Nonsingular Varieties

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25 July, 2017

1 Introduction

Nonsingularity is the algebrogeometric equivalent of differentiability in a suitably defined sense. Over $\mathbb{C}$, for instance, the nonsingular varieties correspond to complex manifolds. Over fields of positive characteristic, various complications arise, and notions distinguishing such terms as *smooth*, *normal*, *nonsingular*, etc. are of vital importance. For the sake of simplicity, we restrict ourselves to working over $\mathbb{C}$, although we occasionally mention ways in which the sequel applies to varieties over arbitrary fields. We begin by defining nonsingularity of affine varieties and giving examples before applying an important result of Zariski to extend the theory to arbitrary varieties.

All rings will be taken to be noetherian and commutative with identity.

2 Algebraic Preliminaries

We must first introduce a few algebraic preliminaries. The notion of an algebraic derivative is central.

Definition 1. Let A be a k-algebra, M an A-module. A map $d : A \to M$ such that for all $u, v \in A$, $a \in k$,

- $d(u + v) = d(u) + d(v)$
- $d(a) = 0$
- $d(uv) = ud(v) + d(u)v$

is called a *derivation*.

We care about derivations $d : A \to A$, where A is a finite type k algebra, in particular when d is an *algebraic partial derivative*. This will serve as our motivating example

Example 2. Let $f \in k[x_1, ..., x_n]$. Then we can define

$$df = \frac{\partial f}{\partial x_i}$$

where

$$\frac{\partial}{\partial x_i}(x_j) = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$

is our normal notion of derivative. Of course, in fields in which we can apply analysis, such as $\mathbb{R}$ or $\mathbb{C}$, this definition agrees with the analytic notion; the advantage here is that this definition greatly extends the analytic one to more general fields. Note that over fields of positive characteristic, some notions may not match our intuition. For instance if $\operatorname{char} k = p > 0$, then $d(x^p) = px^{p-1} = 0$, thus yielding a nonconstant polynomial with derivative 0.

The next notion that we need is that of a regular local ring. For the sake of completeness, we recall a few facts from earlier lectures.

Definition 3. A ring R is *local* if and only if it has a unique maximal ideal, $\mathfrak{m}$ (recall that Zorn's lemma implies that all nontrivial rings have at least one maximal ideal). The *Krull Dimension* of a ring R is

$$\dim R = \max_{\mathfrak{p} \subset R} (\operatorname{ht} \mathfrak{p})$$

or equivalently is the dimension of $\operatorname{Spec} R$ as a topological space.

We recall the following result without proof:

Proposition 4. *Let $(R, \mathfrak{m})$ be a local ring. Let S be the set of ideals of definition, I, of R, i.e., there exist $m, n \in \mathbb{N}$ such that $\mathfrak{m}^m \subset I \subset \mathfrak{m}^n$. Let $\delta(I)$ be the minimal number of generators of I. Then*

$$\dim R = \min_{I \in S} \delta(I)$$

Definition 5. If $(R, \mathfrak{m})$ is a local ring, then let κ be its quotient field, i.e., $R/\mathfrak{m} = \kappa$. Then $\mathfrak{m}/\mathfrak{m}^2$ is naturally endowed with a κ vector space structure. We say that R is *regular local* (or just *regular*) if

$$\dim_\kappa \mathfrak{m}/\mathfrak{m}^2 = \dim R$$

Remark 6. Note that using Proposition 4, because $\mathfrak{m}$ in particular is an ideal of definition, we get that in general $\dim_\kappa \mathfrak{m}/\mathfrak{m}^2 \geq \dim R$. Thus regularity occurs when this dimension is minimal. A theorem from commutative algebra states that a ring is regular local if and only if it is a UFD.

Remark 7. For the experts, note that we can extend our definition of regularity to include all Noetherian rings and drop the local assumption. To do this, we prove that regular local rings are exactly those with finite projective dimension. This property is stable under localization so we get a well defined concept of regularity if we say that a ring R is regular if and only if for all primes $\mathfrak{p} \subset R$, $R_\mathfrak{p}$ is regular local.

Example 8. Note that if $A = k[x_1, ..., x_n]$ then $A_{(x_1,...,x_n)}$ is regular local. To see this, note that $\dim A = n$. In fact, we can localize at any closed point of $\mathbb{A}^n_k$ to get a regular local ring.

