Representation Theory and Combinatorics
$\mathfrak{sl}_2$ and Applications

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1 Introduction

Representation theory is a fascinating field of math with applications to group theory, number theory, algebraic geometry, physics, and much more, as well as being an interesting study in its own right. We will be restricting our focus specifically to the representations of the Lie algebra $\mathfrak{sl}_2$. This lecture (series) will be in two parts. We will begin by setting up the representation theory of $\mathfrak{sl}_2$, in particular finding all of the irreducible representations and computing the representation ring. No background in representation theory is assumed. The second part of the series will be devoted to applying our knowledge of representation theory to some problems in combinatorics. The mathematician Richard P. Stanley was instrumental in the development of these techniques, exemplified by [Sta89], with results from [Alm] also covered. Partly because the primary aim of these two lectures is to apply representation theory to combinatorics, and partly because of the richness of the field, we will be completely unable to do the discipline justice in these two lectures. The interested reader is referred to [FH].

2 Lie Algebras and Representations

2.1 Definitions and Basics

We work exclusively over $\mathbb{C}$ with finite dimension.

Definition 1. A *Lie algebra* is a vector space V over $\mathbb{C}$ equipped with a map $[\cdot,\cdot] : V \times V \to V$ called the *Lie bracket* that satisfies

- Bilinearity, i.e, for any $\alpha, \beta \in \mathbb{C}$, $X, X', Y \in V$, we have

$$[\alpha X + \beta X', Y] = \alpha[X, Y] + \beta[X', Y]$$
$$[Y, \alpha X + \beta X'] = \alpha[Y, X] + \beta[Y, X']$$

- Alternating on V, i.e, for all $X \in V$, $[X, X] = 0$
- Jacobi Condition, i.e., for all $X, Y, Z \in V$,

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

The *dimension* of the Lie algebra V is just its dimension as a vector space over $\mathbb{C}$. A *morphism* of Lie algebras V, W is a linear map $\rho : V \to W$ that is compatible with the bracket, i.e.,

$$\rho([X, Y]) = [\rho(X), \rho(Y)]$$

Remark 2. Note that we can combine the first two conditions to get anticommutativity, i.e., for all $X, Y \in V$, $[X, Y] = -[Y, X]$. This is because we have

$$0 = [X + Y, X + Y] = [X, X] + [X, Y] + [Y, X] + [Y, Y] = [X, Y] + [Y, X]$$

On the other hand, the Jacobi condition guarantees that all associative Lie algebras are trivial in that the bracket is just the 0 map.

Example 3. Let $V = \mathbb{C}^3$ and let $[X, Y] = X \times Y$ the cross product. It is easy to see that this forms a Lie Algebra.

Example 4. The classical examples all come from matrices. We define $\mathfrak{gl}_n = M_n(\mathbb{C})$ with the bracket $[X, Y] = XY - YX$. It is easy to check that this forms a Lie algebra.

The theory of Lie algebras is rich, but we will be restricting ourselves to discussion of their *representations*.

Definition 5. A *representation* of a Lie algebra $\mathfrak{g}$ is a morphism to $\mathfrak{gl}(V)$ with dimension defined as the dimension of V. A *subrepresentation* is a subspace $V \subset \mathfrak{gl}_n$ that is stable under the action of $\mathfrak{g}$. Given two representations V and W, we can form their direct sum $V \oplus W$ and their tensor $V \otimes W$ such that if $v \in V$, $w \in W$ and $X \in \mathfrak{g}$ then

$$X \cdot (v, w) = (X \cdot v, X \cdot W)$$
$$X \cdot (v \otimes w) = (X \cdot V) \otimes (X \cdot W)$$

An *irreducible representation* is a representation that has only the trivial representation as a subrepresentation. A *semisimple representation* is a representation that is decomposable as a direct sum of irreducible representations. A *morphism* of representations is a linear map $\phi : V \to W$ such that for all $X \in \mathfrak{g}$, $v \in V$, $X \cdot \phi(v) = \phi(X \cdot v)$.

An easy example of a representation is taking the identity map on $\mathfrak{g}_n$. We recall two key theorems without proof.

Theorem 6. *Let $\mathfrak{g}$ be a semisimple Lie algebra. Then any representation of $\mathfrak{g}$ is semisimple.*

Proof. See [FH, Theorem 9.19] ■

Theorem 7 (Lie)**.** *Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a solvable, semisimple Lie algebra. Then there exists a nonzero $v \in V$ such that v is an eigenvector for all $X \in \mathfrak{g}$.*

Proof. See [FH, Theorem 9.11] ■

We now narrow our focus to a particular Lie algebra, $\mathfrak{sl}_2$.

