Riemann-Roch on Surfaces

Adam Block

May 2017

1 Introduction

Classically, the most important theorem regarding classification questions of curves in algebraic geometry is that of Riemann-Roch. There are generalizations of this theorem to higher dimensions, although they become less useful as we shall see. In the following we follow in the footsteps of [Har] to develop a generalization to surfaces, while some of the ideas and applications follow in the footsteps of [Ful] and [EH]. The key theorem to be proved is as follows:

Theorem 1. *Let X be a surface and D a divisor on X. Let K be the canonical class, $l(D) = \dim_k H^0(X, \mathcal{O}_X(D))$, $s(D) = \dim_k H^1(X, \mathcal{O}_X(D))$ and the arithmetic genus of X, $p_a = \chi(\mathcal{O}_X) - 1$. Then*

$$l(D) - s(D) + l(K - D) = \frac{1}{2} D.(D - K) + 1 + p_a$$

For the following, unless explicitly stated otherwise, k is an algebrically closed field, X, is a surface, i.e., a nonsingular 2-dimensional projective variety (integral scheme) over k, a curve on X is an effective (Weil) divisor, and a point is a closed point (of X). We will reserve K for a canonical divisor. Note that we will be using the term curve interchangeably to refer to the divisor and to the subscheme, but context should make clear to which we refer.

2 Intersection Number

Before we can prove Theorem 1, we need to define the product on divisors. As we shall see, this will be thought of as an intersection number and will give us as an application a classical theorem of Bezout. It is outside the scope of the current paper to introduce the Chow ring, but [EH] and [Ful] define this product as the product in the Chow ring, which allows for greater generality. First, we need to gather a few results.

2.1 Bertini's Theorem

Much of the following rests upon the ability to move curves so that they intersect transversally. The fact that this can be done rests on Bertini's theorem. While the theorem holds for higher dimensions, we only use the case of surfaces.

Theorem 2 (Bertini)**.** *Let X be a nonsingular variety with a fixed closed immersion into $\mathbf{P}_k^n$ such that $\dim X > 2$. Then there exists a hyperplane H such that $X \not\subset H$ and $H \cap X$ is a nonsingular variety.*

Proof. For the existence of H such that $Y = H \cap X$ is regular at every point, see [Har, pp. 179-180]. To see that Y is a variety, by and application of Serre duality we get by [Har, pp. 244-245] that Y is connected because it is a closed subset of codimension 1 of X, a variety of dimension at least 2, which is the support of an effective ample divisor, namely H, and it is regular, so the stalks are all regular local rings, which are domains by [Mat, p. 106] so Y is a nonsingular variety. ■

Remark 3. In point of fact, Bertini proved more than this, saying that the set of hyperplanes H satisfying the conclusion of his theorem is open and dense in the complete linear system $|H|$ if considered as a projective space over k. We do not use explicitly use this result.

In some sense, Theorem 2 allows us to 'move' curves so that they intersect transversally. This transversal intersection is critical for the definition of intersection number.

2.2 Definitions and Basic Applications

Any curve on a surface X are divisors. We wish to define an intersection number on these divisors that obeys rules that extend our notion of intersection from our geometric intuition. First, although referenced above, we need a rigorous definition of transversal intersection:

Definition 4. Let C, D be two curves on X and let $P \in C \cap D$ be a point of intersection. Then if f, g are the local equations defining C and D respectively, we say that C and D meet transversally at P if the maximal ideal $\mathfrak{m}_P$ in $\mathcal{O}_{X,P}$ is generated by f and g.

An elementary consequence of the definition is that if C and D meet transversally at P, then they are both nonsingular at P because if $(f, g) = \mathfrak{m}_{X,P}$ then (f) generates $\mathfrak{m}_{D,P}$ and conversely for g and these local rings are both one dimensional so regularity follows.

Our intuition would suggest that if C and D intersect transversally at r distinct points then their intersection number should be r. Moreover, it would be nice if the product did not depend on rational equivalence and was additive. Rigorously, we would like a product $\operatorname{Div} X \times \operatorname{Div} X \to \mathbb{Z}$ that satisfies the following properties assuming C and D are two curves:

1. If C and D intersect transversally in r points, then $C.D = r$
2. $C.D = D.C$
3. $(C + C').D = C.D + C'.D$
4. If $C \sim C'$ and $D \sim D'$ then $C.D = C'.D'$

We will soon show that there exists a unique such product. Before we do this, we need a lemma:

Lemma 5. *If C is an irreducible and nonsingular curve on X and D is a curve that meets C transversally at every point. Then*

$$|C \cap D| = \deg_C(\mathcal{O}_X(D) \otimes \mathcal{O}_C)$$

Proof. Recall that if D is a divisor then the ideal sheaf for D is $\mathcal{O}_X(-D)$. Thus we get an exact sequence