Example 9. As an example of a local ring that is not regular, consider $A = (k[x, y]/(x^3 - y^2))_{(x,y)}$. Note that $\dim A = 1$ because $\dim k[x, y] = 2$ and we are quotienting out by one element that is not in the minimal prime. However, $\dim_\kappa \mathfrak{m}/\mathfrak{m}^2 = 2$ with $\{x, y\}$ as a basis.

Recall that if $X = V(f_1, ..., f_r) \subset \mathbb{A}^n$, then the local ring of X at the point $P = (a_1, ..., a_n)$ is defined to be $\mathcal{O}_P = (k[x_1, ..., x_n]/(f_1, ..., f_r))_{(x_1-a_1,...,x_n-a_n)}$. We are now ready to proceed to nonsingularity.

3 Nonsingularity

Before we define nonsingularity in general, we will consider the affine case. Our definition of nonsingularity is motivated by our notion of smoothness of manifolds. Thus a natural choice is to use a derivative and check when the Jacobian has full rank. This will be our starting point. Let $X \subset \mathbb{A}^n_k$ be an algebraic set defined by the ideal $I = (f_1, ..., f_t)$. We define the *Jacobian* at the point P, $J_P = (\frac{\partial f_i}{\partial x_j}(P))$. Note that this agrees with our intuition coming from the study of manifolds.

Definition 10. If X is an affine variety of dimension r as above, then for $P \in X$, we say that X is *nonsingular* at P if $\operatorname{rk} J_P = n - r$. If X is not nonsingular at P, then we say that X is singular at P. If X is nonsingular at all points P, then we say that X is nonsingular. Otherwise, X is singular.

As a first example, let us consider $X = V(x^3 - y^2) \subset \mathbb{A}^2$. Then we get that if $P = (a, b)$, then

$$J_P = \begin{bmatrix} 3a^2 & -2b \end{bmatrix}$$

Note that as long as a, b are not both 0, then $\operatorname{rk} J_P = 1 = 2-1$ and X is nonsingular at P. But for $P = (0, 0)$, $J_P = 0$ and X is singular. As an example of a nonsingular variety, consider $X = V(x^2 - y)$.

The above definition is all very well, but it has a few key drawbacks. First, it is not a priori obvious that the singularity of a point is independent of the choice of generators of I, although row reductions yield a solution to this quite quickly. More importantly, it is not at all clear that nonsingularity is independent of affine embedding and that nonsingularity behaves well with respect to isomorphisms. Perhaps most seriously, our definition does not readily lend itself to generalizing to non affine varieties. The following result is the key theorem that allows us to do all of these.

Theorem 11. *Let $X \subset \mathbb{A}^n$ be an affine variety. Then X is nonsingular at P if and only if $\mathcal{O}_P$ is a regular local ring.*

Proof. Let $P = (a_1, ..., a_n) \in k^n$ and let $\mathfrak{p}$ be the corresponding maximal ideal in $A = k[x_1, ..., x_n]$ (i.e. $\mathfrak{p} = (x_1 - a_1, ..., x_n - a_n)$). We define a derivation $\theta_P : A \to k^n$ by

$$f \mapsto (\frac{\partial f}{\partial x_1}(P), ..., \frac{\partial f}{\partial x_n}(P))$$

We note two things. First, $\theta_P(x_i - a_i) = (0, ..., 1, ..., 0) \in k^n$, with a 1 in the i^{th} spot and zeros elsewhere. Second, $\theta((x_i - a_i)(x_j - a_j) = (0, 0, ..., x_j - a_j, ..., x_i - a_i, ..., 0)|_{x_i = a_i, x_j = a_j} = 0$. Thus, $\theta_P(\mathfrak{p}^2) = 0$. Thus we see that θ_P induces an isomorphism $\theta : \mathfrak{p}/\mathfrak{p}^2 \to k^n$ of vector spaces.