2.2 The Representation Theory of $\mathfrak{sl}_2$

We begin with a definition.

Definition 8. Let $\mathfrak{sl}_2$ be the set of traceless 2×2 matrices with bracket

$$[X, Y] = XY - YX$$

Note that this is clearly a Lie algebra because it inherits its bracket from that in Example 4. We wish to study the representation theory of $\mathfrak{sl}_2$. We first state without proof that $\mathfrak{sl}_2$ is semisimple. Thus we have

Corollary 9. *All representations of $\mathfrak{sl}_2$ are semisimple.*

Proof. This is a special case of Theorem 6. ■

Thus it suffices to consider the irreducible representations of $\mathfrak{sl}_2$. First, we would like a nice way to think about $\mathfrak{sl}_2$. We know that it is a vector space (of dimension 3) so it has a basis. Note that a representation of $\mathfrak{sl}_2$ is uniquely determined by its action on this basis. Let us choose

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$
$$F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$
$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

A quick computation shows the following relations:

$$[E, F] = H$$
$$[H, E] = 2E$$
$$[H, F] = -2F$$

Thus, a dimension n representation of $\mathfrak{sl}_2$ is uniquely determined by choosing three $n \times n$ matrices that satisfy the above relations. Through abuse of notation, we will identify the elements of the above basis of $\mathfrak{sl}_2$ and its image in the representation.

We now begin some computations. Let $v \in V$ be an eigenvector for H with eigenvalue λ:

$$Hv = \lambda v$$

We now consider

$$H(Ev) = (HE)v = ([H, E] + EH)v = (2E + EH)v = 2Ev + E(\lambda v) = (\lambda + 2)Ev$$

Thus we note that Ev is also an eigenvector for H with eigenvalue $\lambda + 2$! Similarly,

$$H(Fv) = ([H, F] + FH)v = (-2F + FH)v = (\lambda - 2)v$$

Let V_λ be the eigenspace of H with eigenvalue λ. Then we see by above that

$$E : V_\lambda \to V_{\lambda+2}$$
$$F : V_\lambda \to V_{\lambda-2}$$

Now, by Theorem 7, we have that H has at least one eigenvalue, but it has a finite number of eigenvalues by finiteness of dimension. Let λ be the maximal eigenvalue and let $v \in V$ be nonzero such that $Hv = \lambda v$. Let us define $W_v = \text{Span}\{v, Fv, F^2v, ...\}$. Then we have that $Ev = 0$ because λ is maximal. Then we show

Proposition 10. *Let W be defined as above. Then W is an irreducible representation of $\mathfrak{sl}_2$. Moreover, all irreducible representatcions of $\mathfrak{sl}_2$ arise in this way.*

Proof. If W_v is a representation, it is clear that it is irreducible. Moreover, given that W_v is a representation, then any representation V clearly contains a W_v for some nonzero $v \in V$ so if V is irreducible then $V = W_v$. Thus it suffices to show that the actions of E, F, H on W_v preserve W_v. Clearly $F(F^n v) = F^{n+1}v \in W_v$. We saw above that $HF^n v = (\lambda - 2n)v$. Thus it suffices to show that $EF^n v \in W_v$. We show by induction that $EF^n v = (n-1)(\lambda - n)F^{n-1}v \in W_v$. For $n = 0$, we have $Ev = 0$ as above. Suppose the result holds for $n - 1$. Then we have

$$EF^n v = EFF^{n-1}v = [E, F]F^{n-1}v + FEF^{n-1}v$$
$$= HF^{n-1}v + F(n-2)(\lambda - n + 1)F^{n-2}v = (\lambda - 2(n-1) + (n-2)(\lambda - n + 1)F^{n-1}v = (n-1)(\lambda - n)F^{n-1}v$$

Thus E, F, H take W_v to W_v and so W_v is a representation. ■

Note that $EF^n v = 0$ if and only if $(\lambda - n)(n-1) = 0$. By the finiteness of dimension we must have λ is a positive integer. Thus, letting V'_j be the one dimensional space spanned by w such that $Hw = jw$, we get that for some n,

$$W_v = \bigoplus_{i=0}^{n} V'_{n-2i}$$

Let V_n be the irreducible representation with maximal eigenvalue (we will call these *weights* from now on) n. Then we have proven

Theorem 11. *Each and every irreducible representation of $\mathfrak{sl}_2$ is uniquely determined by its maximal weight $n-1$ and is termed V_n. Then $\dim V_n = n$ and V_n is the sum of one dimensional weight spaces each having weight $n-1-2i$ for $0 \leq i \leq n-1$. All finite dimensional representations of $\mathfrak{sl}_2$ are sums of V_n's and can be thought of as*

$$W = \bigoplus_j V_{n_j}^{\oplus a_j}$$

Definition 12. Let W be a representation of $\mathfrak{sl}_2$. A weight space, W_i of weight i is the eigenspace of H each of whose vectors has eigenvalue n_i. For the remainder of the work, after fixing a representation W, we reserve $d_i = \dim W_i$

