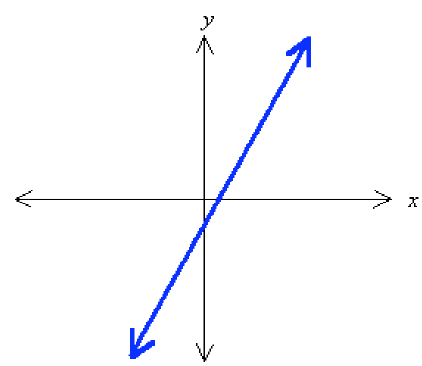
This relationship between x and its corresponding y values, produces a collection of pairs of points (x, y), namely

$$(-2, -5)$$
$$(-1, -3)$$
$$(0, -1)$$
$$(1, 1)$$
$$(2, 3)$$
$$(3, 5)$$

Since each of these pairs of numbers can be the coordinates of a point on the plane, it is natural to ask what this collection of ordered pairs would look like if we graphed them. The result is something like this:



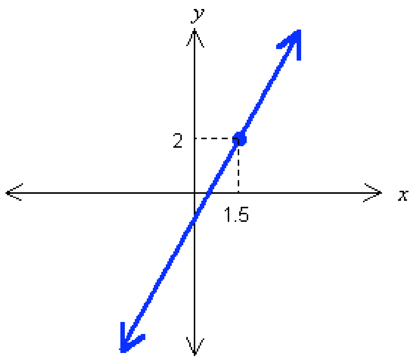
The points seem to fall in a straight line. Now, our choices for x were quite arbitrary. We could just as well have picked other values, including non-integer values. Suppose we picked many more values for x, like 2.7, 3.14, etc. and added them to our graph. Eventually the points would be so crowded together that they would form a solid line:



The arrows on the ends of the line indicate that it goes on forever, because there is no limit to what numbers we could choose for x. We say that this line is the *graph* of the function $y = 2x - 1$.

If you pick any point on this line and read off its x and y coordinates, they will satisfy the equation

$y = 2x - 1$. For example, the point (1.5, 2) is on the line:



and the coordinates $x = 1.5$, $y = 2$ satisfy the equation $y = 2x - 1$:

$$2 = 2(1.5) - 1$$

Note: This graph turned out to be a straight line only because of the particular function that we used as an example. There are many other functions whose graphs turn out to be various curves.

C. Straight Lines

Linear Equations in Two Variables

The equation $y = 2x - 1$ that we used as an example for graphing functions produced a graph that was a straight line. This was no accident. This equation is one example of a general class of equations that we call *linear equations in two variables*. The two variables are usually (but of course don't have to be) x and y. The equations are called *linear* because their graphs are straight lines. Linear equations are easy to recognize because they obey the following rules:

1. The variables (usually x and y) appear only to the first power
2. The variables may be multiplied only by real number constants
3. Any real number term may be added (or subtracted, of course)
4. Nothing else is permitted!

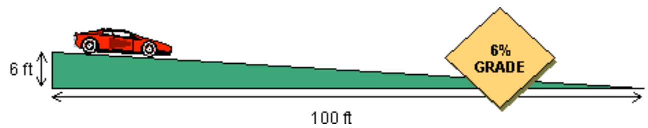
* This means that any equation containing things like x^2, y^2, $1/x$, xy, square roots, or any other function of x or y is not linear.

Describing Lines

Just as there are an infinite number of equations that satisfy the above conditions, there are also an infinite number of straight lines that we can draw on a graph. To describe a particular line we need to specify two distinct pieces of information concerning that line. A specific straight line can be determined by specifying two distinct points that the line passes through, or it can be determined by giving one point that it passes through and somehow describing how "tilted" the line is.

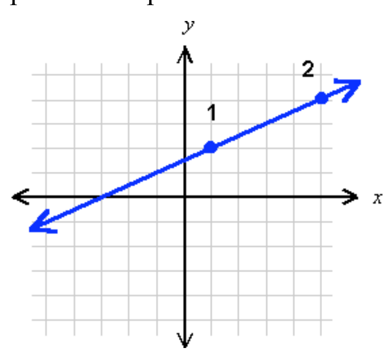
Slope

The *slope* of a line is a measure of how "tilted" the line is. A highway sign might say something like "6% grade ahead." What does this mean, other than that you hope your brakes work? What it means is that the ratio of your drop in altitude to your horizontal distance is 6%, or 6/100. In other words, if you move 100 feet forward, you will drop 6 feet; if you move 200 feet forward, you will drop 12 feet, and so on.

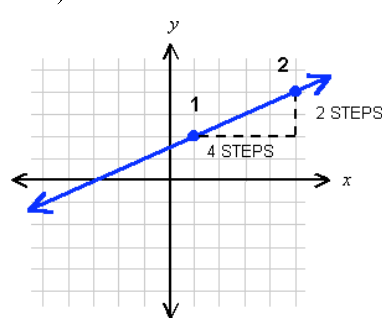


We measure the slope of lines in much the same way, although we do not convert the result to a percent.

Suppose we have a graph of an unknown straight line. Pick any two different points on the line and label them point 1 and point 2:



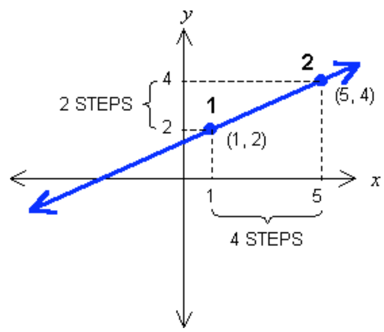
In moving from point 1 to point 2, we cover 4 steps horizontally (the x direction) and 2 steps vertically (the y direction):



Therefore, the ratio of the change in altitude to the change in horizontal distance is 2 to 4. Expressing it as a fraction and reducing, we say that the slope of this line is

$$\frac{2}{4} = \frac{1}{2}$$

To formalize this procedure a bit, we need to think about the two points in terms of their *x* and *y* coordinates.



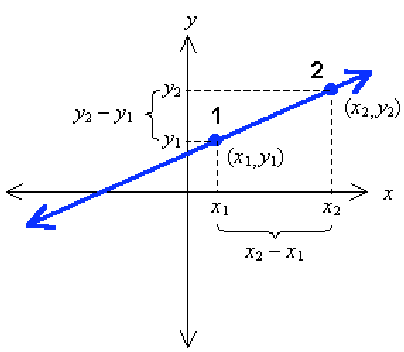
Now you should be able to see that the horizontal displacement is the difference between the *x* coordinates of the two points, or

$$4 = 5 - 1,$$

and the vertical displacement is the difference between the *y* coordinates, or

$$2 = 4 - 2.$$

In general, if we say that the coordinates of point 1 are ($x1$, $y1$) and the coordinates of point 2 are ($x2$, $y2$),

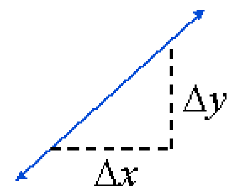


then we can define the slope *m* as follows:

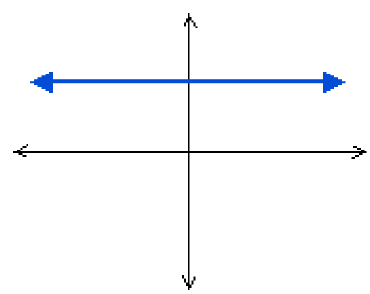
$$m = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1}$$

where ($x1$, $y1$) and ($x2$, $y2$) are any two distinct points on the line.

- It is customary (in the US) to use the letter *m* to represent slope. No one knows why.
- It makes *no difference* which two points are used for point 1 and point 2. If they were switched, both the numerator and the denominator of the fraction would be changed to the opposite sign, giving exactly the same result.
- Many people find it useful to remember this formula as "slope is rise over run."
- Another common notation is $m = \Delta y/\Delta x$, where the Greek letter delta (Δ) means "the change in." The slope is a *ratio* of how much *y* changes per change in *x*:



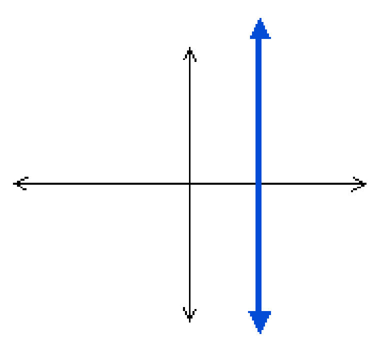
Horizontal Lines



A horizontal line has zero slope, because there is no change in *y* as *x* increases. Thus, any two points will have the same *y* coordinates, and since $y1 = y2$,

$$m = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0}{x_2 - x_1} = 0$$

Vertical Lines



A vertical line presents a different problem. If you look at the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

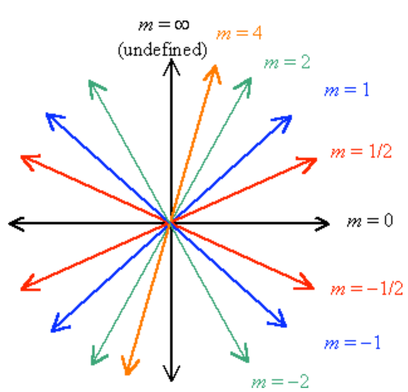
you see that there is a problem with the denominator. It is not possible to get two different values for $x1$ and $x2$, because if x changes then you are not on the vertical line anymore. Any two points on a vertical line will have the *same* x coordinates, and so $x2 - x1 = 0$. Since the denominator of a fraction cannot be zero, we have to say that **a vertical line has undefined slope**. Do not confuse this with the case of the horizontal line, which has a well-defined slope that just happens to equal zero.

Positive and Negative Slope

The x coordinate increases to the right, so moving from left to right is motion in the positive x direction. Suppose that you are going uphill as you move in the positive x direction. Then both your x and y coordinates are increasing, so the ratio of rise over run will be positive—you will have a positive increase in y for a positive increase in x. On the other hand, if you are going downhill as you move from left to right, then the ratio of rise over run will be negative because you *lose* height for a given positive increase in x. The thing to remember is:
As you go from left to right,

- Uphill = Positive Slope
- Downhill = Negative Slope

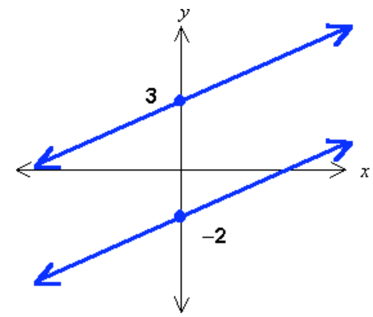
And of course, no change in height means that the line has zero slope.



Some Slopes

Intercepts

Two lines can have the same slope and be in different places on the graph. This means that in addition to describing the slope of a line we need some way to specify exactly where the line is on the graph. This can be accomplished by specifying one particular point that the line passes through. Although any point will do, it is conventional to specify the point where the line crosses the y-axis. This point is called the *y-intercept*, and is usually denoted by the letter b. Note that every line except vertical lines will cross the y-axis at some point, and we have to handle vertical lines as a special case anyway because we cannot define a slope for them.



