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POSITIVE RECURRENCE OF PIECEWISE ORNSTEIN–UHLENBECK PROCESSES AND COMMON QUADRATIC LYAPUNOV FUNCTIONS[1]

BY A. B. DIEKER AND XUEFENG GAO

Georgia Institute of Technology

We study the positive recurrence of piecewise Ornstein–Uhlenbeck (OU) diffusion processes, which arise from many-server queueing systems with phase-type service requirements. These diffusion processes exhibit different behavior in two regions of the state space, corresponding to "overload" (service demand exceeds capacity) and "underload" (service capacity exceeds demand). The two regimes cause standard techniques for proving positive recurrence to fail. Using and extending the framework of common quadratic Lyapunov functions from the theory of control, we construct Lyapunov functions for the diffusion approximations corresponding to systems with and without abandonment. With these Lyapunov functions, we prove that piecewise OU processes have a unique stationary distribution.

1. Introduction. Since the pioneering paper of Halfin and Whitt (1981), and particularly within the last 10 years, there has been a surge of interest in diffusion approximations for queueing systems with many servers. These queueing systems model customer contact centers with hundreds of servers. Empirical study in Brown et al. (2005) suggests that the service time distribution is far from exponential. Despite past and foreseeable advances in computer hardware and architectures, the sheer size of such systems prohibits exact (numerical) calculations even when the arrival process is Poisson and the service time distribution is of phase type. Diffusion approximations such as piecewise Ornstein–Uhlenbeck (OU) processes can be used to approximate the queue length process. Such approximations are rooted in many-server heavy traffic limits proved in Puhalskii and Reiman (2000) and Dai, He and Tezcan (2010). These approximations are remarkably accurate in predicting system performance measures, sometimes for systems with as few as 20 servers [see He and Dai (2011)].

For a diffusion approximation to work, it is critical to know whether the approximating diffusion process has a unique stationary distribution. In this paper we prove that, under some natural conditions, this is the case for piecewise OU processes. Thus, this paper provides a solid mathematical foundation for He and

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Dai (2011), who devise an algorithm to numerically compute the stationary distribution of a piecewise OU process.

A standard technique for proving stability of queueing systems is to first establish the stability of a so-called fluid model and then to appeal to general theory for establishing stochastic stability [see, e.g., Dupuis and Williams (1994), Dai (1995), Stolyar (1995)]. However, this theory is restricted to systems with nonnegative fluid levels which are attracted to the origin. The fluid analog of a piecewise Ornstein–Uhlenbeck process does not possess this property. As an alternative to the fluid model framework, the family of *quadratic* Lyapunov functions is a natural choice for establishing positive recurrence. Indeed, due to diffusive properties of piecewise Ornstein–Uhlenbeck processes, if a quadratic Lyapunov function can be shown to stabilize the fluid model, it simultaneously and directly establishes stochastic stability, that is, the positive recurrence of piecewise OU processes. As a result of working with quadratic forms as Lyapunov functions, several key results from linear algebra lie at the heart of our main results. We were unable to devise an equally powerful framework without using this algebraic machinery.

Piecewise OU processes exhibit different behavior in two regions of the state space, corresponding to "overload" and "underload." The two regions are separated by a hyperplane, which corresponds to "critical load." In each of the two regions, a piecewise OU process can be thought of as a first-order linear differential equation with stochastic noise. A standard technique in proving its positive recurrence is to use a quadratic Lyapunov function to prove stability of such first-order linear differential equations. However, the two different regions of a piecewise OU process pose considerable challenges to apply this methodology. A natural approach would be to "paste together" two quadratic Lyapunov functions from the two regions, but our attempts in this direction have failed. In fact, it is well known that a diffusion with two stable regimes can lead to an instable hybrid system [see Yin and Zhu (2010) for related examples]. In Blondel and Tsitsiklis (2000), the stability of a switched linear system is discussed from the perspective of complexity theory.

Using the interpretation of the diffusion parameters in terms of a many-server queueing system, our main results can be formulated as follows: (1) For a slightly underloaded system without abandonment, we show that there exists a quadratic Lyapunov function which yields the desired positive recurrence using the Foster–Lyapunov criterion (Theorem 2). In general, this quadratic Lyapunov function is not explicit and nonunique. (2) We show that no quadratic Lyapunov function can satisfy the Foster–Lyapunov criterion for systems with abandonment. (3) We construct a suitable nonquadratic Lyapunov function to prove positive recurrence for systems with abandonment (Theorem 3).

The main building blocks for these two types of Lyapunov functions are so-called common quadratic Lyapunov functions (CQLFs), which are widely used in the theory of control. Such functions play an important role in the stability analysis for deterministic linear systems, with different dynamics in different parts of the state space (or, more generally, operating under a switching rule). They are called *common* quadratic Lyapunov functions since they serve as a quadratic Lyapunov function in each part of the state space. There is a vast body of literature on CQLFs and related theory [see the survey Shorten et al. (2007) for details]. Although quadratic Lyapunov functions are ubiquitous in the literature on queueing systems Dai and Prabhakar (2000), Gamarnik and Momčilović (2008), Tassiulas and Ephremides (1992), to our knowledge, our paper is the first to exploit CQLFs in this context.

As mentioned in the section on open problems of Shorten et al. (2007), it is of considerable interest to determine simple conditions for the existence of CQLFs. Theorem 1, which is our main technical contribution in this space, establishes such a result in the context of M-matrices and rank-1 perturbations. The theorem shows that existence of a CQLF is guaranteed after merely verifying that certain vectors are nonnegative. It is a first result of this kind. Its proof relies on a delicate analysis involving Chebyshev polynomials, as well as on an extension of recent work of King and Nathanson (2006) and Shorten et al. (2009) summarized in Proposition 3 below.

