Large Orders in Small Markets: On Optimal Execution with Endogenous Liquidity Supply
Online Appendix

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Abstract

We solve a Stackelberg game where a large uninformed seller executes optimally, fully cognizant of the response of Cournot-competitive market makers. The game therefore endogenizes both demand and supply of liquidity. The closed-form solution yields several insights. First, stealth trading is both privately and socially costly because market makers incur additional cost not knowing when execution ends. Second, the presence of a large seller does not unambiguously benefit other participants. Market makers benefit only if there is enough risk-absorption capacity or if the execution period is short. Other investors benefit only when the seller sells at high enough intensity.
Proof of claims in Table 1. Recall that $A^*$ is the unique positive solution to

$$\eta - \beta A = \frac{8\delta \lambda A^2(1 + \delta A)}{(N + 1 + 2\delta A)^2}. \quad (1)$$

The left hand side of (1) is strictly decreasing in $A$, whereas the right hand side of (1) is strictly increasing in $A$. Notice that $(A, \theta)$ satisfies the equation

$$H(A, \theta) = 0, \text{ where } H(A, \theta) = \frac{8\delta \lambda A^2(1 + \delta A)}{(N + 1 + 2\delta A)^2} + \beta A - \eta. \quad (2)$$

- Dependence on $\eta$: Taking partial derivatives in (2) yields

$$\frac{\partial H}{\partial A} = \frac{8\delta \lambda(N + 1 + 2N\delta A)}{(N + 1 + 2\delta A)^3} + \beta > 0, \quad \frac{\partial H}{\partial \eta} = -1 < 0.$$  

Hence, we have $\frac{\partial A^*}{\partial \eta} = -\frac{\partial H}{\partial \eta} |_{A=A^*} > 0$. Moreover, the bid-ask spread $2A_{\theta} = \frac{2(1+2\delta A^*)}{N+1+2\delta A^*} \omega = 2\omega - \frac{2N}{N+1+2\delta A^*} \omega$, and the conditional price pressure $B_{\theta} = \frac{2A^*}{N+1+2\delta A^*} = \frac{2}{N+1+2\delta}$ are strictly increasing in $A^*$, so they are also increasing in $\eta$.

- Dependence on $\lambda$: Taking partial derivatives in (2) yields

$$\frac{\partial H}{\partial A} = \frac{8\delta \lambda(N + 1 + 2N\delta A)}{(N + 1 + 2\delta A)^3} + \beta > 0, \quad \frac{\partial H}{\partial \lambda} = \frac{8\delta A^2(1 + \delta A)}{(N + 1 + 2\delta A)^2} > 0.$$  

Hence, we have $\frac{\partial A^*}{\partial \lambda} = -\frac{\partial H}{\partial \lambda} |_{A=A^*} < 0$. Recall that we have already shown that the bid-ask spread $2A_{\theta} = \frac{2(1+2\delta A^*)}{N+1+2\delta A^*} \omega = 2\omega - \frac{2N}{N+1+2\delta A^*} \omega$ and the conditional price pressure $B_{\theta} = \frac{2A^*}{N+1+2\delta A^*} = \frac{2}{N+1+2\delta}$ are strictly increasing in $A^*$, so they are decreasing in $\lambda$.

- Dependence on $N$: first of all, let us treat $N$ as a continuous variable taking values in the space of positive real numbers. Taking partial derivatives in (2) yields

$$\frac{\partial H}{\partial A} = \frac{8\delta \lambda(N + 1 + 2N\delta A)}{(N + 1 + 2\delta A)^3} + \beta > 0, \quad \frac{\partial H}{\partial N} = -\frac{16A^2(1 + \delta A)}{(N + 1 + 2\delta A)^3} < 0.$$  

So $\frac{\partial A^*}{\partial N} = -\frac{\partial H}{\partial N} |_{A=A^*} |_{(N)} > 0$, i.e., $A^*$ is strictly increasing with $N$. We cannot yet make conclusions about sensitivities of the bid-ask spread and conditional price pressure to $N$, because they both depend on $N$ and $A^*$. To proceed, we consider a change of variables that

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1This is because its derivative with respect to $A$ is $\frac{8\delta \lambda(N+1+2N\delta A)}{(N+1+2\delta A)^3} > 0$. 
expresses $N$ in terms of $A^*$:

$$N \equiv N(A^*) = A^* \sqrt{\frac{8\delta \lambda (1 + \delta A^*)}{\eta - \beta A^*}} - 1 - 2\delta A^*. \tag{3}$$