Same Slopes, Different y-Intercepts

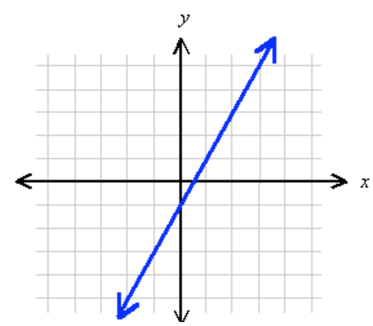
Equations

The equation of a line gives the mathematical relationship between the x and y coordinates of any point on the line.

Let's return to the example we used in graphing functions. The equation

$$y = 2x - 1$$

produces the following graph:



This line evidently has a slope of 2 and a y intercept equal to –1. The numbers 2 and –1 also appear in the equation—the coefficient of x is 2, and the additive constant is –1. This is not a coincidence, but is due to the standard form in which the equation was written.

Standard Form (Slope-Intercept Form)
If a linear equation in two unknowns is written in the form

$$y = mx + b$$

where m and b are any two real numbers, then the graph will be a straight line with a slope of m and a y intercept equal to b.

Point-Slope Form

As mentioned earlier, a line is fully described by giving its slope and one distinct point that the line passes through. While this point is customarily the y intercept, it does not need to be. If you want to describe a line with a given slope m that passes through a given point ($x1$, $y1$), the formula is

$$y - y_1 = m(x - x_1)$$

To help remember this formula, think of solving it for m:

$$m = \frac{y - y_1}{x - x_1}$$

Since the point (x, y) is an arbitrary point on the line and the point ($x1$, $y1$) is another point on the line, this is nothing more than the definition of slope for that line.

Two-Point Form
Another way to completely specify a line is to give two different points that the line passes through. If you are given that the line passes through the points ($x1$, $y1$) and ($x2$, $y2$), the formula is

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

This formula is also easy to remember if you notice that it is just the same as the point-slope form with the slope m replaced by the definition of slope,

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Systems of Linear Equations

A. The Solutions of a System of Equations

A system of equations refers to a number of equations with an equal number of variables. We will only look at the case of two linear equations in two unknowns. The situation gets much more complex as the number of unknowns increases, and larger systems are commonly attacked with the aid of a computer.

A system of two linear equations in two unknowns might look like

$$\begin{cases} 2x + 4y = 3 \\ x - 3y = 1 \end{cases}$$

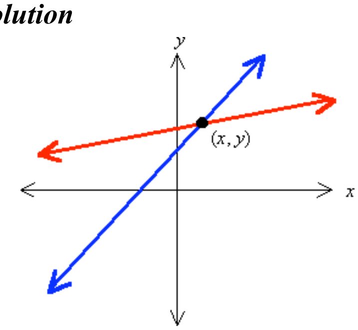
This is the standard form for writing equations when they are part of a system of equations: the variables go in order on the left side and the constant term is on the right. The bracket on the left indicates that the two equations are intended to be solved simultaneously, but it is not always used.

When we talk about the *solution* of this system of equations, we mean the values of the variables that make both equations true at the same time. There may be many pairs of x and y that make the first equation true, and many pairs of x and y that make the second equation true, but we are looking for an x and y that would work in *both* equations. In the following pages we will look at algebraic methods for finding this solution, if it exists.

Because these are linear equations, their graphs will be straight lines. This can help us visualize the situation graphically. There are three possibilities:

1. Independent Equations

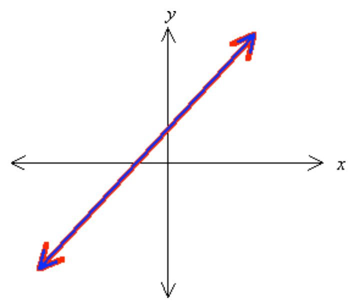
- ***Lines intersect***
- ***One solution***



In this case the two equations describe lines that intersect at one particular point. Clearly this point is on both lines, and therefore its coordinates (x, y) will satisfy the equation of either line. Thus the pair (x, y) is the one and only solution to the system of equations.

2. Dependent Equations

- ***Equations describe the same line***
- ***Infinite number of solutions***



Sometimes two equations might look different but actually describe the same line. For example, in

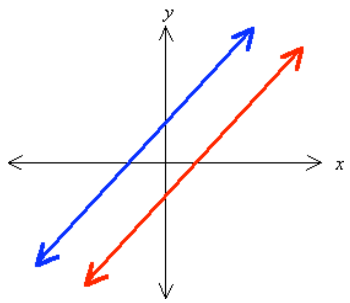
The second equation is just two times the first equation, so they are actually equivalent and would both be equations of the same line. Because the two equations describe the same line, they have *all* their points in common; hence there are an infinite number of solutions to the system.

- ***Attempting to solve gives an identity***

If you try to solve a dependent system by algebraic methods, you will eventually run into an equation that is an *identity*. An identity is an equation that is always true, independent of the value(s) of any variable(s). For example, you might get an equation that looks like $x = x$, or $3 = 3$. This would tell you that the system is a dependent system, and you could stop right there because you will never find a unique solution.

3. Inconsistent Equations

- ***Lines do not intersect (Parallel Lines; have the same slope)***
- ***No solutions***



If two lines happen to have the same slope, but are not identically the same line, then they will never intersect. There is no pair (x, y) that could satisfy both equations, because there is no point (x, y) that is simultaneously on both lines. Thus these equations are said to be *inconsistent*, and there is no solution. The fact that they both have the same slope may not be obvious from the equations, because they are not written in one of the standard forms for straight lines. The slope is not readily evident in the form we use for writing systems of equations. (If you think about it you will see that the slope is the negative of the coefficient of x divided by the coefficient of y).

- ***Attempting to solve gives a false statement***

By attempting to solve such a system of equations algebraically, you are operating on a false assumption—namely that a solution exists. This will eventually lead you to a *contradiction*: a statement that is obviously false, regardless of the value(s) of the variable(s). At some point in your work you would get an obviously false equation like $3=4$. This would tell you that the system of equations is inconsistent, and there is no solution.□

Solution by Graphing

For more complex systems, and especially those that contain non-linear equations, finding a solution by algebraic methods can be very difficult or even impossible. Using a graphing calculator (or a computer), you can graph the equations and actually see where they intersect. The calculator can then give you the coordinates of the intersection point. The only drawback to this method is that the solution is only an approximation, whereas the algebraic method gives the exact solution. In most practical situations, though, the precision of the calculator is sufficient. For more demanding scientific and engineering applications there are computer methods that can find approximate solutions to very high precision.

B. Addition Method

The whole problem with solving a system of equations is that you cannot solve an equation that has two unknowns in it. You need an equation with only one variable so that you can isolate the variable on one side of the equation. Both methods that we will look at are techniques for eliminating one of the variables to give you an equation in just one unknown, which you can then solve by the usual methods.

The first method of solving systems of linear equations is the addition method, in which the two equations are added together to eliminate one of the variables.

Adding the equations means that we add the left sides of the two equations together, and we add the right sides together. This is legal because of the Addition Principle, which says that we can add the same amount to both sides of an equation. Since the left and right sides of any equation are equal to each other, we are indeed adding the same amount to both sides of an equation.

Consider this simple example:

Example:

$$\begin{cases} 3x+2y=4 \\ 2x-2y=1 \end{cases}$$

If we add these equations together, the terms containing y will add up to zero ($2y$ plus $-2y$), and we will get

$$\begin{array}{r} 3x+2y=4 \\ \underline{2x-2y=1} \\ 5x+\ \ 0=5 \end{array}$$

or

$$5x = 5$$

$$\boldsymbol{x = 1}$$

However, we are not finished yet—we know x, but we still don't know y. We can solve for y by substituting the now known value for x into either of our original equations. This will produce an equation that can be solved for y:

$$3x+2y=4$$
$$3(1)+2y=4$$
$$3+2y=4$$
$$2y=1$$
$$y=\frac{1}{2}$$

Now that we know both *x* and *y*, we can say that the solution to the system is the pair (1, 1/2).

This last example was easy to see because of the fortunate presence of both a positive and a negative 2*y*. One is not always this lucky. Consider

Example:

$$\begin{cases} x+2y=3 \\ 3x+4y=2 \end{cases}$$

Now there is nothing so obvious, but there is still something we can do. If we multiply the first equation by -3, we get

$$\begin{cases} -3x-6y=-9 \\ 3x+4y=2 \end{cases}$$

(Don't forget to multiply every term in the equation, on both sides of the equal sign). Now if we add them together the terms containing *x* will cancel:

$$\begin{array}{r} -3x-6y=-9 \\ \underline{3x+4y=\ \ 2} \\ -2y=-7 \end{array}$$

or

$$y=\frac{7}{2}$$

As in the previous example, now that we know *y* we can solve for *x* by substituting into either original equation. The first equation looks like the easiest to solve for *x*, so we will use it:

$$x+2y=3$$
$$x+2\left(\frac{7}{2}\right)=3$$
$$x+7=3$$
$$x=-4$$

And so the solution point is (-4, 7/2).

Now we look at an even less obvious example:

Example:

$$\begin{cases} 5x-2y=6 \\ 2x+3y=10 \end{cases}$$

Here there is nothing particularly attractive about going after either the *x* or the *y*. In either case, both equations will have to be multiplied by some factor to arrive at a common coefficient. This is very much like the situation you face trying to find a least common denominator for adding fractions, except that here we call it a Least Common Multiple (LCM). As a general rule, it is easiest to eliminate the variable with the smallest LCM. In this case that would be the *y*, because the LCM of 2 and 3 is 6. If we wanted to eliminate the *x* we would have to use an LCM of 10 (5 times 2). So, we choose to make the coefficients of *y* into plus and minus 6. To do this, the first equation must be multiplied by 3, and the second equation by 2:

$$(3)5x-(3)2y=(3)6$$
$$(2)2x+(2)3y=(2)10$$

or

$$\begin{array}{r} 15x-6y=18 \\ \underline{4x+6y=20} \\ 19x\qquad=38 \end{array}$$

Now adding these two together will eliminate the terms containing *y*:

or

$$x=2$$

We still need to substitute this value into one of the original equation to solve for *y*:

$$2x+3y=10$$
$$2(2)+3y=10$$
$$4+3y=10$$
$$3y=6$$
$$y=2$$

Thus the solution is the point (2, 2).

C. Substitution Method

When we used the Addition Method to solve a system of equations, we still had to do a substitution to solve for the remaining variable. With the substitution method, we solve one of the equations for one variable in terms of the other, and then substitute that into the other equation. This makes more sense with an example:

Example:

$$2y + x = 3 \qquad (1)$$

$$4y - 3x = 1 \qquad (2)$$

Equation 1 looks like it would be easy to solve for x, so we take it and isolate x:

$$2y + x = 3$$

$$x = 3 - 2y \qquad (3)$$

Now we can use this result and substitute 3 - 2y in for x in equation 2:

$$\begin{aligned} 4y - 3x &= 1 \\ 4y - 3(3 - 2y) &= 1 \\ 4y - 9 + 6y &= 1 \\ 10y - 9 &= 1 \\ 10y &= 10 \\ y &= 1 \end{aligned}$$

Now that we have y, we still need to substitute back in to get x. We could substitute back into any of the previous equations, but notice that equation 3 is already conveniently solved for x:

$$\begin{aligned} x &= 3 - 2y \\ x &= 3 - 2(1) \\ x &= 3 - 2 \\ x &= 1 \end{aligned}$$

And so the solution is (1, 1).