To conclude this Introduction, we mention a body of work on the recurrence of multidimensional Ornstein–Uhlenbeck type processes by Sato, Watanabe and Yamazato (1994), Sato et al. (1996), which differ from the processes studied here. The processes studied in these references are driven by Lévy processes and they have a linear drift coefficient. As a result, their multidimensional processes do not possess the critical feature of the processes we study here, namely, a piecewise linear drift coefficient.

This paper is organized as follows. Section 2 discusses the required background on piecewise OU process and positive recurrence. Section 3 is devoted to common quadratic Lyapunov functions. Section 4 summarizes the main results and Section 5 contains the proofs of the main results. The proof of Proposition 3, which mainly uses existing methodology from the theory of control, is given in Appendix A. Appendix B shows that no quadratic Lyapunov function can work in the Foster–Lyapunov criterion if abandonment is allowed.

Notation. All random variables and stochastic processes are defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ unless otherwise specified. For some $d \in \mathbb{N}$, $\mathbb{R}^d$ denotes the d-dimensional Euclidean space. The space of functions $f : \mathbb{R}^K \to \mathbb{R}$ that are twice continuously differentiable is denoted by $C^2(\mathbb{R}^K)$. We use ∇ to denote the gradient operator. Given $x \in \mathbb{R}$, we set $x^+ = \max\{x, 0\}$. All vectors are envisioned as column vectors. For a K-dimensional vector u, we use u_k to denote its kth entry and we write $|u|$ for its Euclidean norm. We also write u' for its transpose. For two K-dimensional vectors u and v, we write $u' \geq v'$ ($u' > v'$) if $u_k \geq v_k$ ($u_k > v_k$) for each $k = 1, 2, \dots, K$. The inner product of u and v is denoted by $u'v$, which is $\sum_{k=1}^K u_k v_k$. Given a $K \times K$ matrix M, we use M' to denote its transpose and M_{ij} for its (i, j)th entry. We write $M > 0$ ($M < 0$) if M is a positive (negative) definite matrix and $M \geq 0$ ($M \leq 0$) if it is a positive (negative) semi-definite matrix. Let the matrix norm of M be $|M| = \sum_{ij} |M_{i,j}|$, where $|M_{ij}|$ is the absolute value of M_{ij}. We reserve I for the $K \times K$ identity matrix and e for the K-dimensional vector of ones.

2. Piecewise OU processes and positive recurrence. This section introduces the piecewise Ornstein–Uhlenbeck (OU) processes studied in this paper, and discusses preliminaries on positive recurrence.

2.1. *Piecewise OU processes.* We first define M-matrices. We call a matrix nonnegative when each element of the matrix is nonnegative.

DEFINITION 1 (M-matrix). A $K \times K$ matrix R is said to be an M-matrix if it can be expressed as $R = sI - N$ for some $s > 0$ and some nonnegative matrix N with the property that $\rho(N) \leq s$, where $\rho(N)$ is the spectral radius of N. The matrix R is nonsingular if $\rho(N) < s$.

We next define piecewise Ornstein–Uhlenbeck (OU) processes, which are special diffusion processes. Let $\{W(t)\}$ be a standard Brownian motion in any dimension. A K-dimensional diffusion process Y is the strong solution to a stochastic differential equation of the form

$$dY(t) = b(Y(t))\,dt + \sigma(Y(t))\,dW(t),$$

where the drift coefficient $b(\cdot)$ and the diffusion coefficient $\sigma(\cdot)$ have appropriate sizes and satisfy the following Lipschitz continuity condition: there exists some $C > 0$ such that

$$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq C|x - y| \qquad \text{for all } x, y \in \mathbb{R}^K. \tag{2.1}$$

For a real-valued function $V \in C^2(\mathbb{R}^K)$, the generator G of Y applied to V is given by, for $y \in \mathbb{R}^K$,

$$GV(y) = (\nabla V(y))' b(y) + \frac{1}{2} \sum_{i,j} (\sigma\sigma')_{ij}(y) \frac{\partial^2 V}{\partial y_i\, \partial y_j}(y). \tag{2.2}$$

We refer to Rogers and Williams [(2000), Chapter V], for more details on diffusion processes.

DEFINITION 2 (Piecewise OU processes). Let p be a K-dimensional probability vector, e be the K-dimensional vector of ones and let R be a $K \times K$ nonsingular M-matrix. For $\alpha, \beta \in \mathbb{R}$, a K-dimensional diffusion process Y is called a piecewise Ornstein–Uhlenbeck (OU) process if it has drift coefficient

$$b(y) = -\beta p - R\bigl(y - p(e'y)^+\bigr) - \alpha p(e'y)^+, \tag{2.3}$$

and diffusion coefficient $\sigma(y) \equiv \sigma$ for all $y \in \mathbb{R}^K$, such that $\sigma\sigma'$ is a $K \times K$ nonsingular matrix.

As in Dai, He and Tezcan (2010), we call this process a *piecewise* OU process since the drift coefficient is affine (hence, OU process) yet it differs on each side of the hyperplane $\{y \in \mathbb{R}^K : e'y = 0\}$ (hence, piecewise). Indeed, for $e'y \geq 0$ we have $b(y) = -\beta p - R(I - pe')y - \alpha p(e'y)$ while for $e'y \leq 0$ we have $b(y) = -\beta p - Ry$. In conjunction with $\sigma(y) \equiv \sigma$, this implies the Lipschitz continuity condition (2.1). As a consequence, the piecewise OU process Y is well-defined as a diffusion process.