The above expression is obtained from (1) by “solving” for $N$. Since $A^*$ is strictly increasing in $N$, we know that, as $A^*$ increases from $A^*(N = 1)$ to $A^*(N = \infty) = \frac{\eta}{\beta}$, $N$ will increase from 1 to $\infty$. Now, we can express the bid-ask spread as a function of $A$ only:

$$2A_\theta = \frac{2(1 + 2\delta A^*)}{N + 1 + 2\delta A^*} \omega = 2\omega - \frac{2N}{N + 1 + 2\delta A^*} \omega = \omega \sqrt{\frac{(\eta - \beta A^*)(\frac{1}{A^*} + 2\delta)^2}{2\delta \lambda (1 + \delta A^*)}}.$$

Notice that the final expression has a numerator that decreases with $\delta$ and a denominator that increases with $A$ when $A^* < \frac{\eta}{\beta}$. Hence, we may conclude that the bid-ask spread is strictly decreasing in $A^*$. Because $A^*$ increases with $N$, we conclude that the bid-ask spread is strictly decreasing in $N$.

As for the conditional price pressure, we obtain by a similar argument that

$$B_\theta = \frac{2A^*}{N + 1 + 2\delta A^*} \sqrt{\frac{(\eta - \beta A^*)}{2\delta \lambda (1 + \delta A^*)}}.$$

It is clear that the above expression is decreasing in $A^*$ for all $A^* \in (0, \frac{\eta}{\beta})$. Hence, the conditional price pressure is strictly decreasing in $N$.

- Dependence on $\beta$: Taking partial derivatives in (2) yields

$$\frac{\partial H}{\partial A} = \frac{8\delta \lambda (N + 1 + 2N \delta A)}{(N + 1 + 2\delta A)^3} + \beta > 0, \quad \frac{\partial H}{\partial \beta} = A > 0.$$

Hence, we have $\frac{\partial A^*}{\partial \beta} = -\frac{\partial H}{\partial \beta} \big|_{A=A^*} < 0$. Recall that we have already shown that the bid-ask spread $2A_\theta = \frac{2(1 + 2\delta A^*)}{N + 1 + 2\delta A^*} \omega = \omega - \frac{2N}{N + 1 + 2\delta A^*} \omega$ and the conditional price pressure $B_\theta = \frac{2A^*}{N + 1 + 2\delta A^*} = \frac{2}{\frac{N+1}{N+1+2\delta}}$ are strictly increasing in $A^*$, so they are decreasing in $\beta$.

- Dependence on $\delta$: Taking partial derivatives in (2) yields

$$\frac{\partial H}{\partial A} = \frac{8\delta \lambda (N + 1 + 2N \delta A)}{(N + 1 + 2\delta A)^3} + \beta > 0, \quad \frac{\partial H}{\partial \delta} = \frac{8\lambda A^2 (1 + \delta A)}{(N + 1 + 2\delta A)^2} > 0.$$

Hence, we have $\frac{\partial A^*}{\partial \delta} = -\frac{\partial H}{\partial \delta} \big|_{A=A^*} < 0$, i.e., $A^*$ is strictly decreasing in $\delta$. On the other hand, $\alpha = \delta A^*$ and $\delta$ solves a slightly different equation $K(\alpha, \delta) = \frac{8k\alpha^2(1+\alpha)}{(N+1+2\alpha)^2} + \beta \alpha - \delta \eta$. It is
straightforward to verify that $\frac{\partial K}{\partial \alpha} > 0$ and $\frac{\partial K}{\partial \delta} < 0$, so using the same argument as above, we know that the product $\delta A^\ast$ is strictly increasing in $\delta$. Recall that we have already shown that the bid-ask spread $2A_\theta = \frac{2(1+2\delta A^\ast)}{N+1+2\delta A^\ast} \omega = 2\omega - \frac{2N}{N+1+2\delta A^\ast} \omega$ is strictly increasing in $\delta A^\ast$, so we can conclude that the bid-ask spread is strictly increasing in $\delta$. As for the conditional price pressure $B_\theta = \frac{2A^\ast}{N+1+2\delta A^\ast}$, we notice that its numerator is strictly decreasing in $\delta$ (because $A^\ast$ is), and its denominator is strictly increasing in $\delta$ (because $\delta A^\ast$ is). Hence, we can conclude that the conditional price pressure is decreasing in $\delta$.

- Dependence on $\omega$: the results are obvious.

This completes the proof. □