Definition 13. A sequence $\{a_1, ..., a_n\}$ is *unimodal* if there exists some $k \in \{1, 3, ..., n\}$ such that

$$a_1 \leq a_2 \leq ... \leq a_k \geq a_{k+1} \geq a_{k+2} \geq ... \geq a_n$$

Such a sequence is *symmetric* if $a_i = a_{n-i}$.

Corollary 14. *For any representation W of $\mathfrak{sl}_2$, the sequences $\{d_{2i}\}_{i\in\mathbb{Z}}$ and $\{d_{2i+1}\}_{i\in\mathbb{Z}}$ are unimodal and symmetric about 0.*

Proof. By Theorem 11, the statement holds for irreducible representations. By Corollary 9, W is semisimple. But the sequences corresponding to a direct sum is the sum of the sequences so the result holds in general. ∎

We note that Corollary 14 is the foundation of our applications to combinatorics. The technique involves associating a combinatorial object to a representation of $\mathfrak{sl}_2$ with the sequence that we want to count being associated to the d_i. Then Corollary 14 gives us symmetry and unimodality. Before we begin the combinatorics, though, we need to introduce another concept.

2.3 The Representation Ring

Temporarily returning to the general theory, we quickly prove an important lemma.

Lemma 15 (Schur)**.** *If V, W are irreducible representations and $\phi : V \to W$ is a morphism, then either $\phi = 0$ or ϕ is an isomorphism.*

Proof. Given a morphism of representations $\phi : V \to W$ it is easy to see that the kernel and image are both subrepresentations. But V, W are irreducible so if the kernel is not all of V, in which case ϕ is constant, then ϕ is injective and the image must be all of W by irreducibility. ∎

Recall by Theorem 6 that any representation W of a semisimple Lie algebra $\mathfrak{g}$ can be expressed as a (direct) sum

$$W = \sum w_i W_i$$

For $w_i \in \mathbb{N} \cup \{0\}$ and W_i distinct irreducible representations. From Lemma 15, it is clear that such an expression is unique. Thus it may behoove one to study the commutative monoid generated by the irreducible representations, or, if one likes a little bit more structure, to study the free $\mathbb{Z}$-module generated by the irreducible representations. Let R be this abelian group.

Remark 16. For lovers of category theory, note that Lemma 15 and Theorem 6 imply that R is exactly the *Grothendieck group* associated to the category of representations of $\mathfrak{g}$, where the abelian group is generated on isomorphism classes of representations, $[V]$, and quotiented out by the relations $[V] + [V'] - [V \oplus V']$. Thus $R = K_0(\mathbb{C}[\mathfrak{sl}_2])$.

There is a natural way to impose a ring structure on R. If V, V' are two representations of $\mathfrak{g}$, then let $[V][V'] = [V \otimes V']$. This turns R into a ring.

Definition 17. Given a Lie algebra $\mathfrak{g}$, the *representation ring* of $\mathfrak{g}$, called R is the free abelian group generated by isomorphism classes of representations quotiented out by the additive relation on short exact sequences, and endowed with a product through the tensor product. Note that R is a commutative, associative ring.

Remark 18. For $\mathfrak{g}$ a semisimple Lie algebra, the discussion above states that if $\{W_i\}$ is the set of irreducible representations, then

$$R = \bigoplus \mathbb{Z} W_i$$

Combining the above results, we get

Proposition 19. *Let $\mathfrak{g} = \mathfrak{sl}_2$. Then we get that the representation ring*

$$R = \mathbb{Z} \oplus \bigoplus_{n \in \mathbb{N}} \mathbb{Z} V_n$$

where V_n is the unique irreducible representation of dimension n.

Note that there is a bijective correspondence between finite dimensional representations of $\mathfrak{sl}_2$ and elements of R with all coefficients nonnegative integers. From now on, fix our Lie algebra to be $\mathfrak{sl}_2$ and R to be the representation ring.