$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0$$

Tensoring with $\mathcal{O}_C$ is exact because $\mathcal{O}_C$ is invertible, so we get

$$0 \longrightarrow \mathcal{O}_X(-D) \otimes \mathcal{O}_C \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_{C\cap D} \longrightarrow 0$$

Thus we have that $\mathcal{O}_X(-D) \otimes \mathcal{O}_C$ is the sheaf on C associated to $C \cap D$. The intersection is transversal so the divisor $C \cap D$ is just the sum of each point of intersection, each with multiplicity 1, so the degree of the divisor is just the number of points of intersection. ∎

Now we are prepared to prove that an intersection product exists and is unique.

Theorem 6. *There exists a unique pairing* $\operatorname{Div} X \times \operatorname{Div} X \to \mathbb{Z}$ *that satisfies axioms (1)-(4) above.*

Proof. To prove uniqueness, fix some ample divisor H. Then by definition of ample, there exists some $n > 0$ such that $C + nH$, $D + nH$, and nH are all very ample. By an elementary application of Theorem 2, we can choose nonsingular $C' \in |C + nH|$, $D' \in |D + nH|$, $E, F \in |nH|$ such that D' is transversal to C', E' is transversal to D' and F' is transversal to both C' and E'. Thus we get after modding out by rational equivalence

$$[C] = [C'] - [E']$$
$$[D] = [D'] - [F']$$

and by additivity, constancy on rational equivalence class, symmetry, and evaluation on transversal intersections, we get $C.D = |C' \cap D'| - |C' \cap F| - |E \cap D'| + |E \cap F|$ which is constant, so if such a function exists it is unique.

For existence, if C and D are divisors, then let $C' \in |C|$ and $D' \in |D|$ transversal, which exist by an elementary application of Theorem 2 and define $C.D = C'.D' = |C' \cap D'|$. For a proof that this is well defined, see [Har, pp. 259-260] or, for a more general result about multiplication in the Chow ring and how this applies, see [EH, pp. 19 - 21] or [Ful, pp. 93-97]. ■

Definition 7. The product described above in Theorem 6 is the intersection product and will be denoted for two curves C, D on X as $C.D \in \mathbb{Z}$.

While Definition 7 satisfies the intuition that we might expect of an intersection number, it has the drawback that it is not so easy to calculate. In order to aid us in this, we must introduce another, more local concept of intersection. At some point of intersection $P \in C \cap D$, we have local functions f, g that cut out C and D respectively. Then $f, g \in \mathcal{O}_{X,P}$ and $\mathcal{O}_{X,P}$ is an algebra over k, so we can define $(C.D)_P = \dim_k \mathcal{O}_{X,P}/(f,g)$. Note that $(C.D)_P$ is always finite because the Nullstellensatz gives us that there exists an $r > 0$ such that $\mathfrak{m}_P^r \subset (f,g)$ and $\mathfrak{m}_P^i/\mathfrak{m}_P^{i+1}$ is finite by results from last semester; we know that $\mathcal{O}_{X,P}/\mathfrak{m}_P^i$ is finite dimensional so $(C.D)_P$ is contained in a finite dimensional vector space. With this local notion of 'multiplicity' we can find an easier way to compute the intersection number:

Proposition 8. *If C, D are curves on X not sharing any irreducible components, then*

$$C.D = \sum_{P \in C \cap D} (C.D)_P$$

Proof. As in the proof of Lemma 5, we have an exact sequence

$$0 \longrightarrow \mathcal{O}_X(-D) \otimes \mathcal{O}_C \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_{C \cap D} \longrightarrow 0$$

Note that the structure sheaf at each point $P \in sp(C \cap D)$ is just $\mathcal{O}_{X,P}/(f,g)$. Thus, by gluing, we find

$$\dim_k H^0(X, \mathcal{O}_{C \cap D}) = \sum_{P \in C \cap D} (C.D)_P$$

Note that $\mathcal{O}_{C \cap D}$ is acyclic so $\chi(X, \mathcal{O}_{C \cap D}) = \dim_k H^0(X, \mathcal{O}_{C \cap D})$. The Euler characteristic is additive on exact sequences(as proven in a homework problem earlier in the course), so

$$\dim_k H^0(X, \mathcal{O}_{C \cap D}) = \chi(X, \mathcal{O}_{C \cap D}) = \chi(X, \mathcal{O}_C) - \chi(X, \mathcal{O}_X(-D) \otimes \mathcal{O}_C)$$

Now it is simply a matter of checking that the right hand side above satisfies properties (1) - (4) and by uniqueness in Definition 7 the result follows. ■

Note that a self intersection number of a curve, $C^2 = C.C$ is well defined in Definition 7 but that the above Proposition 8 does not help us compute it for one of the hypotheses is the lack of a common irreducible component; a condition that C obviously does not have with itself. On the other hand, in some cases this can be useful. Letting K denote a representative of the canonical class, i.e., $\mathcal{O}_X(K) = \omega_X$, we can consider the number K^2 as an invariant of the surface X. We see in [Har] the example of $X = \mathbf{P}^2$, yielding a $K^2 = 9$.