As a rule, the substitution method is easier and quicker than the addition method when one of the equations is very simple and can readily be solved for one of the variables.

Exponents and Roots

A. *Exponents*

Definition

In x^n, x is the *base*, and n is the *exponent* (or *power*)

We defined positive integer powers by

$$x^n = x \cdot x \cdot x \cdot \ldots \cdot x \text{ (} n \text{ factors of } x\text{)}$$

Properties

The above definition can be extended by requiring other powers (i.e. other than positive integers) to behave like the positive integer powers. For example, we know that

$$x^n \cdot x^m = x^{n+m}$$

for positive integer powers, because we can write out the multiplication.

Example:

$$x^2 x^5 = (x \cdot x)(x \cdot x \cdot x \cdot x \cdot x) = x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x = x^7$$

We now require that this rule hold even if n and m are not positive integers, although this means that we can no longer write out the multiplication (How do you multiply something by itself a negative number of times? Or a fractional number of times?).

We can find several new properties of exponents by similarly considering the rule for dividing powers:

$$\frac{x^m}{x^n} = x^{m-n}$$

This rule is quite reasonable when m and n are positive integers and $m > n$. For example:

$$\frac{x^5}{x^2} = \frac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x} = \frac{x \cdot x \cdot x}{1} = x^3$$

where indeed 5 – 2 = 3.

However, in other cases it leads to situations where we have to define new properties for exponents. First, suppose that $m < n$. We can simplify it by canceling like factors as before:

$$\frac{x^2}{x^5} = \frac{x \cdot x}{x \cdot x \cdot x \cdot x \cdot x} = \frac{1}{x \cdot x \cdot x} = \frac{1}{x^3}$$

But following our rule would give

$$\frac{x^2}{x^5} = x^{2-5} = x^{-3}$$

In order for these two results to be consistent, it must be true that

$$\frac{1}{x^3} = x^{-3}$$

or, in general,

$$x^{-n} = \frac{1}{x^n}$$

- Notice that a minus sign in the exponent does not make the result negative—instead, it makes it the *reciprocal* of the result with the positive exponent.

Now suppose that $n = m$. The fraction becomes

$$\frac{x^n}{x^n}$$

which is obviously equal to 1. But our rule gives

$$\frac{x^n}{x^n} = x^{n-n} = x^0$$

Again, in order to remain consistent we have to say that these two results are equal, and so we define

$$x^0 = 1$$

for all values of x (except $x = 0$, because 0^0 is undefined)

Summary of Exponent Rules

The following properties hold for all real numbers x, y, n, and m, with these exceptions:

1. 0^0 is undefined

2. Dividing by zero is undefined

3. Raising negative numbers to fractional powers can be undefined

$$x^1 = x \qquad \frac{x^m}{x^n} = x^{m-n}$$

$$x^0 = 1 \qquad x^{-n} = \frac{1}{x^n}$$

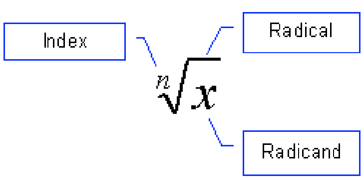
$$x^n \cdot x^m = x^{n+m} \qquad (xy)^n = x^n y^n$$

$$(x^n)^m = x^{nm} \qquad \left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$

B. Roots

Definition

Roots are the inverse of exponents. An *n*th root "undoes" raising a number to the *n*th power, and vice-versa. (The correct terminology for these types of relationships is *inverse functions*, but powers and roots can only be strictly classified as inverse functions if we take care of some ambiguities associated with plus or minus signs, so we will not worry about this yet). The common example is the *square root*, which "undoes" the act of squaring. For example, take 3 and square it to get 9. Now take the square root of 9 and get 3 again. It is also possible to have roots related to powers other than the square. The cube root, for example, is the inverse of raising to the power of 3. The cube root of 8 is 2 because $2^3 = 8$. In general, the *n*th root of a number is written:



$$\sqrt[n]{x} = y \text{ if and only if } y^n = x$$

$$\sqrt[3]{64} = 4 \text{ because } 4^3 = 64$$

We leave the index off the square root symbol only because it is the most common one. It is understood that if no index is shown, then the index is 2.

$$\sqrt{x} = y \text{ if and only if } y^2 = x$$

$$\sqrt{16} = 4 \text{ because } 4^2 = 16$$

Square Roots

The square root is the inverse function of squaring (strictly speaking only for positive numbers, because sign information can be lost)

Principal Root

- Every positive number has two square roots, one positive and one negative

Example: 2 is a square root of 4 because 2 x 2 = 4, but –2 is also a square root of 4 because (–2) x (–2) = 4

To avoid confusion between the two we **define** the $\sqrt{}$ symbol (this symbol is called a *radical*) to mean the **principal** or **positive** square root.

The convention is: For any positive number *x*,

$\sqrt{x}$ is the positive root, and

$-\sqrt{x}$ is the negative root.

If you mean the negative root, use a minus sign in front of the radical.

Example:

$$\sqrt{25} = 5$$

$$-\sqrt{25} = -5$$

Properties

$\left(\sqrt{x}\right)^2 = x$ for all non-negative numbers x

$\sqrt{x^2} = x$ for all non-negative numbers x

However, if *x* happens to be negative, then squaring it will produce a positive number, which will have a positive square root, so

$\sqrt{x^2} = |x|$ for all real numbers x

- You don't need the absolute value sign if you already know that *x* is positive. For example, $\sqrt{4} = 2$, and saying anything about the absolute value of 2 would be superfluous. You only need the absolute value signs when you are taking the square root of a square of a *variable*, which may be positive or negative.
- The square root of a negative number is undefined, because anything times itself will give a positive (or zero) result. $\sqrt{-4} = \textit{undefined}$ (your calculator will probably say ERROR).
- **Note**: Zero has only one square root (itself). Zero is considered neither positive nor negative.

WARNING: Do not attempt to do something like the distributive law with radicals:

$\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$ (**WRONG**) or

$\sqrt{a^2+b^2} \neq a+b$ (**WRONG**).

This is a violation of the order of operations. The radical operates on the *result* of everything inside of it, not individual terms. Try it with numbers to see:

$$\sqrt{9+16} = \sqrt{25} = 5 \text{ (CORRECT)}$$

But if we (incorrectly) do the square roots first, we get

$$\sqrt{9+16} = \sqrt{9} + \sqrt{16} = 3+4 = 7 \text{ (WRONG)}$$

However, radicals do distribute over products:

$$\sqrt{ab} = \sqrt{a}\sqrt{b}$$

and

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

provided that both a and b are non-negative (otherwise you would have the square root of a negative number).

Perfect Squares

Some numbers are perfect squares, that is, their square roots are integers:

0, 1, 4, 9, 16, 25, 36, etc.

It turns out that all other whole numbers have irrational square roots:

$\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}$ etc. are all irrational numbers.

The square root of an integer is either perfect or irrational

C. Simplifying Radical Expressions

$\sqrt{x^2} = |x|$ for all real numbers

$\sqrt{xy} = \sqrt{x}\sqrt{y}$ if both x and y are non-negative, and

$\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$ if both x and y are non-negative, and y is not zero

WARNING: Never cancel something inside a radical with something outside of it:

$\frac{\sqrt{3x}}{3} \neq \sqrt{x}$ **WRONG!** If you did this you would be canceling a 3 with $\sqrt{3}$, and they are certainly not the same number.

The general plan for reducing the radicand is to remove any perfect powers. We are only considering square roots here, so what we are looking for is any factor that is a perfect square. In the following examples we will assume that x is positive.

Example:

$$\sqrt{16x} = \sqrt{16}\sqrt{x} = 4\sqrt{x}$$

In this case the 16 was recognized as a perfect square and removed from the radical, causing it to become its square root, 4.

Example:

$$\sqrt{x^3} = \sqrt{x^2 x} = \sqrt{x^2}\sqrt{x} = x\sqrt{x}$$

Although x3 is not a perfect square, it has a factor of x2, which is the square of x.

Example:

$$\sqrt{x^5} = \sqrt{x^4 x} = \sqrt{x^4}\sqrt{x} = x^2\sqrt{x}$$

Here the perfect square factor is x4, which is the square of x2.

Example:

$$\sqrt{8x^5} = \sqrt{4 \cdot 2 \cdot x^4 \cdot x} = \sqrt{4}\sqrt{x^4}\sqrt{2x} = 2x^2\sqrt{2x}$$

In this example we could take out a 4 and a factor of x2, leaving behind a 2 and one factor of x.

- The basic idea is to factor out anything that is "square-rootable" and then go ahead and square root it.

D. Rationalizing the Denominator

One of the "rules" for simplifying radicals is that you should never leave a radical in the denominator of a fraction. The reason for this rule is unclear (it appears to be a holdover from the days of slide rules), but it is nevertheless a rule that you will be expected to know in future math classes. The way to get rid of a square root is to multiply it by itself, which of course will give you whatever it was the square root of. To keep things legal, you must do to the numerator whatever you do to the denominator, and so we have the rule:

If the Denominator is Just a Single Radical

Multiply the numerator and denominator by the denominator

Example:

$$\frac{3}{\sqrt{x-1}}$$
$$= \frac{3}{\sqrt{x-1}}\left(\frac{\sqrt{x-1}}{\sqrt{x-1}}\right)$$
$$= \frac{3\sqrt{x-1}}{x-1}$$

- **Note**: If you are dealing with an *n*th root instead of a square root, then you need *n* factors of that root in order to make it go away. For instance, if it is a cube root ($n = 3$), then you need to multiply by two more factors of that root to give a total of three factors.

If the Denominator Contains Two Terms

If the denominator contains a square root plus some other terms, a special trick does the job. It makes use of the difference of two squares formula:

$$(a + b)(a - b) = a^2 - b^2$$

Suppose that your denominator looked like $a + b$, where b was a square root and a represents all the other terms. If you multiply it by $a - b$, then you will end up with the square of your square root, which means no more square roots. It is called the *conjugate* when you replace the plus with a minus (or vice-versa). An example would help.