The quantities α, β, R, p on the right-hand side of (2.3) come from the queueing system that gave rise to the piecewise OU diffusion. Their queueing interpretation is as follows: α is the abandonment rate, β is the slack in the arrival rate relative to a critically loaded system while p and R are the parameters of the service-time distribution (assumed to be of phase-type). For more details, we refer to Dai, He and Tezcan (2010).

Throughout the paper, we impose the following assumption.

ASSUMPTION 1. Each component of the row vector $e'R$ is nonnegative, that is,

$$e'R \geq 0'.$$

We now make the connection between piecewise OU processes and many-server queueing models explicit, and we discuss Assumption 1 in this context. For presentational convenience, we do so in the special case of $M/H_2/n + M$ queues. In fact, we consider a sequence of $M/H_2/n + M$ queues indexed by n, where n is the number of (identical) servers, meaning that (1) the arrival process is a Poisson process with some intensity λ^n, (2) the service times have a two-phase hyperexponential distribution, so they are exponential with parameter ν_1 with probability p_1, and exponential with parameter ν_2 with probability $p_2 = 1 - p_1$ and (3) each customer has a patience time which follows an exponential distribution with parameter $\alpha > 0$. Hypergeometric service time distributions are of special interest, since they can be used to model multiclass systems [see Puhalskii and Reiman (2000), Gamarnik and Stolyar (2011)]. To see this, envision two types of customers entering a buffer to seek service. Suppose that the mean service time is 1, that is, $p_1/\nu_1 + p_2/\nu_2 = 1$. We further assume the system is operated under Halfin–Whitt regime, that is, for some $\beta \in \mathbb{R}$,

$$\lim_{n\to\infty} \sqrt{n}\left(1 - \frac{\lambda^n}{n}\right) = \beta.$$

Let $X_1^n(t)$ and $X_2^n(t)$ denote the number of customers of type 1 and 2 in the system at time t. For $i = 1, 2$ and $t \geq 0$, we define

$$\tilde{X}_i^n(t) = \frac{1}{\sqrt{n}}\left(X_i^n(t) - n\frac{p_i}{\nu_i}t\right).$$

As detailed in Dai, He and Tezcan (2010) [see Gamarnik and Goldberg (2011) for a related general result], the "centering" in this expression has been chosen so that, in a sense of weak convergence on the process level,

$$(\tilde{X}_1^n, \tilde{X}_2^n) \Rightarrow (Y_1, Y_2), \qquad n \to \infty,$$

where (Y_1, Y_2) satisfies the following system of stochastic differential equations: for $i = 1, 2$,

$$\begin{aligned} Y_i(t) &= Y_i(0) + W_i(t) - \beta p_i t \\ &\quad - \nu_i \int_0^t \big(Y_i(s) - p_i\big(Y_1(s) + Y_2(s)\big)^+\big)\, ds - \alpha p_i \int_0^t \big(Y_1(s) + Y_2(s)\big)^+\, ds. \end{aligned}$$

Note that $(Y_1(s) + Y_2(s))^+$ represents the (scaled) number of customers waiting in the buffer, and the fraction of type i customers in the buffer is approximately p_i. Thus, the term involving ν_i can be thought of as a service-rate term. Similarly, the terms involving α and β are the abandonment and arrival term, respectively. The randomness in the system is represented by $W = (W_1, W_2)$, which is a driftless Brownian motion with nonsingular covariance matrix

$$\begin{pmatrix} p_1(p_1c^2 - p_1 + 2) & p_1p_2(c^2 - 1) \\ p_1p_2(c^2 - 1) & p_2(p_2c^2 - p_2 + 2) \end{pmatrix}$$

for some constant $c \in \mathbb{R}$. Therefore, $Y = (Y_1, Y_2)$ is a two-dimensional piecewise OU process with drift coefficient

$$b(y) = -\beta p - R\big(y - p(e'y)^+\big) - \alpha p(e'y)^+, \tag{2.4}$$

where the matrix R is given by

$$R = \begin{pmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{pmatrix}.$$

When we apply this procedure to a general phase-type service time distribution with K phases, the corresponding diffusion limit is a K-dimensional piecewise OU process Y. The parameters p and R represent the distribution of the initial phase and phase dynamics, respectively. Each component of the piecewise OU process Y_k approximates the number of phase-k customers in the many-server queueing system, either waiting or in service. Thus $e'Y$ represents the total number of customers in the system after centering and scaling. Thus, whenever $e'Y > 0$ the system is in "overload," that is, there are customers waiting in the buffer, and whenever $e'Y < 0$ the system is in "underload," that is, there are idle servers. We refer readers to Puhalskii and Reiman (2000) and Dai, He and Tezcan (2010) for more details. We remark that the matrix R in these two papers takes the form of $(I - P')\,\mathrm{diag}\{\nu\}$, where P is assumed to be a transient matrix describing the transitions between each service phase, and $\mathrm{diag}\{\nu\}$ is a diagonal matrix with kth diagonal entry given by ν_k, where ν_k is the rate for the sojourn time in phase k. Transience of P corresponds to customers who eventually leave the system after receiving a sufficient amount of service, which implies that $e'R = e'(I - P')\operatorname{diag}\{\nu\} \geq 0$. Therefore, we conclude that in this setting, R is a nonsingular M-matrix and that Assumption 1 is satisfied.