Definition 20. We turn R into a $\mathbb{Z}/2\mathbb{Z}$ graded module by letting

$$R^e = \{\sum a_i V_i | a_i = 0 \quad \text{for } i \text{ odd}\}$$
$$R^o = \{\sum a_i V_i | a_i = 0 \quad \text{for } i \text{ even}\}$$

It is easy to compute the sum of two elements in R, but the product at first seems more difficult. We fix this by citing some plethyism, noting

Proposition 21 (Clebsch-Gordan)**.** *For $m \geq n$, we have*

$$V_m \otimes V_n = V_{m+n-1} \oplus V_{m+n-3} \oplus ... \oplus V_{m-n+1}$$

In particular,

$$V_2 \cdot V_n = V_{n+1} + V_{n-1} \in R$$

so R is generated over $\mathbb{Z}$ by V_2. Thus sending $V_2 \mapsto t$ gives an isomorphism $R \to \mathbb{Z}[t]$.

Proof. This is an easy computation on weight spaces and is left to the reader. ■

Call a representation *homogeneous* if the corresponding element of R is homogeneous. Applying Proposition 21 immediately yields

$$R = R^o \oplus R^e$$
$$R^o \cdot R^o \subset R^e$$
$$R^o \cdot R^e \subset R^o$$
$$R^e \cdot R^e \subset R^e$$

Thus R is a $\mathbb{Z}/2\mathbb{Z}$ graded algebra. For ease of computation, we introduce a formal x such that

$$V_2 = x + x^{-1}$$

Proposition 22. *With definitions as above,*

$$V_n = \frac{x^n - x^{-n}}{x - x^{-1}}$$

Proof. Let us induct. This is trivially true for $n = 2$. Suppose the result holds for V_n. Then, applying Proposition 21, we have

$$V_{n+1} = V_2 V_n - V_{n-1}$$
$$= (x + x^{-1})\frac{x^n - x^{-n}}{x - x^{-1}} - \frac{x^{n-1} - x^{1-n}}{x - x^{-1}}$$
$$= \frac{x^{n+1} - x^{-n+1}}{x - x^{-1}}$$

■

More computations in the representation ring are to follow, and thus our representation theory is not over. We will, however, begin our discussion of combinatorics.

3 Symmetric Unimodal Sequences and $\mathfrak{sl}_2$

We begin by recalling

Definition 23. A sequence $\{a_1, ..., a_n\}$ is *unimodal* if there exists some $k \in \{1, 3, ..., n\}$ such that

$$a_1 \leq a_2 \leq ... \leq a_k \geq a_{k+1} \geq a_{k+2} \geq ... \geq a_n$$

Such a sequence is *symmetric* if $a_i = a_{n-i}$.

The method of applying the above representation theory to problems in combinatorics rests upon Corollary 14. What we wish to do is to turn the set of combinatorial objects of study into a representation of $\mathfrak{sl}_2$, graded appropriately. Then we identify the sequence which we are considering with the dimensions of weight spaces and Corollary 14 implies that this sequence is symmetric and unimodal. We begin with a motivating example involving graphs that does not rely upon computations in the representation ring R. Then, we study R in greater depth and prove a fundamental result relating symmetric, unimodal sequences and representations of $\mathfrak{sl}_2$. We then apply this to some combinatorial problems that seem intuitive but are tricky to prove with only elementary methods.

3.1 A Motivating Example: Graphs

Our motivating example comes from graphs. The method of proof is from Stanley and the author wishes to thank Dr. Daniel Litt for bringing the topic to his attention. For our purposes, a graph is a set G of n points, called vertices and a set of 2 element subsets called edges. Note that this excludes edges that are loops and multiple edges between the same two vertices. An isomorphism of graphs is a bijection $\phi : G \to G$ that sends edges to edges. We define the numbers g_{nk} to be the number of isomorphism classes of graphs with n vertices and k edges. We will see that the sequences $\{g_{n,2i}\}$ and $\{g_{n,2i+1}\}$ are symmetric and unimodal. To do this, we will apply Corollary 14, but we need to associate these to a representation of $\mathfrak{sl}_2$. Let W^n be the vector space over $\mathbb{C}$ with basis graphs on n vertices. Note that we have

$$W^n = \bigoplus_{i=0}^{\binom{n}{2}} W_i^n$$

where W_i^n is the space spanned by graphs with n vertices and k edges. We wish to look at isomorphism classes rather than general graphs, so we need to find a away to reduce the space W. This is the content of the following proposition.