Example:

Given:

$$\frac{x}{2+\sqrt{x}}$$

Multiply numerator and denominator by the conjugate of the denominator:

$$\frac{x}{\left(2+\sqrt{x}\right)}\;\frac{\left(2-\sqrt{x}\right)}{\left(2-\sqrt{x}\right)}$$

Multiply out:

$$\frac{2x - x\sqrt{x}}{4 - x}$$

Quadratic Equations

Definition

$$ax^2 + bx + c = 0$$

a, b, c are constants (generally integers)

Roots

Synonyms: Solutions or Zeros

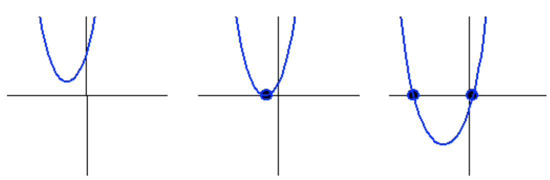
- Can have 0, 1, or 2 real roots

Consider the graph of quadratic equations. The quadratic equation looks like $ax2 + bx + c = 0$, but if we take the quadratic *expression* on the left and set it equal to y, we will have a function:

$$y = ax^2 + bx + c$$

When we graph y vs. x, we find that we get a curve called a *parabola*. The specific values of a, b, and c control where the curve is relative to the origin (left, right, up, or down), and how rapidly it spreads out. Also, if a is negative then the parabola will be upside-down. What does this have to do with finding the solutions to our original quadratic equation? Well, whenever $y = 0$ then the equation $y = ax^2 + bx + c$ is the same as our original equation.

Graphically, y is zero whenever the curve crosses the x-axis. Thus, the solutions to the original quadratic equation ($ax^2 + bx + c = 0$) are the values of x where the function ($y = ax^2 + bx + c$) crosses the x-axis. From the figures below, you can see that it can cross the x-axis once, twice, or not at all.



No Solutions One Solution Two Solutions

Actually, if you have a graphing calculator this technique can be used to find solutions to *any* equation, not just quadratics. All you need to do is

1. Move all the terms to one side, so that it is equal to zero
2. Set the resulting expression equal to y (in place of zero)
3. Enter the function into your calculator and graph it
4. Look for places where the graph crosses the x-axis

Your graphing calculator most likely has a function that will automatically find these intercepts and give you the x-values with great precision. Of course, no matter how many decimal places you have it is still just an approximation of the exact solution. In real life, though, a close approximation is often good enough.

Solving Quadratic Equations

A. Solving by Square Roots

No First-Degree Term

If the quadratic has no linear, or first-degree term (i.e. $b = 0$), then it can be solved by isolating the x^2 and taking square roots of both sides:

$$ax^2 + c = 0$$
$$ax^2 = -c$$
$$x^2 = \frac{-c}{a}$$
$$x^2 = \frac{-c}{a}$$

- You need both the positive and negative roots because $\sqrt{x^2} = |x|$, so x could be either positive or negative.
- This is only going to give a real solution if either a or c is negative (but not both)

B. Solving by Factoring

Solving a quadratic (or any kind of equation) by factoring it makes use of a principle known as the zero-product rule.

Zero Product Rule

If $ab = 0$ then either $a = 0$ or $b = 0$ (or both).

In other words, if the product of two things is zero then one of those two things must be zero, because the only way to multiply something and get zero is to multiply it by zero.

Thus, if you can factor an expression that is equal to zero, then you can set each factor equal to zero and solve it for the unknown.

- The expression *must* be set equal to zero to use this principle
- You can always make any equation equal to zero by moving all the terms to one side.

Example:

Given:

$$x^2 - x = 6$$

Move all terms to one side:

$$x^2 - x - 6 = 0$$

Factor:

$$(x - 3)(x + 2) = 0$$

Set each factor equal to zero and solve:

$$(x - 3) = 0 \quad \textbf{OR} \quad (x + 2) = 0$$

Solutions:

$$x = 3 \textbf{ OR } \quad x = -2$$

No Constant Term

If a quadratic equation has no constant term (i.e. $c = 0$) then it can easily be solved by factoring out the common x from the remaining two terms:

$$ax^2 + bx = 0$$

$$x(ax + b) = 0$$

Then, using the zero-product rule, you set each factor equal to zero and solve to get the two solutions:

$$x = 0 \quad \textbf{or} \quad ax + b = 0$$

$$x = 0 \quad \textbf{or} \quad x = -b/a$$

WARNING: Do not divide out the common factor of x or you will lose the $x = 0$ solution. Keep all the factors and use the zero-product rule to get the solutions.

Trinomials

When a quadratic has all three terms, you can still solve it with the zero-product rule if you are able to factor the trinomial.

- Remember, not all trinomial quadratics *can* be factored with integer constants

If it can be factored, then it can be written as a product of two binomials. The zero-product rule can then be used to set each of these factors equal to zero, resulting in two equations that are both simple linear equations that can be solved for x. See the above example for the zero-product rule to see how this works.

A more thorough discussion of factoring trinomials may be found in the chapter on polynomials, but here is a quick review:

Tips for Factoring Trinomials

1) Clear fractions (by multiplying through by the common denominator)
2) Remove common factors if possible
3) If the coefficient of the $x2$ term is 1, then
4) $x^2 + bx + c = (x + n)(x + m)$, where n and m
   i. Multiply to give c
   ii. Add to give b
5) If the coefficient of the $x2$ term is not 1, then use either
   i. Guess-and Check
   ii. List the factors of the coefficient of the $x2$ term
   iii. List the factors of the constant term
   iv. Test all the possible binomials you can make from these factors
   v. Factoring by Grouping
6) Find the product ac
   i. Find two factors of ac that add to give b
   ii. Split the middle term into the sum of two terms, using these two factors
   iii. Group the terms into pairs
   iv. Factor out the common binomial

C. Solving by Completing the Square

The technique of completing the square is presented here primarily to justify the quadratic formula, which will be presented next. However, the technique does have applications besides being used to derive the quadratic formula. In analytic geometry, for example, completing the square is used to put the equations of conic sections into standard form.
Before considering the technique of completing the square, we must define a perfect square trinomial.

Perfect Square Trinomial

What happens when you square a binomial?

$$(x + a)^2 = x^2 + 2ax + a^2$$

- **Note**: that the coefficient of the middle term ($2a$) is twice the square root of the constant term (a^2)

- Thus the constant term is the square of half the coefficient of x
- Important: These observations only hold true if the coefficient of x is 1.

This means that any trinomial that satisfies this condition is a perfect square. For example,

$$x^2 + 8x + 16$$

is a perfect square, because half the coefficient of x (which in this case is 4) happens to be the square root of the constant term (16). That means that

$$x^2 + 8x + 16 = (x + 4)2$$

Multiply out the binomial $(x + 4)$ times itself and you will see that this works.

The technique of completing the square is to take a trinomial that is not a perfect square, and make it into one by inserting the correct constant term (which is the square of half the coefficient of x). Of course, inserting a new constant term has to be done in an algebraically legal manner, which means that the same thing needs to be done to both sides of the equation. This is best demonstrated with an example.

Example:
Given Equation:

$$x^2 + 6x - 2 = 0$$

Move original constant to other side:

$$x^2 + 6x = 2$$

Add new constant to both sides
(the square of half the coefficient of x):

$$x^2 + 6x + 9 = 2 + 9$$

Write left side as perfect square:

$$(x + 3)^2 = 11$$

Square root both sides
(remember to use plus-or-minus):

$$x + 3 = \pm\sqrt{11}$$

Solve for x:

$$x = -3 \pm \sqrt{11}$$

Notes

- Finds all real roots. Factoring can only find integer or rational roots.
- When you write it as a binomial squared, the constant in the binomial will be half of the coefficient of x.

If the Coefficient of x^2 is Not 1

First divide through by the coefficient, then proceed with completing the square.

Example:
Given Equation:

$$2x^2 + 3x - 2 = 0$$

Divide through by coefficient of x^2:
(in this case a 2)

$$\frac{1}{2}\left(2x^2 + 3x - 2 = 0\right)$$

Move constant to other side:

$$x^2 + \tfrac{3}{2}x - 1 = 0$$

$$x^2 + \tfrac{3}{2}x = 1$$

Add new constant term:
(the square of half the coefficient of x, in this case 9/16):

$$x^2 + \tfrac{3}{2}x + \tfrac{9}{16} = 1 + \tfrac{9}{16}$$

Write as a binomial squared:
(the constant in the binomial is half the coefficient of x)

$$\left(x + \tfrac{3}{4}\right)^2 = \tfrac{25}{16}$$

Square root both sides:
(remember to use plus-or-minus)

$$x + \tfrac{3}{4} = \pm\tfrac{5}{4}$$

Solve for x:

$$x = \frac{-3 \pm 5}{4}$$

Thus:

$$x = 1/2 \quad \text{or} \quad x = -2$$

D. Solving using the Quadratic Formula

The solutions to a quadratic equation can be found directly from the quadratic formula.
The equation

$$ax^2 + bx + c = 0$$

has solutions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The advantage of using the formula is that it always works. The disadvantage is that it can be more time-consuming than some of the methods previously discussed. As a general rule you should look at a quadratic and see if it can be solved by taking square roots; if not, then if it can be easily factored; and finally use the quadratic formula if there is no easier way.

- Notice the plus-or-minus symbol (±) in the formula. This is how you get the two different solutions—one using the plus sign, and one with the minus.
- Make sure the equation is written in standard form before reading off *a*, *b*, and *c*.
- Most importantly, make sure the quadratic expression is equal to zero.

The Discriminant

The formula requires you to take the square root of the expression *b*2 – 4*ac*, which is called the *discriminant* because it determines the nature of the solutions. For example, you can't take the square root of a negative number, so if the discriminant is negative then there are no solutions.

If *b*2 – 4*ac* > 0	There are two distinct real roots
If *b*2 – 4*ac* = 0	There is one real root
If *b*2 – 4*ac* < 0	There are no real roots

Deriving the Quadratic Formula

The quadratic formula can be derived by using the technique of completing the square on the general quadratic formula:

Given:

$$ax^2 + bx + c = 0$$

Divide through by *a*:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Move the constant term to the right side:

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Add the square of one-half the coefficient of *x* to both sides:

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

Factor the left side (which is now a perfect square), and rearrange the right side:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

Get the right side over a common denominator:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Take the square root of both sides (remembering to use plus-or-minus):

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

Solve for *x*:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Basic Concepts of Statistics

Note: This section is not intended to provide a full coverage of statistics. A formal book on statistical methods and applications will be more appropriate for that. This section, instead, intends to provide a quick overview of simple statistical approaches used to establish relationships between data and how these can be used in solving some environmental problems.

Introduction

What is Statistics?

Statistics is the discipline concerned with the collection, organization, and interpretation of numerical data, especially as it relates to the analysis of population characteristics by inference from sampling. It addresses all elements of numerical analysis, from study planning to the presentation of final results. Statistics, therefore, is *more than a compilation of computational techniques*. It is a means of *learning* from data, a way of *viewing* information, and a *servant* of all science.