2.2. *Positive recurrence and Lyapunov functions*. In this section, we recall the definitions and the criteria for positive recurrence and exponential ergodicity in the context of general diffusion processes.

Let $\mathbb{E}_\pi$ be the expectation operator with respect to a probability distribution π.

DEFINITION 3 (Positive recurrence and stationary distribution). For a K-dimensional diffusion process Y, we say that Y is positive recurrent if for any $y \in \mathbb{R}^K$ and any compact set C in $\mathbb{R}^K$ with positive Lebesgue measure, we have

$$\mathbb{E}(\tau_C | Y(0) = y) < \infty,$$

where $\tau_C = \inf\{t \geq 0 : Y(t) \in C\}$ is the hitting time of the set C. We call a probability distribution π on $\mathbb{R}^K$ a stationary distribution for Y if for every bounded continuous function $f : \mathbb{R}^K \to \mathbb{R}$,

$$\mathbb{E}_\pi[f(Y(t))] = \mathbb{E}_\pi[f(Y(0))] \qquad \text{for all } t \geq 0.$$

In the following, we assume that the diffusion coefficient of the diffusion process Y is uniformly nonsingular. That is, there exists some $c \in (0, \infty)$ such that for all $y \in \mathbb{R}^K$ and $a \in \mathbb{R}^K$,

$$a'\sigma(y)\sigma(y)'a \geq ca'a. \tag{2.5}$$

The next result gives a sufficient criterion for positive recurrence of diffusion processes [see Khasminskii (2011), Sections 3.7, 4.3 and 4.4 and Meyn and Tweedie (1993), Section 4]. Uniqueness of the stationary distribution follows from Peszat and Zabczyk (1995) in view of condition (2.5).

PROPOSITION 1 (Foster–Lyapunov criterion). *Let Y be a diffusion process satisfying* (2.5). *Suppose that there exists a nonnegative function $V \in C^2(\mathbb{R}^K)$ and some $r > 0$ such that, for any $|y| > r$,*

$$GV(y) \leq -1.$$

In addition, suppose that $V(y) \to \infty$ as $|y| \to \infty$. Then Y is positive recurrent and has a unique stationary distribution. The function V is called a Lyapunov function.

We now introduce the concept of exponential ergodicity. For any positive measurable function $f \geq 1$ and any signed measure m, we write $\|m\|_f = \sup_{|g| \leq f} |m(g)|$.

DEFINITION 4 (Exponential ergodicity). Suppose that the diffusion process Y is positive recurrent and that it has a unique stationary distribution π. Given a function $f \geq 1$, we say that Y is f-exponentially ergodic if there exists a $\gamma \in (0, 1)$ and a real-valued function B such that for all $t > 0$ and $y \in \mathbb{R}^K$,

$$\|P^t(y, \cdot) - \pi(\cdot)\|_f \leq B(y)\gamma^t,$$

where P^t is the transition function of Y. If $f \equiv 1$, we simply say that Y is exponentially ergodic.

For $f \geq 1$, we have $\|P^t(y, \cdot) - \pi(\cdot)\|_1 \leq \|P^t(y, \cdot) - \pi(\cdot)\|_f$, and we deduce that f-exponential ergodicity implies exponential ergodicity. The following result gives a criterion for exponential ergodicity [see Meyn and Tweedie (1993), Section 6].

PROPOSITION 2. *Suppose that Y is a diffusion process with a unique stationary distribution. If there is a nonnegative function $V \in C^2(\mathbb{R}^K)$ such that $V(y) \to \infty$ as $|y| \to \infty$ and for some $c > 0, d < \infty$,*

$$GV(y) \leq -cV(y) + d \qquad \textit{for any } y \in \mathbb{R}^K,$$

then Y is $(V + 1)$-exponentially ergodic.

3. Common quadratic Lyapunov functions. In this section we introduce common quadratic Lyapunov functions (CQLFs). Such functions play a central role in the stability analysis of deterministic switched linear systems, which is discussed in Section 3.2. We use CQLFs as building blocks to construct Lyapunov functions to prove positive recurrence of piecewise OU processes. At this point it is best to distinguish CQLFs for switched linear systems from the Lyapunov functions in the context of the Foster–Lyapunov criterion. We connect these two concepts in Section 4.

3.1. *Background and definitions.* Quadratic Lyapunov functions form a cornerstone of stability theory for ordinary differential equations. Consider the linear system $\dot{y}(t) = By(t)$ where $y(t) \in \mathbb{R}^K$, $B \in \mathbb{R}^{K\times K}$ is a fixed real matrix and $\dot{y}(t)$ is the derivative of y with respect to t. For $Q \in \mathbb{R}^{K\times K}$, the quadratic form L given by $L(y) = y'Qy$ for $y \in \mathbb{R}^K$ is called a quadratic Lyapunov function for the matrix B if Q is positive definite and $QB + B'Q$ is negative definite. In this case, there exists a constant $C > 0$ such that

$$\frac{d}{dt}L(y(t)) = y(t)'(QB + B'Q)y(t) \leq -CL(y(t)) < 0 \qquad \text{for all } t \geq 0,$$

and thus we can conclude that $L(y(t)) \leq e^{-Ct}L(y(0))$. This implies that $L(y(t)) \to 0$ as $t \to \infty$, thus $y(t) \to 0$ as $t \to \infty$. It is standard fact in Lyapunov stability theory that the existence of a quadratic Lyapunov function L is equivalent to all eigenvalues of B having negative real part [Berman and Plemmons (1994), Section 6.2].