Proposition 24. *There is a natural action of the symmetric group S_n on W^n that preserves the grading, namely a permutation simply acting on the vertices. Let $U_k^n = (W_k^n)^{S_n}$ denote the subspace invariant under this action. Then $g_{nk} = \dim U_k^n$.*

Proof. Let

$$\phi = \sum_{\sigma \in S_n} \sigma$$

Then we see that $U_k^n = \phi(W_k^n)$. To see this, note that the image of ϕ is clearly invariant under the action of the symmetric group. Moreover, any $v \in W_k^n$ invariant under the action of S_n is such that $\phi(v) = (n!)v$. Clearly there is a bijection from the isomorphism classes to the basis for U_k^n defined by $\{\phi(v)\}$ for $v \in V$ representative elements. ■

We define two operators for each $i < j \in \{1, 2, ..., n\}$:

$$a_{ij}(g) = \begin{cases} g \cup \{i,j\} & \text{if } \{i,j\} \text{ is not an edge of } g \\ 0 & \text{otherwise} \end{cases}$$

$$b_{ij}(g) = \begin{cases} g \setminus \{i,j\} & \text{if } \{i,j\} \text{ is an edge of } g \\ 0 & \text{otherwise} \end{cases}$$

Observe that if $\{i,j\} \neq \{i',j'\}$, then a_{ij} and $b_{i',j'}$ commute and so $[a_{ij}, a_{i',j'}] = 0$. We also note that

$$[a_{ij}, b_{ij}] = \begin{cases} g & \text{if } \{i,j\} \in g \\ -g & \text{otherwise} \end{cases}$$

We now define

$$E = \sum_{1 \le i < j \le n} a_{ij}$$
$$F = \sum_{1 \le i < j \le n} b_{ij}$$

Then, combining the above observations, we get that

$$[E, F] = \sum_{1 \le i < j \le n} a_{ij} b_{ij} g - b_{ij} a_{ij} g = \sum_{\{i,j\} \in g} g - \sum_{\{i,j\} \notin g} g$$
$$= kg - (\binom{n}{2} - k)g = (2k - \binom{n}{2})g$$

Let us define for $g \in W_k^n$

$$Hg = (2k - \binom{n}{2})g$$

We have seen that $[E, F] = H$. We need to check

$$[H, E] = 2E$$
$$[H, F] = -2F$$

Let $g \in W_k^n$. Note

$$E : W_k^n \to W_{k+1}^n$$
$$F : W_k^n \to W_{k-1}^n$$

Then let us check

$$[H,E]g = HEg - EHg = (2(k+1) - \binom{n}{2})Eg - (2k - \binom{n}{2})Eg = 2Eg$$
$$[H,F]g = HFg - FHg = (2(k-1) - \binom{n}{2})Fg - (2k - \binom{n}{2})Fg = -2Fg$$

Thus we have $[H,E]=2E$, $[H,F]=-2F$ and $[E,F]=H$; by the discussion preceding Proposition 10, we have defined a representation of $\mathfrak{sl}_2$.

Note that the actions of E,F,H commute with the action of S_n. Let $N=\binom{n}{2}$. Thus

$$U = \bigoplus_{k=0}^{N} U_k^n$$

is a representation of $\mathfrak{sl}_2$ under the action of E,F,H. Applying Corollary 14 and Proposition 24, we have proven the following:

Theorem 25. *Let g_{nk} be the number of isomorphism classes of graphs on n vertices with k edges. Let $N=\binom{n}{2}$. Then the sequences*

$$\{g_{n,2i}\}, \quad \{g_{n,2i+1}\}$$

are symmetric and unimodal with $g_{n,k}=g_{n,N-k}$.

3.2 Computations in R

Before we begin, we assume the notion of

Definition 26. A polynomial $f=a_0+a_1x+...+a_nx^n$ is *symmetric* (resp. *unimodal*) if the corresponding sequence $\{a_i\}$ is symmetric (resp. unimodal).

In some sense, our main result relates symmetric unimodal polynomials to representations of $\mathfrak{sl}_2$.

Theorem 27. *There is a bijection from the set of symmetric unimodal polynomials $f=a_0+...+a_nx^n \in \mathbb{Z}[x]$ such that $a_0>0$ and the set of homogeneous elements in R. This bijection preserves multiplication and is given by*

$$f \mapsto x^{-\deg f} f(x^2)$$

Moreover, we have the correspondence

$$\{f : 2|\deg f\} \leftrightarrow R^e$$
$$\{f : 2\nmid \deg f\} \leftrightarrow R^o$$

Proof. Let $f=a_0+a_1x+...+a_nx^n$ be a symmetric unimodal polynomial with positive coefficients. Then we see that

$$\begin{aligned} f &\mapsto x^{-n}(a_0+a_1x^2+...+a_0x^{2n}) \\ &= a_0(x^n+x^{n-2}+...+x^{-n}) + (a_1-a_0)(x^{n-2}+x^{n-4}+...+x^{2-n}) \\ &\quad +...+(a_i-a_{i-1})(x^{n-2i}+x^{n-2i-2}+...+x^{2i-n})+... \\ &= a_0V_{n+1}+(a_1-a_0)V_{n-1}+...+(a_i-a_{i-1})V_{n+1-2i}+... \end{aligned}$$