In a simplistic way, we can say that Statistics boils down to two approaches: exploration and adjudication. The purpose of exploration is to uncover patterns and clues within data sets. Adjudication, on the other hand, serves to determine whether the uncovered patterns are valid and can be generalized. Both approaches are as important and none can be minimized in the statistical process of data analysis. Statistics is a great quantitative tool to help make any method of enquiry more meaningful and particularly as objective as possible. However, one must avoid falling in the trap of the "black hole of empiricism" whereby data are analyzed with the hopes of discovering the fundamental "laws" responsible for observed outcomes. One must first establish an explanatory protocol of what these laws/processes can be and then use Statistics (among other tools) to test the appropriateness, and sometimes exactness, of such explanations. This pre-formulation of plausible explanations is at the core of the "scientific method" and is called "hypothesis formulation". Hypotheses are established as educated hunches to explain observed facts or findings and should be constructed in ways that can lead to anticipatory deductions (also called predictions). Such predictions should of course be verifiable through data collection and analysis. This is probably where Statistics come most in handy in helping judge the extent to which the recovered data agree with the established predictions (although Statistics also contributes substantially to formulation of test protocols and how data might be collected to verify hypotheses).

Statistics thus seeks to make each process of the scientific method (observation, hypothesis formulation, prediction, verification) more ***objective*** (so that things are observed as they are, without falsification according to some preconceived view) and ***reproducible*** (so that we might judge things in terms of the degree to which observations might be repeated).

It is not the scope of this short introduction to go over the range of statistical analyses possible. In fact, this text explores only selective issues related to statistics leaving room for true course in statistics (applied or theoretical) to develop all concepts more fully. Below we will talk succinctly about variables, summary statistics, and the evaluation of linear relationships between two variables.

A. Measurement

To perform statistical operations we need an object of analysis. For this, number (or codes) are used as the quantitative representation of any specific observation. The assignment of number or codes to describe a pre-set subject is called ***measurement***. Measurements that can be expressed by more than one value during a study are called ***variables***. Examples of variables are AGE of individuals, WEIGHT of objects, or NAME of species. Variables only represent the subject of the measurement, not any intrinsic value or code. Variables can be classified according to the way in which they are encoded (i.e. numeric, text, date) or according to which scale they are measured. Although there exists many ways to classify measurement scales, three will be considered here:

- Nominal (qualitative, categorical)
- Ordinal (semi-quantitative, "ranked")
- Scale (quantitative, "continuous", interval/ratio)

Nominal variables are categorical attributes that have no inherent order. For example SEX (male or female) is a nominal variable, as is NAME and EYECOLOR.

Ordinal variables are ranked-ordered characteristic and responses. For example an opinion graded on a 1-5 scale (5 = strongly agree; 4 = agree; 3 = undecided; 2 = disagree; 1 = strongly disagree). Although the categories can be put in ascending (or descending) order, distances ("differences") between possible responses are uneven (i.e. the distance between "strongly agree" and "agree" is not the same as the distance between "agree" and "undecided"). This makes the measurement ordinal, and *not* scaled.

Scale variables represent quantitative measurements in which differences between possible responses are uniform (or continuous). For example LENGTH (measured in centimeters) is a scale measurement. No matter how much you cut down the measurement into a smaller fraction (i.e. a tenth of a centimeter) the difference between on measurement and the next still remains the same (i.e. the difference between 3 centimeters and 2 centimeters or 3 millimeters and 2 millimeters is the same as that between 2 cm and 1 cm or 2 mm and 1 mm).

Notice that each step up the measurement scale hierarchy takes on the assumptions f the step below it and then adds another restriction. That is, nominal variables are named categories. Ordinal variables are named categories that can be put into logical order. Scale variables are ordinal variables that have equal distance between possible responses.

Data Quality

Something must be said about the quality of data used. A statistical analysis is only as good as its data and interpretative limitations may be imposed by the quality of the data rather than by the analysis. In addressing data quality, we must make a distinction between ***measurement error*** and ***processing error***. Measurement error is represented by differences between the "true" quality of the object observed (i.e the true length of a fish) and what appears during data collection (the actual scale measurement collected during the study). Processing errors are errors that occur during data handling (i.e. wrong data reporting, erroneous rounding or transformation). One must realize that errors are inherent to any measurement and that trying to avoid them is virtually impossible. What must be done is characterize these errors and try minimizing in the best way possible.

Population and Sample

Most statistical analyses are done to learn about a specific ***population*** (the total number of trouts in a specific river, the concentration of a contaminant in a lake's total sediment bed). The population is thus the universe of all possible measurements in a defined unit. When the population is real, it is sometimes possible to obtain information on the entire population. This type of study is called ***census***. However, performing a census is usually impractical, expensive, and time-consuming, if not downright impossible. Therefore, nearly all statistical studies are based on a subset of the population, which is called ***sample***. Whenever possible, a probabilility sample should be used. A probability samples is a sample in which a) every population member (item) has known probability of being sampled, b) the sample is drawn by some method of chance consistent with these probabilities, and c) selection probabilities are considered when making estimates from the samples.

B. Central Tendencies

Although we do not go over frequency distribution of a data set here, we need to develop some concise statement about the data distribution as a whole. To do this, we need *numerical summary measures* of the data ("summary statistics"). The most commonly used descriptive statistics are measures of central tendency. Taken together, such measures provide a great deal of information about the data set but most importantly, they attempt to locate the middle or center point in a group of data.

However, before we can develop and analyze these concepts quantitatively, we need to define an important mathematical symbol for the determination of central tendencies (and other statistics): the ***summation notation***.

The Greek letter Σ (a capital sigma) is used to designate a mathematical summation. We use the summation notation to write the sum of the values of a variable. The summation sign can be read as "the sum of." The expression Σx_i means that we should sum all the values of x_i (i.e., $x_i + x_2 + ... x_n$), where n is the total number of observations in the data set.

For example, let's consider a simple data set consisting of the following ten age values:

21	42
5	11
30	50
28	57
24	52

In discussing these data, let:

n, represent sample size (i.e. $n = 10$)
X, represent the *variable* (i.e. age)
x_i, represent the *value* of the i^{th} observation (i.e. $x_1 = 21$)

The symbol Σ (capital "sigma") is the summation sign, indicating that all values should be added. For the illustrative data set $\Sigma x_i = x_1 + x_2 + x_3 + \ldots + x_{10} = 21 + 42 + 5 + 11 + 30 + 50 + 28 + 27 + 24 + 52 = 290$.

To use the summation notation, you should realize that the summation sign is always followed by a symbol or mathematical expression. To compute $\Sigma(x\text{-}1)^2$, your first task is to calculate all of the $(x\text{-}1)^2$ values and then sum the results.

In general, the best strategy for using summation notation is to proceed as follows:

1) Identify the symbol or expression following the summation sign.

2) Use the symbol or expression as a column heading and list in the column all of the values corresponding to the symbol or all of the values calculated for the expression.

3) Finally, you sum the values in the column.

Mean
When mentioned without specification, the terms *mean* refers to the *arithmetic average* of the data set. Statisticians refer to to two different types of means (arithmetic averages): the ***population mean*** and the ***sample mean***.

The ***population mean*** (μ: pronounced "mu") is:

$$\mu = \frac{\sum x_i}{N} = \frac{1}{N}\sum x_i$$

Where $\sum x_i$ represents the sum of all values in the population and N represents the population size. For example, assuming the sum of all age values ($\sum x$) of a population of 600 individuals (N) is 17,703, then the population mean (μ) is = 17,703/600 = 29.505.

Although knowledge of the population mean is valuable, it is often difficult (if not impossible) to get information on the entire population. This forces us to study the population mean indirectly, through the sample mean. The ***sample mean*** ($\bar{x}$: pronounced "*x* bar") is:

$$\bar{x} = \frac{\sum x_i}{n} = \frac{1}{n}\sum x_i$$

Where $\sum x_i$ represents the sum of all values in the sample and n represents the sample size. For our illustrative data set above, $\sum x = 290$ and $n = 10$. Therefore, the sample mean ($\bar{x}$) is = 290/10 = 29.0 (since we rarelyhave data on all possible values of a population, $\bar{x}$ is usually calculated instead of μ).

- The mean of a distribution represents its ***gravitational center*** (where the distribution would balance if placed on a "numerical scale".
- The ***population mean*** is often called the "expected value", because if you were to select one observation at random from the population, the population mean would provide a reasonable expectation of that value.
- The ***sample mean*** is a) a good reflection of individual values drawn at random from the sample, b) a good reflection of individual values drawn at random from the population, and c) a good estimate of the population mean.

Median
Sometimes an extreme value, called an ***outlier***, will have a disproportionate influence on the mean and thus may affect how well the mean represents the central tendency of the data (i.e. average income of families; or average price of homes in a city). In that case, the ***median*** is a much better indicator of central tendency of such data sets. The median is thus the value that is greater than or equal to the half of the values in the data set. The median has a depth of

$$\frac{n+1}{2}$$

where the depth in an ordered array is the distance from the lowest value to any point in the array. In other words, the median is the 50th percentile in the distribution of the data since half of the observations fall above it and half below it. For an illustrative example let's consider the following ordered array:

5 11 21 24 27 28 30 42 50 52

The median has a depth of (10+1)/2 = 5.5. So the median falls right in between the 5th and 6th numbers in the ordered array (27 and 28, respectively). The median is thus the average between 27 and 28 = 27.5. When n is odd, the depth of the median will be an integer. For the following data set:

4 7 8 11 12

$n = 5$ and the median has a depth of (5+1)/2 = 3. Therefore the median is the 3rd datum and = 8.

Although the median conveys less precise information than the mean, it is sometimes in preference to the mean when the data set contains extreme numbers that "skew" the distribution towards one tail (see discussion below).

Mode
The final measure of central tendency is the ***mode***. The mode is simple the data value that occurs most often (with greatest frequency) in any distribution. The distribution can have more than one mode (if for example there are two numbers with equally large frequency, then the distribution is called bimodal). When each value of a data set occurs only once, then the data set has no mode. When data sets are small to moderate size, the mode is rarely used.

Comparison of Central Tendency Parameters
The mean, median, and mode are equivalent when the distribution is unimodal and symmetrical (Fig. 1 below).

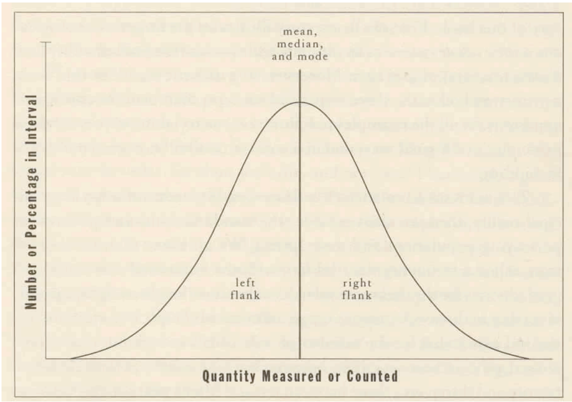


Fig. 1. Symmetric frequency distribution of a data set (from Gould, 1996; "Full House: The spread of excellence from Plato to Darwin"; Random House Inc. NY, NY).