The following definition, tailored to our setting in order to allow for a singular matrix, plays an important role in our analysis. Other versions can be found in Shorten and Narendra (2003) and Shorten et al. (2007). Recall that an eigenvalue of a matrix is called (geometrically) simple if its corresponding eigenspace is one-dimensional.

DEFINITION 5 (CQLF). Let $B_1 \in \mathbb{R}^{K\times K}$ have all eigenvalues with negative real part and let $B_2 \in \mathbb{R}^{K\times K}$ have all eigenvalues with negative real part except for a simple zero eigenvalue. For $Q \in \mathbb{R}^{K\times K}$, the quadratic form L given by $L(y) = y'Qy$ for $y \in \mathbb{R}^K$ is called a common quadratic Lyapunov function (CQLF) for the pair (B_1, B_2) if Q is positive definite and

$$QB_1 + B_1'Q < 0,$$
$$QB_2 + B_2'Q \le 0.$$

3.2. *The CQLF existence problem.* The CQLF existence problem for a pair of matrices has its roots in the study of stability criteria for switched linear systems. These systems have the form $\dot{y}(t) = B(\tau)y(t)$, where $B(\tau) \in \{B_1, B_2\}$ with $B_i \in \mathbb{R}^{K\times K}$ for $i = 1, 2$ and where the switching function τ may depend on both y and t. The existence of a CQLF for the pair (B_1, B_2) guarantees that all solutions of the systems are bounded under arbitrary switching function τ. The CQLF existence problem is also closely related to the Kalman–Yacubovich–Popov lemma in the development of adaptive control algorithms and the Lur'e problem in nonlinear feedback analysis. For more details consult Kalman (1963), Boyd et al. (1994) and the recent survey paper by Shorten et al. (2007). For an arbitrary matrix pair, no simple analytic and verifiable conditions are known for the pair to admit a CQLF. In the special case where the difference of the matrices has rank one, King and Nathanson (2006) shows that if both B_1 and B_2 are Hurwitz, that is, all eigenvalues of the matrices B_1, B_2 have negative real part, then there exists a positive definite matrix Q such that $QB_1 + B_1'Q < 0$ and $QB_2 + B_2'Q < 0$ if and only if the matrix product B_1B_2 has no real negative eigenvalues. Note that in this case, both B_1 and B_2 are nonsingular. A similar CQLF existence result has been obtained in Shorten et al. (2009) when one of the matrices (B_1 or B_2) is singular.

We now state a result on the CQLF existence problem for a pair of matrices with one of them being singular. It is essentially the main theorem in Shorten et al. (2009) but we relax their assumptions. Let $B \in \mathbb{R}^{K\times K}$ be a real matrix and let $g, h \in \mathbb{R}^K$. The proposition below is stated in Shorten et al. (2009) under the assumptions that (B, g) is controllable, meaning that the vectors $g, Bg, B^2g, \dots$ span $\mathbb{R}^K$, and that (B, h) is observable, meaning that the vectors $h, B'h, (B')^2h, \dots$ span $\mathbb{R}^K$. Using techniques from King and Nathanson (2006), we show that these assumptions are unnecessary and we state the result in its full generality here. A proof is given in Appendix A.

PROPOSITION 3. *Suppose that all eigenvalues of matrix B have negative real part and all eigenvalues of $B - gh'$ have negative real part, except for a simple zero eigenvalue. Then there exists a CQLF for the pair $(B, B - gh')$ if and only if the matrix product $B(B - gh')$ has no real negative eigenvalues and a simple zero eigenvalue.*

4. Main results.

In this section, we present our results on positive recurrence of the piecewise OU process Y. Key to these results is the following theorem, which uses Proposition 3 to establish the existence of a CQLF for certain matrix pairs. Recall the definitions of R, p and e from Definition 2 in Section 2.1, and note that we are working under Assumption 1.

THEOREM 1. *There exists a CQLF for both the pair $(-R, -R(I - pe'))$ and the pair $(-R, -(I - pe')R)$.*

By Theorem 1, there exists a CQLF L for the pair $(-R, -R(I - pe'))$ and another CQLF $\widetilde{L}$ for the pair $(-R, -(I - pe')R)$. Typically there are many CQLFs corresponding to these pairs, that is, L and $\widetilde{L}$ are not unique. Note that L and $\widetilde{L}$ are closely related in the following sense. If the CQLF L for the pair $(-R, -R(I - pe'))$ is given by $L(y) = y'Qy$ for some $Q > 0$ and for all $y \in \mathbb{R}^K$, then one readily checks that the quadratic form $\widetilde{L}$ given by $\widetilde{L}(y) = y'(R'QR)y$ for $y \in \mathbb{R}^K$ is a CQLF for the pair $(-R, -(I - pe')R)$. We remark that, apart from special cases, the CQLFs from Theorem 1 are not explicit.

We know from Theorem 1 that there exists a CQLF L for the pair $(-R, -R(I - pe'))$, where L is given by $L(y) = y'Qy$ for some $Q > 0$ and for all $y \in \mathbb{R}^K$. We are able to use the quadratic form L as a Lyapunov function in the Foster–Lyapunov criterion of Proposition 1 to prove the following result.