By unimodality, we have $a_{i+1}-a_i \geq 0$ so this is a representation. Moreover, given a representation W of $\mathfrak{sl}_2$, let $W=\bigoplus c_iV_i$. The representation is finite dimensional so let N be the maximal i such that $c_i \neq 0$. Then let $a_0=c_N$ and let $a_i=c_N+c_{N-1}+...+c_{N-2i}$. These constructions are clearly inverse to each other. To see that this map preserves multiplication, note that

$$fg \mapsto x^{-\deg(fg)}fg(x^2) = (x^{-\deg f}f(x^2))(x^{-\deg g}g(x^2))$$

The last statement is obvious given the above. ■

While a somewhat easy result, Theorem 27 has deep implications. One easy result is

Corollary 28. *The product of symmetric unimodal polynomials is symmetric unimodal.*

Proof. Trivial given Theorem 27. ■

We now change tack a bit and do some computations that will eventually justify the work we have done above.

Recall that $[n] = 1 - q^n$ and that

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{[n][n-1]....[n-r+1]}{[r][r-1]...[1]} = \frac{(1-q^n)(1-q^{n-1})....(1-q^{n-r+1})}{(1-q)...(1-q^r)}$$

is the generalized binomial coefficient. We define the symmetric Gaussian polynomial as

$$G_{n,r}(X,Y) = \frac{(X^n - Y^n)(X^{n-1} - Y^{n-1})...(X^{n-r+1} - Y^{n-r+1})}{(X-Y)...(X^r - Y^r)}$$

So we see that $G_{n,r}(1,q) = \begin{bmatrix} n \\ r \end{bmatrix}$. It is easy to see by induction that the following equations hold

$$\prod_{j=1}^{n}(1 + X^{n-j}Y^{j-1}t) = \sum_{r=0}^{n}(XY)^{\frac{r(r-1)}{2}} G_{n,r}(X,Y)t^r \tag{1}$$

$$\prod_{j=0}^{n}(1 - X^{n-j}Y^{j-1}t)^{-1} = \sum_{r=0}^{\infty} G_{n+r,r}(X,Y)t^r \tag{2}$$

The proofs of these two generating functions are easy but tedious; the curious reader is referred to [Alm82]. Now, we introduce two members of $R[x][\![t]\!]$,

$$\lambda(V) = \sum_{r\geq 0}(\bigwedge^r V)t^r$$
$$\sigma(V) = \sum_{r\geq 0}(\mathrm{Sym}^r\, V)t^r$$

Now, we prove the following.

Proposition 29. *Let V be a representation of $\mathfrak{sl}_2$. Then the following identities hold*

$$\lambda(V_{n+1}) = \prod_{j=0}^{n}(1 + x^{n-2j}t)$$
$$\sigma(V_{n+1}) = \prod_{j=0}^{n}(1 - x^{n-2j}t)^{-1}$$

Proof. We prove the first identity. The proof of the second is almost identical, but the confused reader is referred either to [AF, III.1.3] or to [Alm82, 1.6], although there is a slight error in the latter.

We induct on n. First note that

$$\lambda(V_2) = 1 + V_2 t + \bigwedge^2 V_2 t = 1 + (x + x^{-1})t + t^2 = (1 + xt)(1 + x^{-1}t)$$

Thus our base case holds. Now suppose that the result holds for all $1 \leq m \leq n$. Recall

$$\bigwedge^p(V \oplus W) = \bigoplus_{i+j=p} \bigwedge^i V \otimes \bigwedge^j V$$

Thus we have

$$\lambda(V_{n+1} \oplus V_{n-1}) = \lambda(V_{n+1})\lambda(V_{n-1})$$

so

$$\lambda(V_{n+1}) = \lambda(V_{n-1})^{-1}\lambda(V_{n+1} \oplus V_{n-1})$$

Recall from Proposition 21 that $V_n \otimes V_2 = V_{n+1} \oplus V_{n-1}$. Considering the weight spaces, it is apparent that

$$\bigwedge^r (V_n \otimes V_2) = \bigwedge^r V_n \oplus (\bigwedge^{r-1} V_n) \otimes V_2 \oplus (\bigwedge^{r-2} V_n) \otimes (\bigwedge^2 V_2)$$

Thus we get that $\lambda(V_n \otimes V_2) = \lambda(V_n)\lambda(V_2)$. Thus, we have

$$\lambda(V_{n+1}) = \lambda(V_{n-1})^{-1}\lambda(V_n \otimes V_2) = \lambda(V_{n-1})^{-1}\lambda(V_n)\lambda(V_2)$$

Applying the inductive hypothesis and rearranging yields us the result. ∎

Putting together all of the above, we get the following result.