However, in an ***asymmetric distribution***, the data falls more on one side of the center (or middle) than on the other side. In that case skewness exists in the data. Skewness will be either ***positive*** (extended "tail" of extreme data to the right of the central tendency) or ***negative*** (extended "tail" of extreme data to the left of the central tendency). If data are strongly skewed, the mean is not a good measure of central tendency as it gets disproportionately "pulled" towards the outliers. In asymmetry the median is approximately one-third the distance between the mean and the mode:

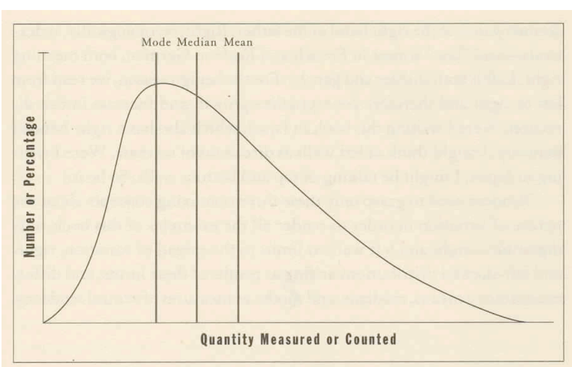


Fig. 2. Asymmetric frequency distribution of a data set: Positive skewness (from Gould, 1996).

Measure of Dispersion

A useful descriptive statistic complementary to the measures of central tendency is the ***measure of dispersion***. The measure of dispersion tells how much the data do or do not cluster around the mean. The ***standard deviation*** (*syn*: "root mean square") is the most common measure of dispersion and is simply the average square ***deviation*** of the data from the mean. The deviation of a data point is its difference from the mean:

$$\text{deviation}_i = x_i - \bar{x}$$

Although the sum of deviations may seem like a good basis for a measure of spread (dispersion around the mean), the sum of the deviations will always be eual to zero. Therefore, the sum of the deviation **cannot** be used to measure spread. Instead, statisticians *square* the deviations before summing them up. This statistics, known as ***sum of squares (SS)*** is:

$$SS = \sum_{i=1}^{n} (x_i - \bar{x})^2$$

The variance of the data can now be calculated. The ***population standard deviation*** (σ), for example, is the square root of the population variance:

$$\sqrt{\sigma^2} = \sqrt{\frac{SS}{N}}$$

However, since we rarely have data on the entire population, we usually must calculate the ***sample standard deviation (s)***, which is the square root of the sample variance:

$$\sqrt{s^2} = \sqrt{\frac{SS}{n-1}}$$

Interpreting standard deviation is not as easy as, say, interpreting a mean. One thing to keep in mind is that big standard deviations are associated with big data spreads and small standard deviations are associated with small data spreads. One way to interpret the standard deviation is to indicate the percent data that is within a specified number of standard deviations of the mean. There are two rules for applying this approach: when the data is distributed *normally,* and when it is non normally distributed.

Normal distribution

- About 68% of all values lie within 1 standard deviation from the mean.
- About 95% of all values lie within 2 standard deviations from the mean.
- Nearly all values will lie within 3 standard deviations from the mean.

Non-normal distribution

- At least 75% of all values lie within 2 standard deviations from the mean.
- At least seven-eighths of all values will lie within 3 standard deviations from the mean.

C. Correlations

Correlations quantify the extent to which two quantitative (continuous) variables, *X* and *Y*, "go together". When high values of *X* are associated with high values of *Y*, a positive correlation is said to exist. When high values of *X* are associated with low values of *Y*, a negative correlation is said to exist.

The first step in determining if a relationship exists between two variables is to plot them in the form of a scatter plot. Let's consider the following data presented in Table 1.

Year	Atmos. CO_2 (ppmv)	World population (billions)
1791	279.7	0.9
1816	283.8	1.0
1843	287.4	1.1
1854	288.2	1.2
1869	289.3	1.3
1887	292.3	1.5
1899	295.8	1.7
1909	299.2	1.8
1921	301.6	2.0
1943	307.9	2.3
1953	312.7	2.7
1960	316.9	3.0
1965	320.1	3.3
1970	325.7	3.7
1975	331.2	4.1
1980	338.7	4.5
1985	345.9	4.9
1990	354.2	5.3
1995	360.9	5.7
2000	369.4	6.1

Both human populations levels and the concentrations of CO_2 in the atmosphere have shown exponential growth in the last two centuries (see figure one below), and such growth has been considered as indicators of global change (See Aber and Melillo, 2001: *Terrestrial Ecosystems*; Academic Press, London, UK).

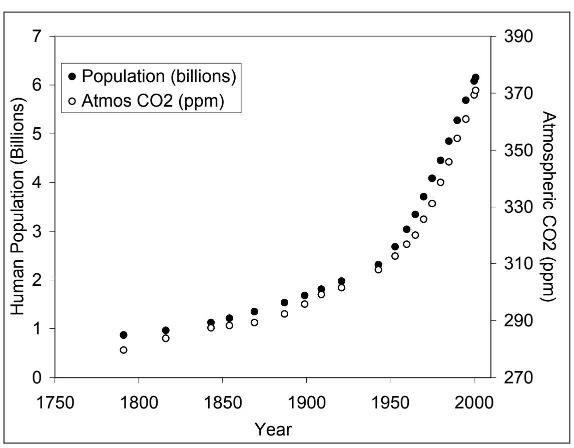


Fig. 3. Human population and atmospheric CO_2 concentrations from 1750 to 2001. (Population data from the US Census Bureau: http://www.census.gov/ipc/www/world.html and http://www.census.gov/ipc/www/worldhis.html; CO_2 data from Keeling and Whorf: http://cdiac.esd.ornl.gov/trends/co2/sio-mlo.htm and Neftel et al. http://cdiac.esd.ornl.gov/trends/co2/siple.htm)

On a first look, the two variables seem to behave similarly and may indicate a relationship between them. The first step in determining if there exists such relationship between the variables is to plot the data in the form of scatter plot. Figure 4 below indeed reveals that high values of *Y* are related to high values of *X*. That is, as the total amount of people on Earth increases, the atmospheric CO_2 concentration also increases.

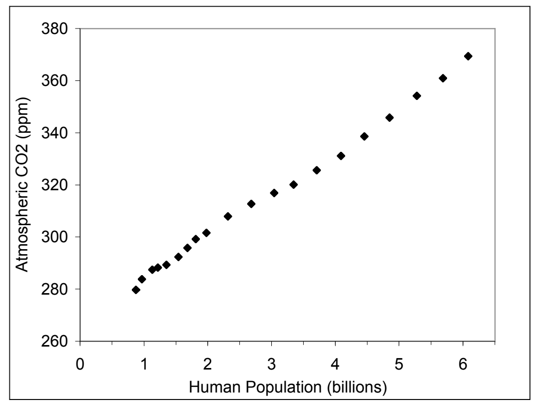
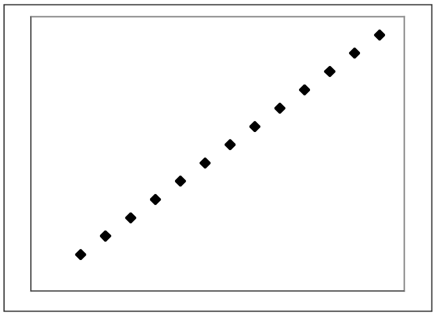


Fig. 4. Human population *vs*. atmospheric CO_2 concentrations from 1750 to 2001

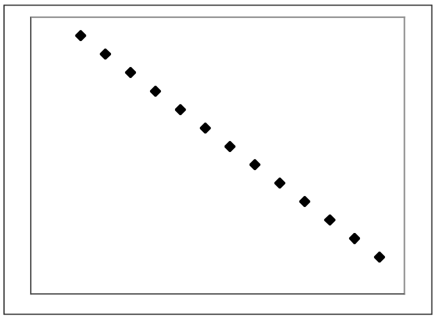
In general, a scatter plot may reveal either

- a positive relationship between the two studied variables (high values of *X* associated to high values of *Y*)
- a negative relationship between the two studied variables (high values of *X* associated to low values of *Y*)
- no relationship

Clearly, Figure 4 shows a strong positive relationship between human population and atmospheric CO_2. It is said that the two variables are ***correlated***. Such correlation between two variables can be quantified with a unit-free statistics called Pearson's ***correlation coefficient***, denoted *r*. When all points on a scatter plot fall directly on an upward incline, $r = +1$. When all points on a scatter plot fall directly on a downward incline, $r = -1$.

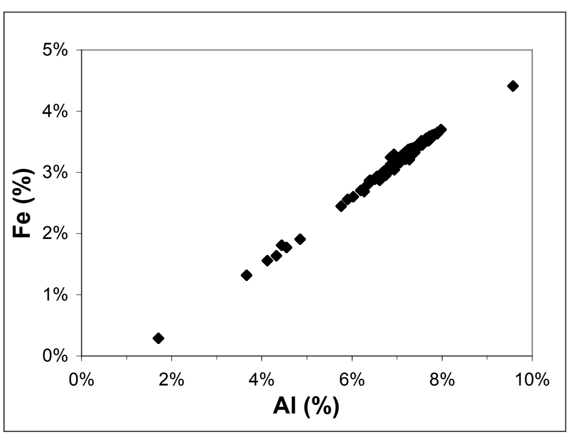


A. A perfect positive correlation ($r = +1$)

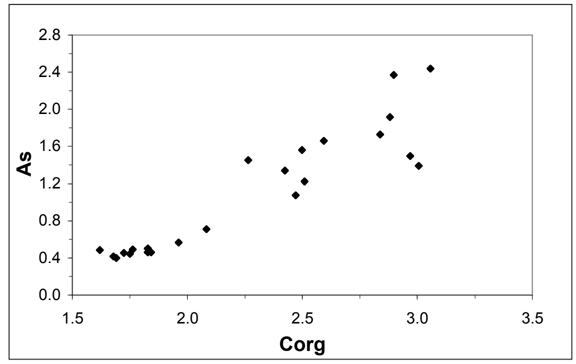


B. A perfect negative correlation ($r = -1$)

We quantify the strength of the correlation by the degree to which data adhere to an imaginary trend line that passes through the data. Strong correlations are associated with scatter clouds that adhere closely to the imaginary trend line. Weak correlations are associated with scatter clouds that adhere weakly to the imaginary trend line. The stength of the correlation is quantified by the value of *r*. The closer *r* is to +1, the stronger the positive correlation. The closer *r* is to -1, the stronger the negative correlation. Examples of strong and weak correlations are shown below:



A. Strong positive correlation (real relationship between iron and aluminum in lake sediments)



B. Weak positive correlation (real relationship between natural organic matter and arsenic in lake sediments)

However god the relationship appears between two variable, one needs to quantify the correlation strength (note: impressions of strength are subjective and will be influenced by axis scaling).