THEOREM 2. *If $\alpha = 0$ and $\beta > 0$, then the piecewise OU process Y is positive recurrent and has a unique stationary distribution.*

For $\alpha > 0$, no quadratic function can serve as a Lyapunov function in the Foster–Lyapunov criterion to prove positive recurrence of the piecewise OU process Y (see Appendix B for details). Despite this fact, still relying on Theorem 1, we overcome this difficulty in Section 5.3 by constructing a suitable nonquadratic Lyapunov function. Specifically, there exists a CQLF $\widetilde{L}$ for the pair $(-R, -(I - pe')R)$ by Theorem 1, where $\widetilde{L}$ is given by $\widetilde{L}(y) = y'\widetilde{Q}y$ for some $\widetilde{Q} > 0$ and for all $y \in \mathbb{R}^K$. A suitable approximation to the function f, given by, for all $y \in \mathbb{R}^K$,

$$f(y) = (e'y)^2 + \kappa \widetilde{L}\big(y - p(e'y)^+\big) \qquad \text{for some large constant } \kappa,$$

provides the desired nonquadratic Lyapunov function in the Foster–Lyapunov criterion to prove positive recurrence of Y when $\alpha > 0$. Note that, in queueing terminology, the vector $y - p(e'y)^+$ relates to the customers in service, and not to those in the buffer. We therefore need the extra term $(e'y)^2$. Applying Proposition 2 with the same nonquadratic Lyapunov function yields exponential ergodicity of Y for $\alpha > 0$. We use a smooth approximation of f as a Lyapunov function in the Foster–Lyapunov criterion of Proposition 1 instead of using f directly since $f \notin C^2(\mathbb{R}^K)$. This leads to the following result.

THEOREM 3. *If $\alpha > 0$, then the piecewise OU process Y is positive recurrent and has a unique stationary distribution. Moreover, Y is exponentially ergodic.*

5. Proof of the main results.

5.1. *Proof of Theorem 1.*

PROOF. We only establish the existence of a CQLF for the pair $(-R, -R(I - pe'))$, since the existence of a CQLF for the other pair $(-R, -(I - pe')R)$ follows directly. Since $-R - (-R(I - pe')) = -Rpe'$ is a rank-one matrix, in view of Proposition 3, we need to check three conditions:

(a) All eigenvalues of $-R$ have negative real part.
(b) All eigenvalues of $-R(I - pe')$ have negative real part except for a simple zero eigenvalue.
(c) The matrix product $R^2(I - pe')$ has no real negative eigenvalues and a simple zero eigenvalue.

We first prove (a) and (b). It is known that all eigenvalues of a nonsingular M-matrix have positive real part, and all eigenvalues of a singular M-matrix have nonnegative real part [see Berman and Plemmons (1994), Chapter 6]. Since R is a nonsingular M-matrix, we immediately get (a). For (b), it is clear that $-R(I - pe')$ has a simple zero eigenvalue. We notice that $(I - pe')R = R - pe'R$ where $e'R \geq 0'$ by Assumption 1, p is a nonnegative vector and R is a nonsingular M-matrix, so the off-diagonal elements of $(I - pe')R$ are nonpositive. Using this in conjunction with the fact that both $I - pe'$ and R are M-matrices, we find that $(I - pe')R$ is also an M-matrix and all its eigenvalues have nonnegative real part [see Berman and Plemmons (1994), Exercise 5.2]. Thus we get (b) after a similarity transform.

We now concentrate on proving (c). The key ingredient of the proof is an identity for Chebyshev polynomials. Suppose that $R^2(I - pe')$ has a real negative eigenvalue $-\lambda$ with $\lambda > 0$, and write v for the corresponding left eigenvector, thus we have $v'R^2(I - pe') = -\lambda v'$. Right-multiplying by p on both sides, we obtain $v'p = 0$ and the following equality:

$$\begin{aligned} 0 &= v'R^2(I - pe') + \lambda v' = v'R^2(I - pe') + \lambda v'(I - pe') \\ &= v'(R^2 + \lambda I)(I - pe'). \end{aligned} \tag{5.1}$$

Since R is a nonsingular M-matrix having only eigenvalues with positive real part, the matrix $(R^2 + \lambda I)$ is invertible for all $\lambda > 0$. Also, by the fact that p is a nonnegative probability vector with $e'p = 1$, we deduce the matrix $(I - pe')$ has an eigenvalue 0 and the corresponding left eigenvector must be in the form of ce' for some $c \neq 0$. Thus, it follows from (5.1) that $v' = ce'(R^2 + \lambda I)^{-1}$ for some $c \neq 0$. We show below that $e'(R^2 + \lambda I)^{-1}$ is a positive vector for all $\lambda > 0$, that is,

$$e'(R^2 + \lambda I)^{-1} > 0' \qquad \text{for all } \lambda > 0. \tag{5.2}$$

This yields a contradiction in view of $v'p = 0$. By definition of a nonsingular M-matrix, R is of the form $sI - N$, where N is a nonnegative matrix with $\rho(N) < s$ and $e'R \geq 0$ by Assumption 1. Inequality (5.2) thus states that for all $\lambda > 0$ and for every nonnegative matrix N with $\rho(N) < s$ and $se' \geq e'N$,

$$e'\big((sI - N)^2 + \lambda I\big)^{-1} > 0'.$$

Equivalently, we show the following inequality: for all $y \in (0, 1)$ and for every nonnegative matrix N with $\rho(N) < 1$ and $e' \geq e'N$,

$$e'\big(y(I - N)^2 + (1 - y)I\big)^{-1} > 0'. \tag{5.3}$$

Therefore, to show (c), it suffices to prove (5.3) for fixed N and $y \in (0, 1)$.