Proposition 30. *We have that*

$$\bigwedge^r V_n = G_{n,r}(x, x^{-1})$$
$$\operatorname{Sym}^r V_{n+1} = G_{n+r,r}(x, x^{-1})$$

Proof. This follows immediately by letting $X = x$ and $Y = x^{-1}$ in Equations (1) and (2) and using Proposition 29. ∎

Remark 31. The above results are actually special cases of a much more general result involving Schur Functors. In particular, there are nice ways of expressing a genral Schur functor in R and the symmetric and exterior powers are special cases of Schur functors. There are many beautiful results involving Schur functors, all of which lie outside the scope of this talk. The interested reader is referred to [Alm], [FH], [Alm82], or [AF].

Corollary 32 (Generalized Hermite Reciprocity)**.** *For any $n, r \in \mathbb{N}$, we have*

$$\operatorname{Sym}^r V_{n+1} \cong \operatorname{Sym}^n V_{r+1}$$

Proof. Trivial given Proposition 30. ∎

Remark 33. Putting $r = 1$ into the above yields the well-known fact that $V_n = \operatorname{Sym}^{n-1} V_2$. It might be interesting for the reader to consider what the actions of E, F, H are when the V_n are considered as homogeneous parts of $\mathbb{C}[x, y]$. In fact, we can consider the operators

$$E = x\frac{\partial}{\partial y}$$
$$F = y\frac{\partial}{\partial x}$$
$$H = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$$

acting on $\operatorname{Sym}^{n-1} V_2$ and it is easy to see that this is an irreducible representation of $\mathfrak{sl}_2$ and so isomorphic to our V_n by Theorem 11.

We now apply the above exposition to a surprisingly hard, if simply stated, problem.

3.3 Partitions

The study of integer partitions has a fairly long history and appears in many surprising places. We do not have the time to explore much of this theory at all unfortunately, but we do present an interesting fact about partitions whose proof has no elementary counterpart of which the author knows.

Definition 34. A *partition* of an integer n is a tuple $\lambda = (\alpha_1, \alpha_2, ..., \alpha_r)$ of nonnegative integers such that $\alpha_1 + \alpha_2 + ... + \alpha_r = n$ and $\alpha_i \geq \alpha_{i+1}$. We define the *length* of a partition to be k such that $\alpha_k > 0$ and $\alpha_{k+1} = 0$. We let the *maximal part* of λ is α_1.

Counting partitions is in general a hard problem. If $p(n)$ is the number of partitions of n, it is a rather difficult fact that

$$p(n) \to \exp(\pi\sqrt{\frac{2n}{3}})$$

asymptotically as $n \to \infty$. We define $p_{kl}(n)$ to be the number of partitions of n with length at most k and maximal part at most l. We wish to show that for fixed k, l that the sequence $\{p_{kl}(n)\}$ is symmetric and unimodal. We begin by proving a theorem.

Proposition 35. *For any $n, r \in \mathbb{N}$, the polynomials $\begin{bmatrix} n \\ r \end{bmatrix}$, $\begin{bmatrix} n+r \\ r \end{bmatrix}$ are symmetric and unimodal.*

Proof. By Proposition 30 and Theorem 27 we have that the polynomial $\begin{bmatrix} n \\ r \end{bmatrix}$ corresponds to $\bigwedge^r V_n$ and $\begin{bmatrix} n+r \\ r \end{bmatrix}$ corresponds to $\mathrm{Sym}^r V_{n+1}$. ■

Remark 36. There are many proofs of the fact that $\begin{bmatrix} n \\ r \end{bmatrix}$ is symmetric and unimodal, but this result is not so easy to do in an elementary way. The above proof has the advantage that after the representation theory has been set up, there is essentially no work that has to be done, making this the most elegant of all the proofs of this fact.

We are now ready to prove the desired result.

Theorem 37. *For fixed $k, l \in \mathbb{N}$, the sequence $\{p_{kl}(n)\}$ is symmetric and unimodal.*

Proof. We see the following by showing that

$$\begin{bmatrix} k+l \\ k \end{bmatrix} = \sum_{n=0}^{\infty} p_{kl}(n) q^n$$

By Proposition 35, if we have the above then we are done. Let

$$P(k,l) = \sum_{t=0}^{\infty} p_{kl}(n) q^n$$

We will show that $P(k,l) = \begin{bmatrix} k+l \\ k \end{bmatrix}$. We begin by letting $P'(d,r) = P(r, d-r)$ and we will show that $P'(d,r) = \begin{bmatrix} d \\ r \end{bmatrix}$. Recall the following trivially verifiable properties of the q-binomial

$$\begin{bmatrix} d \\ r \end{bmatrix} = \begin{bmatrix} d \\ d-r \end{bmatrix}$$