Calculation of Correlation Coefficient
To calculate the correlation coefficient, one needs three different sums of squares:

A. Sum of squares for variable *X*. This statistics quantifies the spread of variable *X*. Its formula is:

$$SS_{XX} = \sum_{i=1}^{n} (x_i - \bar{x})^2$$

For the illustrative data in Table 1 above, n = 20; $\bar{x}$ = 2.9 billion humans; $SS_{XX} = (0.9 - 2.9)^2 + (1 - 2.9)^2 + \ldots + (6.1 - 2.9)^2 = 54.0$

B. Sum of squares for variable *Y*. This statistics quantifies the spread of variable *Y*. Its formula is:

$$SS_{YY} = \sum_{i=1}^{n} (y_i - \bar{y})^2$$

For the illustrative data in Table 1 above $\bar{y} = 315.0$ ppm CO_2; $SS_{YY} = (279.7 - 315.0)^2 + (283.8 - 315.0)^2 + \ldots + (369.4 - 315.0)^2 = 14246.5$

C. Sum of the cross-products. This statistics is analogous to the other sums of squares except that it quantifies the extent to which the two variables go together or apart. Its formula is:

$$SS_{XY} = \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})$$

For the illustrative data in Table 1 above, SS_{XY} = (0.9 - 2.9)*(279.7 – 315.0) + (1 - 2.9)*(283.8 – 315.0) + … + (6.1 - 2.9)*(369.4 - 315.0) = 875.7

The correlation coefficient thus becomes:

$$r = \frac{SS_{XY}}{\sqrt{(SS_{XX})(SS_{YY})}}$$

For the illustrative data set:

$$r = \frac{875.7}{\sqrt{(54.0)(14246.5)}} = 0.9981$$

Here, the correlation coefficient is positive (indicates a positive relationship) and very close to 1 (indicates a very strong relationship between the two variables). Although the application of this statistic seems quite straightforward, it turns out that the correlation coefficient has no inherent value, and in the exception of strong relationships as in the case presented, r is hard to use to determine correlational strength. Another statistics is much more useful: the ***coefficient of determination***. The coefficient of determination is he square of the correlation coefficient (r^2). For the illustrative data set, $r^2 = (0.998)^2 = 0.996$. This statistic quantifies the proportion of the variance of one variable that is explained by the other. The illustrative coefficient of determination of 0.99 suggests that >99% of the variability in the global atmospheric CO_2 concentrations in the last 200 years is explained by human population growth.

D. Bivariate Linear Regression

Introduction

One must be very careful however in how the previous statistical approach is used. It is evident from the data that the two variables studied (atmospheric CO_2 and human population) are indeed strongly correlated and that the variation in one explains that of the other. But does this mean a true ***functional dependency***? Functional dependency assumes that the magnitude of the dependent variable (Y) is determined in part by the magnitude of the independent variable (X). For example, in studying the relationship between blood pressure and age, it is reasonable to assume a functional dependency, i.e. a person's blood pressure depends on their age (most of the time). This is not to suggest age is the only factor responsible for increases in blood pressure, and is not to suggest a causal relationship is proven, but that age is one possible determinant of blood pressure. **One must be particularly careful in interpreting relationships (even strong ones)**.

In the example above, the observed relationship between atmospheric CO_2 and human population may be used to suggest that there exists a functional dependency between these two variables through, for example, increased respiration of human populations (and thus larger releases of CO_2 to the atmosphere). This, of course appears immediately as a ludicrous statement that seems impossible to sustain with a straight face. Indeed, although there seems to be a functional dependency between these two variables it is an indirect one, whereby human population increase has lead large-scale environmental changes such as fossil fuel combustion and deforestation that themselves have lead to increases fluxes of CO_2 (and other gases) to the atmosphere.

More dangerous yet is the potential for ***spurious correlations***, the type of strong relationship between two variables that can be completely explained by independent arguments rather than a true functional dependency (we call these arguments *nested* relationships). One example of such spurious correlation is presented below. Consider the growth of two populations in the state of Georgia: humans and storks. One can see from Figure 5 that there exists a strong relationship between these two variables and by running the previous exercise one obtains a value of 0.99 for the coefficient of determination (r^2).

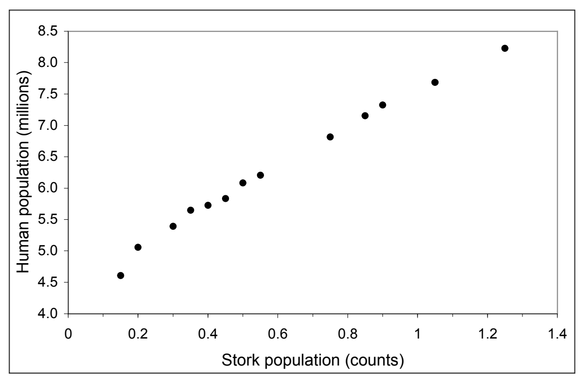


Fig. 4. Human *vs*. wood stork (*Mycteria americana*) populations from 1970 to 2000 in the state of Georgia. Human population data from the Georgia Statistic System: http://www.georgiastats.uga.edu/guide.html; Stork population data from the USGS North American Breeding Bird Survey: http://www.mbr-pwrc.usgs.gov/bbs/bbs00.html)

This suggests that the variation in stork population in the last 30 years in Georgia can explain most of the variation in human population. In other words, this relationship can help demonstrate that the increase in the human population has been made possible thanks to the increase of the stork population in that state (which, if you believe Walt Disney's "*Dumbo*", points to the primordial role of these graceful birds in bringing human babies to their final destination…) If there is isn't a shred of evidence in this past argument, one must also accept that the observed relationship is somewhat suspicious. By all means, one might expect a very opposite effect of increased human population (and thus land use expansion and habitat degradation, added to increases in contaminant releases) on stork populations. Indeed, Stork numbers presented here represent sightings along a predetermined transect, which is not equivalent to a full census of the stork population in Georgia. This might just represent an increased effort from the surveyors. Although a true increase in the population cannot be excluded (maybe through conservation efforts and reinstallation of individuals in the population). In any extent, a "perfect" relationship does not by any means sustain causality in the variables studied.

Linear Regression Analysis
As with correlation, ***regression*** is used to analyze the relationship between two continuous (scale) variables. However, regression is better suited to study the functional dependency between factors. Also, the "products" (output) of regression and correlation analyses differ. Put it very simply, with regression, you are predicting the average change in the dependent variable *Y* per unit independent variable *X*, whereas with correlation you are describing the fit of the two-dimensional scatter (spread) around a trend line. To illustrate regression, let's use the same illustrative example as in the previous section (global human population *vs.* atmospheric CO_2).

Regression Model
The simplest functional relationship between two variables is that of a linear relationship. Yu might remember from algebra (see section above) that a line is described by its intercept and slope:

$$y = ax + b$$

where y is the dependent variable, x is the independent variable, a is the slope of the line (also called m), and b is the intercept (where the line crosses the Y-axis).

If all data were to fall on a straight line, drawing a line that connects the data would be a trivial matter. However, because we are dealing with statistical scatter, choosing a line is not an easy matter. To determine the ***best fitting*** line for the data set, let's assume:

- $\hat{y}$ represents the ***predicted*** value of Y
- a represents the slope of the line
- b represents the intercept of the line

The regression equation is:

$$\hat{y} = ax + b$$

Because of random scatter, each data point will be a certain distance from the line. These distances, are called ***error terms*** or ***residuals***. To illustrate the principles of regression, let's use a data set of chemical parameters in lake sediments: aluminum and lignin content along the depth of a sediment profile (0-40 cm). (Lignin is a natural organic component exclusive to vascular plants and which is used as a tracer for terrigenous inputs to aquatic systems).

Sediment depth (cm)	Al (%)	Lignin (mg/g)
0.5	7.0	0.56
3.5	7.3	0.69
4.5	7.3	0.48
5.5	7.4	0.77
7.5	7.4	1.16
8.5	7.1	0.86
15	6.9	0.91
23	7.0	1.00
25	7.3	0.76
31	6.9	1.36
33	6.3	2.13
35	6.2	2.01
37	6.0	3.60
39	5.8	3.14

The *residuals* for this illustrative data set represented by the vertical lines in Figure 5 below:

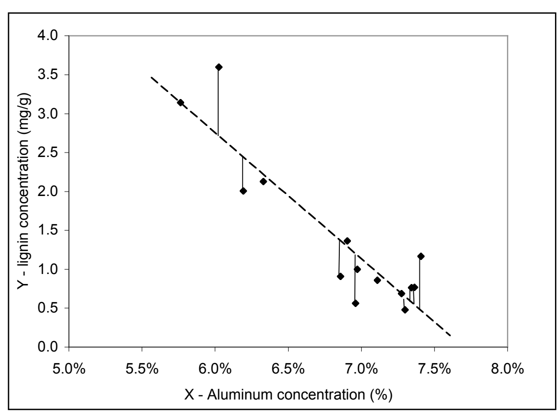


Fig. 5. Lignin concentrations (natural organic matter exclusive to vascular plants) *vs.* aluminum concentrations in lake sediments.

The strategy for determining the line is to select the intercept (b) and slope (a) that minimizes the sum of squared residuals. This is called the ***least squares line***. The slope of the least squares line is given by the equation:

$$a = \frac{SS_{XY}}{SS_{XX}}$$

Where SS_{XY} is the sum of the cross-products and SS_{XX} is the sum of squares for the variable *X*. (remember that formulas for these sums of squares were presented in the previous section).

For the illustrative data, $SS_{XY} = -6.26$ and $SS_{XX} = 3.89$ (you should test this by calculating these two parameters). Therefore the slope *a* is:

$$a = \frac{-6.26}{3.89} = -1.61$$

The intercept of the line is:

$$b = \bar{y} - a\bar{x}$$

where $\bar{y}$ is the average value of the variables *Y*, $\bar{x}$ is the average value of the variables *X*, and *a* is the slope of the line. Hence for the illustrative data set,

$$b = 1.4 - (-1.61 \times 6.8) = 12.5$$

The regression model for the illustrative data set is therefore:

$$\hat{y} = (-1.61x) + 12.5$$

Interpretation
In the previous section, we've learned how to calculate the correlation coefficient (r) and the coefficient of determination (r^2). The correlation coefficient for this data set is –0.91, which states that there exists a rather strong negative relationship between the two variables (in other words, any increase in the independent variable, *X*, yields lower values of the dependent variable, *Y*). The coefficient of determination is 0.82 and states that 82% of the variability of *Y* is explained by the variability of *X*. To assess if this is indeed a relationship built on functional dependency one must know something about the system of study (and here geochemical principles).

In short, aluminum is an abundant element in the earth's crust and is a predominant component of small sized minerals operationally defined as clays and mineralogically called aluminosilicates. Generally, the percentage (or relative amount) of aluminum increases inversely with respect to the size of solid particles (i.e. sand fractions, >2mm, will have very low amount of aluminum, whereas clays, <2μm, will tend to have higher relative proportions of aluminum). Lignin, an organic biomolecule exclusive to land plants, tends to occur in high concentrations in woody tissues and in lower amounts in soft tissues (i.e. leaves) and small plant fragments. In a soil, for example, very small particles (i.e. clays) will absorb small quantities of organic matter including small amounts of lignin components (from plant fragments). In contrast, large plant macro-debris will be more enriched in lignin components (but associated with sands that are depleted in aluminum). In any extent, the relationship observed in the studied lake sediments suggest that the bottom of the lake receives a changing proportion of large sandy particles enriched in lignin (but depleted in aluminum) and small clayey material enriched in aluminum (but depleted in woody fragments). This relationship is indeed functional and points to changes in the lake's drainage basin that lead to variations in material inputs to its bottom (i.e. due to variations in storm activities or other natural or human-based process). Hence, it becomes clear that to demonstrate the validity of an observed correlation, one must be able to establish some sort of functional dependency between the variables (whether direct or indirect), and thus know something about the system of study.