Now let $\mathfrak{q} = (f_1, ..., f_t) \subset A$ be a prime ideal generated by the f_i and let $X = V(\mathfrak{q}) \subset \mathbb{A}^n$. By definition of θ_P, we have that $\operatorname{rk} J_P = \dim_k \theta_P(\mathfrak{q})$ when $\theta_P(\mathfrak{q}) \subset k^n$ is viewed as a k-vector space. We can now apply the isomorphism θ and note that $\theta^{-1} \circ \theta_P(\mathfrak{q}) = (\mathfrak{q} + \mathfrak{p}^2)/\mathfrak{p}^2 \subset \mathfrak{p}/\mathfrak{p}^2$ because we are simply taking $\mathfrak{q}$ and modding out by $\mathfrak{p}^2$. We note that the local ring of X at P is just

$$\mathcal{O}_{X,P} = (A/\mathfrak{q})_{\mathfrak{p}}$$

with maximal ideal $\mathfrak{m} = (\mathfrak{q} + \mathfrak{p})/\mathfrak{q}$. Thus we get that $\mathfrak{m}^2 = (\mathfrak{q} + \mathfrak{p}^2)/\mathfrak{q}$. Modding out, we get that $\mathfrak{m}/\mathfrak{m}^2 = \mathfrak{p}/\mathfrak{q} + \mathfrak{p}^2$. Now, let $V = \mathfrak{p}/\mathfrak{p}^2$. Then letting $\mathfrak{q}'$ be the image of $\mathfrak{q}$ in $\mathfrak{p}/\mathfrak{p}^2$, we get that $V/\mathfrak{q}' \oplus \mathfrak{q}' = V$ and counting dimensions, we get

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 + \dim_k \mathfrak{p}/(\mathfrak{q} + \mathfrak{p}^2) = \dim_k \mathfrak{p}/\mathfrak{p}^2 = n$$

Thus we have that $\dim_k \mathfrak{m}/\mathfrak{m}^2 + \operatorname{rk} J_P = n$ or, equivalently, $\operatorname{rk} J_p = n - \dim_k \mathfrak{m}/\mathfrak{m}^2 \leq n - \dim \mathcal{O}_{X,P}$ by Remark 6. By a result from an earlier lecture, $\dim \mathcal{O}_{X,P} = r$. By the definition of regularity, the result follows. ■

With Theorem 11, we are able to extend our definition

Definition 12. Let X be a variety, $P \in X$ a point. We say that X is *nonsingular* at P if $\mathcal{O}_P$ is a regular local ring; otherwise we say that X is *singular* at P. We say that X is *nonsingular* if X is nonsingular at all points P; otherwise X is singular.

Remark 13. It is important to note several important things. One is that Theorem 11 immediately resolves all three of our troubles with the naive definition. If $X = V(I)$ is a variety, then it is clear that the particular choice of generators of I chosen are irrelevant. More importantly, if X' is another variety and $\phi : X \to X'$ is an isomorphism, then for all $P \in X$, $\mathcal{O}_{X,P} \cong \mathcal{O}_{X', \phi(P)}$ so X is nonsingular at P if and only if X' is nonsingular at $\phi(P)$. Finally, because the stalk (local ring) at a point is defined for arbitrary algebraic sets, it is clear that this definition is much more general than the naive one.

Example 14. We considered $X = V(y^2 - x^3) \subset \mathbb{A}^2$ above as an example of a variety that is singular at P, the origin. In Example 9, we saw that $\mathcal{O}_{X,P}$ was not regular.

Example 15. Above, we saw that $X = V(y - x^2) \subset \mathbb{A}^2$ is nonsingular. Let $P = (a, a^2) \in X$. Then we have that

$$\mathcal{O}_{X,P} = (k[x,y]/(y - x^2))_{(x-a, y-a^2)} \cong k[x]_{(x-a, x^2-a^2)} \cong k[x]_{(x)}$$

Note that this is clearly a regular local ring.

Example 16. We can even extend our definition to algebraic sets. Let $X = V(xy) \subset \mathbb{A}^2$, the union of two lines meeting each other at the origin. Let $P = (0, 0)$. We have

$$\mathcal{O}_P = (k[x, y]/(xy))_{(x,y)}$$

We have $\dim \mathcal{O}_P = 1$ but $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 2$. This is a special case of the more general fact that the intersection of two or more irreducible components is always singular.

It may be difficult to tell from the examples cited to date, but there are two salient remarks to be made. The first is that demonstrating nonsingularity from the definition is not an easy thing to compute. It is in general not fun to compute the dimension of a ring and to prove exactly what that dimension is; the naive way was much easier, albeit less general. It might behoove us to extend this method to projective varieties, and so we do exactly that.