Our strategy to prove (5.3) is to use a matrix series expansion and connections with Chebyshev polynomials. Chebyshev polynomials of the second kind U_n can be defined by the following trigonometric form:

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta} \qquad \text{for } n = 0, 1, 2, 3, \dots. \tag{5.4}$$

Moreover, for $z \in [-1, 1]$ and $t \in (-1, 1)$, the generating function of U_n is

$$\sum_{n=0}^{\infty} U_n(z)t^n = \frac{1}{1 - 2tz + t^2}. \tag{5.5}$$

Refer to Abramowitz and Stegun (1992), Chapter 22, for more details. The scalar version of the left-hand side of (5.3) admits the following expansion: for $x, y \in (0, 1)$,

$$\frac{1}{y(1-x)^2 + 1 - y} = \sum_{n=0}^{\infty} C_n(y)x^n, \tag{5.6}$$

where $C_n(y) = U_n(\sqrt{y})(\sqrt{y})^n$ for all $n \geq 0$. This can readily be verified with (5.5). In particular, we have

$$C_0(y) = U_0(y) \equiv 1 \qquad \text{for all } y \in (0, 1). \tag{5.7}$$

For fixed $y \in (0, 1)$, the radius of convergence of the power series in (5.6) is larger than 1. Since $\rho(N) < 1$, we immediately obtain that, for $y \in (0, 1)$,

$$\big(y(I - N)^2 + (1 - y)I\big)^{-1} = \sum_{n=0}^{\infty} C_n(y)N^n. \tag{5.8}$$

Let $y \in (0,1)$ be fixed and define θ through $\sqrt{y} = \cos\theta \in (0,1)$. Using the trigonometric form (5.4) of U_n, we can then show by induction that, for any $m \geq 1$,

$$\begin{aligned}\sum_{n=1}^{m} C_n(y) &= \sum_{n=1}^{m} U_n(\sqrt{y})(\sqrt{y})^n \\ &= \sum_{n=1}^{m} \frac{\sin(n+1)\theta}{\sin\theta} \cdot (\cos\theta)^n \\ &= \frac{\cos^2\theta}{\sin^2\theta}[1-(\cos\theta)^{m-1} \cdot \cos(m+1)\theta] > 0.\end{aligned} \tag{5.9}$$

Since N is nonnegative and $e' \geq e'N$, we immediately get $e'N^n \geq e'N^{n+1} \geq 0$ for all $n \geq 0$. Combining this fact with (5.9), we obtain

$$e' \sum_{n=1}^{k} C_n(y) N^n \geq \sum_{n=1}^{k} C_n(y) e' N^k \geq 0' \qquad \text{for all } k \geq 1. \tag{5.10}$$

Therefore, from (5.7), (5.8) and (5.10) we conclude that, for all $y \in (0,1)$,

$$\begin{aligned}e'\big((1-y)I + y(I-N)^2\big)^{-1} &= e' \sum_{n=0}^{\infty} C_n(y) N^n \\ &= \lim_{k\to\infty} e' \sum_{n=1}^{k} C_n(y) N^n + e' \\ &\geq 0' + e' = e' > 0'.\end{aligned}$$

This concludes the proof of (c) and we deduce that there exists a CQLF for the pair $(-R, -R(I-pe'))$.

To prove the existence of a CQLF for the other pair $(-R, -(I-pe')R)$, we note that $-(I-pe')R$ has the same spectrum as $-R(I-pe')$ and the matrix product $R(I-pe')R$ has the same spectrum as $R^2(I-pe')$. Application of Proposition 3 completes the proof of Theorem 1. $\square$

5.2. *Proof of Theorem* 2. In this section we prove Theorem 2. Key to the proof is the CQLF constructed from Theorem 1.

PROOF. If $\alpha = 0$, then from (2.3) we know that Y has the piecewise linear drift

$$b(y) = -\beta p - R\big(y - p(e'y)^+\big).$$

By Theorem 1, there exists a CQLF

$$L(y) = y'Qy, \tag{5.11}$$

where Q is a positive definite matrix such that

$$Q(-R) + (-R)'Q < 0, \tag{5.12}$$

$$Q\big(-R(I - pe')\big) + \big(-(I - ep')R'\big)Q \le 0. \tag{5.13}$$

We claim that given any positive constant $C > 0$, there exists a constant $M > 0$ such that if $|y| > M$,

$$(\nabla L(y))'b(y) \le -C. \tag{5.14}$$

We discuss the cases $e'y < 0$ and $e'y \ge 0$ separately.

Case 1. $e'y < 0$. In this case, we have

$$(\nabla L(y))'b(y) = y'[Q(-R) + (-R)'Q]y - 2\beta p'Qy.$$

By (5.12), the quadratic term dominates if $|y|$ is large. Thus there exists a constant $M_1 > 0$ such that when $e'y < 0$ and $|y| > M_1$,

$$(\nabla L(y))'b(y) \le -C.$$

Case 2. $e'y \ge 0$. In this case, we have

$$(\nabla L(y))'b(y) = y'\big[Q\big(-R(I - pe')\big) + \big(-(I - ep')R'\big)Q\big]y - 2\beta p'Qy. \tag{5.15}$$

To overcome the difficulty caused by the singularity of $-R(I - pe')$, we decompose y as follows:

$$y = ap + \xi, \tag{5.16}$$

where $\xi'p = 0$ and $a \in \mathbb{R}$. Then we have

$$|y|^2 = |ap|^2 + |\xi|^2 \quad \text{and} \quad e'y = a + e'\xi \ge 0. \tag{5.17}$$