An important aspect of regression analysis, aside from establishing a relationship, is the calculation of the slope of the model. The slope has a direct interpretation as *the predicted change in variable Y per unit change in the variable X*. In the case of a linear correlation, the rate of change is constant throughout the range of the data set. For the illustrative example above, the slope of –1.61 suggests that for every additional unit in *X* (percentage of aluminum in sediments), we predict a decrease in *Y* (amount of lignin in sediments) of 1.61 units. The model can also be used to predict values for *Y* given a known value of *X*. For examples, if we were to analyze another sediment sample and obtained an aluminum concentration of 5.5%, the predicted lignin content would be ~3.6 mg/g. Or, we could extrapolate the relationship to $y = 0$ and solve to calculate the predicted amount of aluminum in minerals with no lignin whatsoever. The solution of this equation is 7.71% (you should try to solve it).

This study is by far minimal in its analysis but should act as a primer in starting work with linear relationships between two variables. A more in depth approach will help develop the concepts necessary for quantitative analyses of problem sets.

Operations' Reminder

Powers (Exponents)

$x^a \times x^b = x^{(a+b)}$

$x^a y^a = (xy)^a$

$(x^a)^b = x^{(ab)}$

$x^{(a/b)} = b^{th}$ root of $(x^a) = \sqrt[b]{x^a}$

$x^{(-a)} = \dfrac{1}{x^a}$

$x^{(a-b)} = \dfrac{x^a}{x^b}$

Logarithms

$y = \log_b(x)$ if and only if $x=b^y$

also, $\sqrt[y]{x} = b$

$\log_b(1) = 0$

$\log_b(b) = 1$

$\log_b(x*y) = \log_b(x) + \log_b(y)$

$\log_b(x/y) = \log_b(x) - \log_b(y)$

$\log_b(x^n) = n\log_b(x)$

Warning:

$\log_b(x)*\log_b(y) \neq \log_b(x*y)$

$\dfrac{\log_b(x)}{\log_b(y)} \neq \log_b(\dfrac{x}{y})$

Example:

We want to calculate *y* and express it in scientific notation:

$$y = 17^5$$

Using common logarithms (base 10):

$$\text{Log}_{10}(y) = \text{Log}_{10}(17^5) = 5\text{Log}_{10}(17)$$
$$= 5 * 1.2304$$
$$= 6.1522$$

Thus the answer is

$$y = 10^{6.1522}$$

But this is not very meaningful. It is better to express this as:

$$10^{0.1522} * 10^6$$
$$= 1.42.10^6$$

(of course, calculators can give you 17^5 directly in scientific notation, but it's better to know how this number can be derived).

Natural Logarithms (Ln)

The constant *e* (occasionally called *Napier's constant* in honor of the Scottish mathematician who introduced logarithms) is an infinite decimal. This constant is the base of the natural logarithm and is approximately equal to 2.71828... (no precise decimal fraction can be given, as *e* is an irrational number; *e* = 2.71828 18284 59045 23536 02874…)

e can also be written as the infinite series

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

$e^{\ln(x)} = x$ for all positive *x* and

$\ln e^x = x$ for all real *x*

$\ln e = 1$

Initially, it seems that the base-10 would be more "natural" than base *e*. One reason we call ln(*x*) "natural" is that expressions in which the unknown variable appears as the exponent of *e* occur much more often than exponents of 10 (because of the "natural" properties of the exponential function which allow to describe growth and decay behaviors), and so the natural logarithm is more useful in practice

Example:

To illustrate just one reason for its use, let's assume that a sum *P* (the original principal you invest) earns interest at a rate *r* (% per annum). If this interest is NOT compounded continuously and calculated at the end of each year. Assuming we have a principal of $110 and a 12% annual interest, after one year the total amount available is the sum of the principal and its interest (1 + *r*), or $112. However, by compounding the interest quaterly, we have a factor of $(1 + r/4)^4$, and the total available at the end of the year is: $112.55. In other words, we are getting an extra $0.55 by compounding quaterly. Compounding monthly (12 times), we would get an extra $0.68, and $0.73 by compounding weekly (you should try this out). As the compounding events tend to • the amount will increase by a factor of e^r. So, with a interest rate of 12%, with continuous compounding, the principal grows by $e^{0.12}$ and yields $112.75 at the end of one year. Uncontrolled bacterial growth is similar to continuous compound interest.

Computing Factorials

The factorial of a non-negative integer *n*, written as *n*!, is defined as follows:

$n! = 1 \times 2 \times 3 \times \ldots \times n$ (when $n > 1$)

0! is always equal to 1.

A few examples below should make the computation of a factorial clearer.

Examples of factorial computations:

a) 0! = 1
b) 1! = 1
c) 2! = 1 x 2 = 2
d) 3! = 1 x 2 x 3 = 6
e) 5! = 5 x 4 x 3 x 2 x 1 = 120
f) 10! = 10 x 9 x 8 x 7 x 6 x 5 x 4 x 3 x 2 x 1 = 3,628,800

You will need to know how to compute factorials for your quantitative methods courses. Specifically, you will use factorials to compute probability distributions for binomial random variables (note: binary random variables are variables that can have only one of two values – for example, a success or a failure).

An example of a binary random variable is whether a graduating MPA student is offered a job after they apply for an open position. In this case, the variable "hire" would be a binary random variable because it can only have two values (0 = not hired and 1 = hired). You will learn more about computing probability distributions of binary random variables during the fall and spring semester.

Symbols and Conversion Tables

MATHEMATICAL SYMBOLS	
= equal	± plus or minus (i.e. 11±2 is the range of real numbers between 9 and 13)
≡ identical to	µ proportional to
≈ or ≅ approximately equal to	$\sum_{i=1}^{n} x_i$ sum of the values of x from x_1 to x_n
≠ not equal to	$\prod_{i=1}^{n} x_i$ product of the values of x from x_1 to x_n
> greater than	n! factorial of n (i.e. 4! Is 4x3x2x1 = 24)
≥ greater than or equal to	% percent (or part per hundred.)
>> much greater than	‰ per mil (or part per thousand)
< less than	ppm part per million (i.e. mg per kg)
≤ less than or equal to	ppm(v) part per million by volume (i.e. mg per m^3)
<< much less than	ppb part per billion (i.e. ng per kg)
• infinity	*a*:*b* the ratio of *a* to *b*

Multiples and Submultiples			
Number	Scientific Notation	Name	Common Prefixes
1,000,000,000,000	10^{12}	trillion	tera
1,000,000,000	10^{9}	billion	giga
1,000,000	10^{6}	million	mega
1,000	10^{3}	thousand	kilo
100	10^{2}	hundred	hecto
10	10^{1}	ten	deka
1	10^{0}	one	--
0.1	10^{-1}	tenth	deci
0.01	10^{-2}	hundredth	centi
0.001	10^{-3}	thousandth	milli
0.000001	10^{-6}	millionth	micro
0.000000001	10^{-9}	billionth	nano
0.000000000001	10^{-12}	trillionth	pico

To add or subtract two numbers written in scientific notation, first make sure that both are raised to the same power of ten.

Example: $2.5.10^{5} + 5.10^{4} = 2.5.10^{5} + 0.5.10^{5} = 3.0.10^{5}$

To multiply two numbers written in scientific notation, multiply the first part of both numbers together and then sum up their respective power.

Example: $2.5.10^{5}$ x $5.10^{4} = (2.5.0.5).10^{5+4} = 1.25.10^{9}$

To divide two numbers written in scientific notation, divide the first part of both numbers together and then subtract their respective power.

Example: $2.5.10^{5} \div 5.10^{4} = (2.5\div0.5).10^{5-4} = 5.10^{1}$ (which is actually 50)

LENGTH						
	in	ft	mi	cm	m	km
1 inch (in)	1	0.083	$1.58.10^{-5}$	2.54	0.0254	$2.54.10^{-5}$
1 foot (ft)	12	1	$1.89.10^{-4}$	30.48	0.3048	--
1 mile (mi)	63.36	5.28	1	160,934	1,609	1.609
1 centimeter (cm)	0.394	0.0328	$6.20.10^{-6}$	1	0.01	$1.0.10^{-5}$
1 meter (m)	39.37	3.281	$6.2.10^{-4}$	100	1	0.001
1kilometer (km)	39.37	3,281	0.6214	100,000	1000	1

AREA						
	in^2	ft^2	mi^2	cm^2	m^2	km^2
1 in^2	1	$6.944.10^{-3}$	$2.49.10^{-10}$	6.4516	6.45 e-4	$6.45.10^{-10}$
1 ft^2	144	1	$3.59.10^{-8}$	929	0.0929	$9.29.10^{-8}$
1 mi^2	$4.01.10^{9}$	$2.79.10^{7}$	1	2.59e10	259	2,590
1 cm^2	0.155	1.07e-3	3.86e-11	1	1.0e-4	$1.0.10^{-10}$
1 m^2	1,550	10.764	3.86e-7	10,000	1	$1.0.10^{-6}$
1 km^2	$1.55.10^{9}$	$1.07.10^{7}$	0.3861	1.10^{10}	1,000,000	1

VOLUME								
	in^3	ft^3	yd^3	m^3	qt	liter	barrel	gal(U.S)
1 in^3	1	$5.79.10^{-4}$	$2.14.10^{-5}$	$1.64.10^{-5}$	$1.73.10^{-2}$	0.02	--	$4.33.10^{-3}$
1 ft^3	1,728	1	3.70e-3	0.0283	29.922	28.3	--	7.48
1 yd^3	46656	27	1	0.76	807.89	764.55	--	201.97
1 m^3	61,020	35.315	1.307	1	--	1,000	--	--
1 quart (qt)	57.75	$3.34.10^{-2}$	$1.24.10^{-3}$	$9.46.10^{-4}$	1	0.95	--	0.25
1 liter (l)	61.02	$3.53.10^{-2}$	$1.31.10^{-3}$	$1.0.10^{-3}$	1.06	1	--	0.2642
1 barrel (oil)	--	--	--	--	168	159.6	1	42
1 gallon (U.S)	231	0.13	$4.95.10^{-3}$	$3.78.10^{-3}$	4	3.785	0.02	1

MASS AND WEIGHT			
1 pound 453.6 grams 0.4536 Kg 16 ounces **1 metric ton** 2205 pounds 1000 kilograms	**1 gram** 0.0353 ounce 0.0022 pound **1 kilogram** 2.205 pounds	**1 short pound** 2000 pounds 907.2 kilograms	**1 long ton** 2240 pounds 1008 kilograms

TEMPERATURE		
	Fahrenheit to Celsius Celsius to Fahrenheit	$°F = (9/5 \times °C) + 32$ $°C = 5/9 \times (°F - 32)$
	Kelvin to Celsius Celsius to Kelvin	$°K = °C + 273$ $°C = °K - 273$

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