Proposition 17 (Jacobi Criterion)**.** *Let $X = V_+(I) \subset \mathbb{P}^n$ be a projective variety of dimension r such that $f_1, .., f_t \in S = k[x_0, .., x_n]$ are homogeneous polynomials that minimally generate I. Let $P \in Y$ be given by homogeneous coordinates $[a_0 : ... : a_n]$. Then Y is nonsingular at P if and only if* $\operatorname{rk} J_P = n - r$.

Before we begin the proof, we need a lemma.

Lemma 18 (Euler)**.** *Let $f \in k[x_0, ..., x_n]$ be a homogeneous polynomial of degree d. Then*

$$\sum_{i=0}^{n} x_i \frac{\partial f}{\partial x_i} = df$$

Proof. Note that by linearity and homogeneity, it suffices to prove the result on monomials. Let

$$f = \prod_{i=0}^{n} x_i^{\alpha_i}$$

Then

$$x_i \frac{\partial f}{\partial x_i} = \alpha_i x_i x_i^{\alpha_i - 1} \prod_{j \neq i} x_j^{\alpha_j} = \alpha_i \prod_{j=0}^{n} x_j^{\alpha_j} = \alpha_i f$$

Thus, summing over all i gives

$$\sum_{i=0}^{n} x_i \frac{\partial f}{\partial x_i} = \sum_{i=0}^{n} \alpha_i f = df$$

■

Proof of Proposition 17. We must first note that the rank of J_P is well defined because J_P is not. We define $J_P = (\frac{\partial f_i}{\partial x_j}(\alpha_0, ..., \alpha_n))$. Given $P' \sim P$, then $P' = (\lambda a_0, ..., \lambda a_n)$ for some $\lambda \in k^\times$. If f is homogeneous then so is $\frac{\partial f}{\partial x_j}$ so if we let $\deg f_i = d_i$ then

$$J_{P'} = (\frac{\partial f_i}{\partial f_j}(P')) = (\lambda^{d_i - 1} \frac{\partial f_i}{\partial x_j}(P))$$

So the rows of $J_{P'}$ are the rows of J_P scaled by constants. Thus $\operatorname{rk} J_P = \operatorname{rk} J_{P'}$ and this is a well defined notion.

Now note that because regualrity is a local condition, it suffices to restrict to an affine open. After some reordering of the subscripts, it suffices to show the result on $U_0 = \{z \in \mathbb{P}^n | z_0 \neq 0\}$. Let $y_i = \frac{x_i}{x_0}$. Then X is regular at $P = (x_0 : ... : x_n)$ if and only if $X \cap U_0$ is regular at $P' = (1, y_1, ..., y_n)$. Let

$$J'_P = \left(\frac{\partial}{\partial y_j}(f_i(1, y_1, ..., y_n)) \right)$$

Then we get by Theorem 11 that $X \cap U_0$ is regular at P' if and only if $\operatorname{rk} J'_P = n-r$. Thus it suffices to show that $\operatorname{rk} J'_P = \operatorname{rk} J_P$. Up to the change of variable $\frac{x_i}{x_0} = y_i$, note that J'_P is the right $(n-1) \times t$ submatrix of J_P and that the left most column of J_P is given by $\frac{\partial f_i}{\partial x_0}(P)$. But by Lemma 18,

$$\sum_{j=0}^{n} x_j \frac{\partial f_i}{\partial x_j} = df_i$$

so, solving for $\frac{\partial f_i}{\partial x_0}$, and noting that for any point $P \in X$, we have $f_i(P) = 0$, we get for all $P \in X$,

$$\begin{aligned} \frac{\partial f_i}{\partial x_0} &= d\frac{f_i}{x_0} - \frac{1}{x_0}\sum_{j=1}^{n} x_j \frac{\partial f_i}{\partial x_j} \\ &= -\sum_{j=1}^{n} y_j \frac{\partial f_i}{\partial y_j} \end{aligned}$$

Thus this first column of J_P is a linear combination of the columns in J'_P so $\operatorname{rk} J_P = \operatorname{rk} J'_P$ and we are done. ∎

Example 19. Consider $X = V_+(y^2 z - x^2) \subset \mathbb{P}^2$ at the point $P = [0:0:1]$. Note that this is just embedding $V(y^2 - x^3) \subset \mathbb{A}^2 \hookrightarrow \mathbb{P}^2$ with the point $(0,0) \mapsto [0:0:1]$. Thus we expect X to be singular at P. Indeed, we get

$$J_P = \begin{bmatrix} -3x^2 & 2yz & y^2 \end{bmatrix}$$

so $J_p = 0$ at P as expected.