The above work serves to prove the classical Bezout theorem, after which, as noted many times in [EH], much of the field of intersection theory is based.

Theorem 9 (Bézout)**.** *Let C and D be two curves in $\mathbf{P}_k^2$ of degrees c and d. Then, counting with multiplicity, the number of intersections of C and D is cd.*

Proof. As proven in [Har, pp. 132-133], $\operatorname{Pic} \mathbf{P}^2 = \mathbb{Z}$ generated by a line, say l. Thus, because degree is constant on rational equivalence, we must have $C \sim cl$ and $D \sim dl$. Thus, by Definition 7, $C.D = (cl).(dl) = cd(l.l)$ by additivity and constancy on rational equivalence classes. But two transversal lines intersect in exactly one point which has a linear equation of definition so by Proposition 8 and property (4) above, coupled with Theorem 2, we have $l.l = l.l' = 1$ where l' intersects l transversally. ■

It is tempting to remark that we recover the Fundamental Theorem of Algebra from the case $k = \mathbb{C}$, and C is a line above, but note that the reasoning is circular: we have been assuming throughout that k is algebraically closed!

We now proceed to the proof of Theorem 1.

3 Riemann-Roch on Surfaces

We need one last lemma before we can prove Theorem 1. Above we have a nice intersection product, which we saw was part of Theorem 1, and the left hand side of the equality will be taken care of using Serre duality in a direct analogue of the case on curves. Thus, we need a way to relate the intersection product of a divisor to a geometric property. Thus, the following lemma:

Lemma 10. *Let C be nonsingular curve on X of genus g. Then the following holds:*

$$g = \frac{C.(C+K)}{2} + 1$$

Remark 11. Recall that on a curve, the arithmetic genus $p_a = (-1)^{\dim C}(1-\chi(C, \mathcal{O}_C)) = 1-\chi(C, \mathcal{O}_C)$ and the geometric genus $p_g = \Gamma(C, \omega_C)$ agree as a consequence of Serre duality. This does not hold on varieties of higher dimension. For surfaces, we have $p_a(X) = \chi(X, \mathcal{O}_X) - 1$.

Proof. By the adjunction formula [Har, p. 182], we have $\omega_C \cong (\omega_X \otimes \mathcal{O}_X(C)) \otimes \mathcal{O}_C$ and we have that $\omega_X \otimes \mathcal{O}_X(C) = \mathcal{O}_X(C+K)$. Applying Lemma 5 and Definition 7 as well as property (1) of the intersection product, we get that $\deg_C((\mathcal{O}_X(C+K)) \otimes \mathcal{O}_C) = C.(C+K)$. On the other hand, from Riemann-Roch on curves we get that $\deg \omega_C = 2g-2$. Solving for g, the equality follows. ∎

Now we are ready to prove the main theorem:

Theorem 1. *Let X be a surface and D a divisor on X. Let K be the canonical class, $l(D) = \dim_k H^0(X, \mathcal{O}_X(D))$, $s(D) = \dim_k H^1(X, \mathcal{O}_X(D))$ and the arithmetic genus of X, $p_a = \chi(\mathcal{O}_X) - 1$. Then*

$$l(D) - s(D) + l(K-D) = \frac{1}{2}D.(D-K) + 1 + p_a$$

Proof. We take care of the left hand side of the above equality using Serre duality. We have that $\dim_k H^2(X, \mathcal{O}_X(D)) = \dim_k H^0(X, \omega_X \otimes \mathcal{O}_X(D)^\vee) = \dim_k H^0(X, \mathcal{O}_X(K-D)) = l(K-D)$. By definition, $s(D) = \dim_k H^1(X, \mathcal{O}_X(D))$ so the left hand sum just becomes $\dim_k H^0(X, \mathcal{O}_X(D)) - \dim_k H^1(x, \mathcal{O}_X(D)) + \dim_k H^2(X, \mathcal{O}_X(D)) = \chi(X, \mathcal{O}_X(D))$. On the other side, note that by definition, $1 + p_a = \chi(X, \mathcal{O}_X)$. Thus we wish to prove the following equality

$$\chi(X, \mathcal{O}_X(D)) = \frac{D.(D-K)}{2} + \chi(X, \mathcal{O}_X)$$

Note that both sides only depend on the rational equivalence class of D. Thus, as in the proof of Theorem 6, we can write $D \sim C - E$ where C, E are nonsingular curves. Using the fact that $\mathcal{O}_X(-C)$ is the ideal sheaf defining C, we get the following exact sequences:

$$0 \longrightarrow \mathcal{O}_X(-E) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_E \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}_X(-C) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \longrightarrow 0$$