$$\begin{bmatrix} d \\ r \end{bmatrix} = \begin{bmatrix} d-1 \\ r-1 \end{bmatrix} + q^r \begin{bmatrix} d-1 \\ r \end{bmatrix}$$

$$\begin{bmatrix} d \\ 0 \end{bmatrix} = 1$$

$$\begin{bmatrix} d \\ 1 \end{bmatrix} = 1 + q + q^2 + ... + q^{d-1}$$

Note that it is easy to see that $P'(d,1) = 1 + q + ... + q^{d-1}$ and that $P'(d,0) = 1$. Moreover, it is easy to see by taking the complement of the Young diagram in a $k \times l$ box, that $P'(d,r) = P(d-r, r) = P(r, d-r) =$

$P'(d, d-r)$. (Taking this complement is equivalent to sending the partition $(\lambda_1 \geq ... \geq \lambda_k) \mapsto (l - \lambda_k \geq l - \lambda_{k-1} \geq ... \geq l - \lambda_1)$. Clearly the weight of the image partition is $kl - |\lambda|$ and this is a bijective correspondence). If we show that $P'(d, r) = P'(d-1, r-1) + q^r P'(d-1, r)$ then we are done by induction. To see this last, we note that this is equivalent to showing

$$p_{k,l}(n) = p_{k-1,l}(n-l) + p_{k,l-1}(n)$$

To see this last, we note that any partition with greatest part at most l either has maximal part l or has maximal part less than l. There are $p_{k,l-1}(n)$ partitions in the latter category and $p_{k-l,l}(n-l)$ partitions in the former category. Thus we are done. ■

The above demonstrates the power of Theorem 27, but suppose we wish to find an explicit representation of $\mathfrak{sl}_2$ that gives us our sequence. This, too, falls immediately out of Theorem 37 and Corollary 32. We get the representation $\mathrm{Sym}^k \mathrm{Sym}^l V_2$. To see the explicit construction, let

$$\lambda = (a_1, a_1, .., a_1 \geq a_2, a_2, ... \geq ... \geq a_r, ...a_r)$$

be a partition of n and let n_i be the number of times that a_i appears in λ. Then we let $v_{a_i} \in \mathrm{Sym}^l V_2$ such that $v_{a_i} = x^{a_i} y^{l-a_i}$. Then the correspondence between partitions λ and elements of $\mathrm{Sym}^k \mathrm{Sym}^l V_2$ is given by

$$\lambda \mapsto v_{a_1}^{n_1} v_{a_2}^{n_2} ... v_{a_r}^{n_r}$$

We immediately realize that the action of H on v^λ is just multiplication by $kl - 2|\lambda|$. Thus we get immediately that $p_{kl}(n)$ is simply the dimension of the weight space of the above representation with weight $kl - 2n$, thereby yielding an explicit proof of Theorem 37.

We conclude by mentioning a result that easily falls out of the above discussion. Although the original proof of the result is much different ([Alm, Corollary 3.16]), our method is an elegant way of demonstrating the following fact. The interested reader is advised to prove it himself.

Theorem 38 (Cayley-Sylvester's Fundamental Theorem)**.** *The number of linearly independent homogeneous polynomials of degree k in $l+1$ variables that are invariant under the natural action of $SL_2(\mathbb{C})$ is*

$$p_{kl}(\frac{kl}{2}) - p_{kl}(\frac{kl}{2} - 1)$$

Hint. We do not prove the above theorem, but we provide the following hint. Consider that the only irreducible $SL_2(\mathbb{C})$ module of invariants is V_1. Thus the above number is precisely the coefficient on V_1 in the decomposition of $\mathrm{Sym}^k V_{l+1}$.

3.4 Conclusion

Over the course of this miniseries, we have seen representation theory applied to elementary combinatorial results. One 'serious' application of the theory above is to algebraic geometry. In particular

Theorem 39 (Hard Lefschetz)**.** *Let X be a complex projective variety that locally looks like $\mathbb{C}^n/\Gamma$, where Γ is a finite group. Let $H^*(X)$ be the (singular) cohomology ring and let $\beta_i = \dim_{\mathbb{C}} H^i(X)$. Then the sequences $\{\beta_0, \beta_2, ..., \beta_{2n}\}$ and $\{\beta_1, \beta_3, ..., \beta_{2n-1}\}$ are symmetric unimodal. Thus, $H^i(X) \cong H^{2n-i}(X)$.*

is a direct consequence of techniques developed above. There is a great plethora of results that use the above techniques to prove things that do not seem so related to the original theory. The interested reader is referred to many of Stanley's works, starting with [Sta89].

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