Example 20. One thing to remember is that $0 \notin \mathbb{P}^n$. Thus, consider the variety $X = V_+(x^2+y^2+z^2) \subset \mathbb{P}^2$. Let $P = [a:b:c]$. Then,

$$J_P = 2\begin{bmatrix} a & b & c \end{bmatrix}$$

with full rank equal to 1. Clearly $(0,0,0)$ is on the surface Y with rank 0 if we were to consider $Y = V(x^2+y^2+z^2) \subset \mathbb{A}^3$, but this is not the case for $\mathbb{P}^2$.

The second thing to note is that the set of points that are singular in a variety Y, Y^{sing} is pretty small. We can make this statement rigorous.

Theorem 21. *Let Y be a variety. Then $Y^{sing} \subset Y$ is a proper closed subset.*

Proof. Recall that any variety Y can be covered by affine opens, i.e., there exist Y_i for $1 \leq i \leq m$ such that $Y = \bigcup Y_i$. Clearly, it suffices to prove the result for the affine opens, so assume Y is affine of dimension r. Thus $Y = V(f_1, ..., f_t) \subset \mathbb{A}^n$. By Theorem 11, the singular points of Y are those points P such that $\operatorname{rk} J_P < n-r$. Let $J = (\frac{\partial f_i}{\partial x_j})$. Then we get less than full rank if one of the determinants of one of the $(n-r)\times(n-r)$ submatrices is 0. Let $d_1, ..., d_m \in \mathcal{O}_{\mathbb{A}^n}$. Then one a point $P = (a_1, ..., a_n)$ has J_P less than full rank if and only if $\mathfrak{p} = (x_1 - a_1, ..., x_n - a_n) \supset (d_1, ..., d_m) = I'$. Thus, $Y^{\text{sing}} = V(I) \cap V(I') \subset \mathbb{A}^n$ is closed.

To see that Y^{sing} is proper, we recall that if Y is a variety then it is birational to a hypersurface in $\mathbb{P}^n$. It suffices to check on affine opens; thus it suffices to show the result for $Y = V(f) \subset \mathbb{A}^n$. Then the d_j in the above paragraph become particularly simple: $d_j = \frac{\partial f}{\partial x_j}$. We know that f is irreducible so $r(f) = (f)$ and we can apply the Nullstellensatz to get taht $d_j \in (f)$. But $\deg d_j \leq \deg f - 1$ so $f \nmid d_j$ unless $d_j = 0$. Thus we have that $\frac{\partial f}{\partial x_j} = 0$ for all $1 \leq j \leq n$ and so f is constant, contradicting the fact that Y is a variety. ∎

Remark 22. The above proof actually holds in greater generality. In particular if Y is a variety over k, with $\operatorname{char} k = p > 0$, the result still holds. To see this, note that every step above holds until the part where we assumed $\frac{\partial f}{\partial x_j} = 0$ for all j implies that $f = 0$. In the positive characteristic case, all this means is that $f \in k[x_1^p, ..., x_n^p] = \mathcal{O}_{Y^{(p)}}$. But k is algebraically closed so each coefficient is a p^{th} power. Thus there exists some $g \in \mathcal{O}_Y$ such that $f = g^p$, contradicting the fact that f is irreducible.

Intuitively, Theorem 21 tells us exactly what we wanted: that Y^{sing} is small. In an intuitive sense, the closed sets in the Zariski topology are relatively small and the open sets are relatively big. In the case of curves, we have the profinite topology and so Y^{sing} is actually a finite set. Moreover, the intuition of nonsingular sets being small extends to arbitrary algebraic sets. Even if Y is irreducible, then the extra points in Y^{sing} that come from intersections of irreducible components are of smaller dimension than Y and so are suitably "small" as desired.