And, tensoring with $\mathcal{O}_C$, which is exact because $\mathcal{O}_C$ is invertible, we get

$$0 \longrightarrow \mathcal{O}_X(C-E) \longrightarrow \mathcal{O}_X(C) \longrightarrow \mathcal{O}_X \otimes \mathcal{O}_E \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(C) \longrightarrow \mathcal{O}_X(C) \otimes \mathcal{O}_C \longrightarrow 0$$

But χ is additive on exact sequences so we have, abbreviating slightly so all Euler characteristics are taken over the surface X,

$$0 = \chi(\mathcal{O}_X(C - E)) - \chi(\mathcal{O}_X(C)) + \chi(\mathcal{O}_X(C) \otimes \mathcal{O}_E) - \chi(\mathcal{O}_X) + \chi(\mathcal{O}_X(C)) - \chi(\mathcal{O}_X(C) \otimes \mathcal{O}_C)$$

Cancelling, we get

$$\chi(\mathcal{O}_X(C - E)) = \chi(\mathcal{O}_X) + \chi(\mathcal{O}_X(C) \otimes \mathcal{O}_C) - \chi(\mathcal{O}_X(C) \otimes \mathcal{O}_E)$$

The Riemann-Roch theorem for curves gives

$$\chi(\mathcal{O}_X(C) \otimes \mathcal{O}_C) = \deg(\mathcal{O}_X(C) \otimes \mathcal{O}_C) + 1 - g_C$$
$$\chi(\mathcal{O}_X(C) \otimes \mathcal{O}_E) = \deg(\mathcal{O}_X(C) \otimes \mathcal{O}_E) + 1 - g_E$$

We can now apply Lemma 5 to get $\deg(\mathcal{O}_X(C) \otimes \mathcal{O}_C) = C.C = C^2$ and $\deg(\mathcal{O}_X(C) \otimes \mathcal{O}_E) = C.E$. Thus the above turns into

$$\chi(\mathcal{O}_X(C) \otimes \mathcal{O}_C) = C^2 + 1 - g_C$$
$$\chi(\mathcal{O}_X(C) \otimes \mathcal{O}_E) = C.E + 1 - g_E$$

Now we can apply Lemma 10 to give us

$$g_C = \frac{C.(C + K)}{2} + 1$$
$$g_E = \frac{E.(E + K)}{2} + 1$$

Combining all of this, we get

$$\begin{aligned}
\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X(C - E)) &= \chi(\mathcal{O}_X) + \chi(\mathcal{O}_X(C) \otimes \mathcal{O}_C) - \chi(\mathcal{O}_X(C) \otimes \mathcal{O}_E) \\
&= \chi(\mathcal{O}_X) + C^2 + 1 - g_C - C.E + 1 - g_E \\
&= \chi(\mathcal{O}_X) + C^2 + 1 - (\frac{C.(C + K)}{2} + 1) - C.E + 1 - (\frac{E.(E + K)}{2} + 1) \\
&= \chi(\mathcal{O}_X) + \frac{1}{2}(C.(C - E - K) - E.(C - E - K) = \chi(\mathcal{O}_X) + \frac{1}{2}((C - E).(C - E - K)) \\
&= \chi(\mathcal{O}_X) + \frac{D.(D - K)}{2}
\end{aligned}$$

This is exactly the equality we needed to prove from above, so we are done. ■

Now that we have proven this theorem, there are many avenues open for further exploration. Further generalization of the Riemann-Roch theorem is done in [EH] and in [Ful], but requires more complicated machinery, particularly the notion of Chern classes; of course, due to the larger number of cohomology groups, the theorem tells us less and less about the hypersurfaces as the dimension goes up. For surfaces, as with curves, we could use Theorem 1 to answer some classification questions as well as entering into the realm of enumerative geometry on surfaces, but these extend beyond the scope of the paper.

References

[EH] David Eisenbud and Joe Harris. *3264 and All That.*

[Ful] William Fulton. *Intersection Theory.* 2nd edition.

[Har] Robin Hartshorne. *Algebraic Geometry.*

[Mat] Hideyuki Matsumura. *Commutative Ring Theory.*