Note that $p'[Q(-R(I - pe')) + (-(I - ep')R')Q]p = 0$. Using (5.13), we obtain $p'[Q(-R(I - pe')) + (-(I - ep')R')Q] = 0'$. This immediately implies $p'[Q(-R(I - pe')] = 0$. Since $(I - pe')$ has a simple zero eigenvalue, we have

$$p'Q = be'R^{-1} \qquad \text{for some } b \ne 0.$$

Using this fact, we rewrite the left-hand side of (5.13) as

$$\begin{aligned} &Q\big(-R(I - pe')\big) + \big(-(I - ep')R'\big)Q \\ &\quad = \big((I - ep')R'\big) \cdot \big(-QR^{-1} - (R^{-1})'Q\big) \cdot \big(R(I - pe')\big). \end{aligned} \tag{5.18}$$

After left-multiplying by $(R^{-1})'$ and right-multiplying by R^{-1} in (5.12), we deduce that $[-QR^{-1} - (R^{-1})'Q]$ is a negative definite matrix. Moreover, since $\xi'p = 0$, from (5.16) and (5.18) we know that there exists some $c > 0$ such that

$$\begin{aligned} &y'\big[Q\big(-R(I - pe')\big) + \big(-(I - ep')R'\big)Q\big]y \\ &\quad = y'\big[(I - ep')R' \cdot \big(-QR^{-1} - (R^{-1})'Q\big) \cdot R(I - pe')\big]y \\ &\quad = \xi'\big((I - ep')R'\big) \cdot \big(-QR^{-1} - (R^{-1})'Q\big) \cdot \big(R(I - pe')\big)\xi \\ &\quad \le -c|\xi|^2. \end{aligned} \tag{5.19}$$

Therefore, from (5.15) we have that for any y with $e'y \geq 0$,

$$(\nabla L(y))'b(y) \leq -c|\xi|^2 - 2\beta p'Q\xi - 2\beta a p'Qp \tag{5.20}$$

$$\leq -c|\xi|^2 - 2\beta p'Q\xi + 2\beta p'Qpe'\xi, \tag{5.21}$$

where the second inequality is obtained from (5.17), $\beta > 0$ and $p'Qp > 0$. For $|y|$ large, if $|\xi| \geq r$ for some large constant r, we obtain $(\nabla L(y))'b(y) \leq -C$ since the quadratic term $-c|\xi|^2$ in (5.21) dominates. If $|\xi| < r$ and $|y|$ large, we deduce from (5.17) that a must be positive and large, that is,

$$a \geq \frac{1}{|p|}\sqrt{|y|^2 - r^2}.$$

Hence, the dominating term in (5.20) is $-2\beta a p'Qp$ and we immediately obtain $(\nabla L(y))'b(y) \leq -C$ whenever $|y|$ is large. Therefore, there exists a constant $M_2 > 0$ such that when $e'y \geq 0$ and $|y| > M_2$,

$$(\nabla L(y))'b(y) \leq -C.$$

On setting $M = \max\{M_1, M_2\}$, we immediately get (5.14).

Now set $C = |\sum_{i,j} Q_{ij}(\sigma\sigma')_{ij}| + 1$. Equations (5.11) and (5.14) imply that for $|y| > M$,

$$GL(y) = \sum_{i,j} Q_{ij}(\sigma\sigma')_{ij} + (\nabla L(y))'b(y) \leq -1.$$

The proof of Theorem 2 is complete after applying Proposition 1. $\square$

5.3. *Proof of Theorem* 3. In this section we prove Theorem 3. Throughout this section, C is a generic positive constant which may differ from line to line but is independent of y.

By Theorem 1, there exists a positive definite matrix $\widetilde{Q}$ with $|\widetilde{Q}| = 1$ such that

$$\widetilde{Q}(-R) + (-R)'\widetilde{Q} < 0, \tag{5.22}$$

$$\widetilde{Q}\big(-(I - pe')R\big) + \big(-R'(I - ep')\big)\widetilde{Q} \leq 0. \tag{5.23}$$

We construct a nonquadratic Lyapunov function $V \in C^2(\mathbb{R}^K)$ as follows. Let

$$V(y) = (e'y)^2 + \kappa[y - p\phi(e'y)]'\widetilde{Q}[y - p\phi(e'y)], \tag{5.24}$$

where κ is a positive constant to be decided later and $\phi(x)$ is a real-valued $C^2(\mathbb{R})$ function, approximating $x \mapsto x^+$. Specifically, fix $\varepsilon > 0$ and let

$$\phi(x) = \begin{cases} x, & \text{if } x \geq 0, \\ -\frac{1}{2}\varepsilon, & \text{if } x \leq -\varepsilon, \\ \text{smooth}, & \text{if } -\varepsilon < x < 0. \end{cases}$$

We piece $x \geq 0$ and $x \leq -\varepsilon$ together in a smooth way such that ϕ is in $C^2(\mathbb{R})$, $-\frac{1}{2}\varepsilon \leq \phi(x) \leq x^+$ and $0 \leq \dot{\phi}(x) \leq 1$ for any $x \in \mathbb{R}$, where $\dot{\phi}$ is the derivative of ϕ. This function ϕ evidently exists. Note that $V \in C^2(\mathbb{R}^K)$, but that it is not a CQLF due to its nonquadratic nature. We summarize the key result in the following proposition, which implies Theorem 3.

PROPOSITION 4. *If $\alpha > 0$, there exists a constant $C > 0$ such that when $|y|$ is large enough, we have*

$$(\nabla V(y))' b(y) \le -C|y|^2 \quad \text{and} \quad \left| \frac{\partial^2 V}{\partial y_i \, \partial y_j}(y) \right| \le C|y| \qquad \text{for any } i, j. \tag{5.25}$$

Consequently, when $|y|$ is large,

$$GV(y) \le -C|y|^2 \le -1. \tag{5.26}$$

PROOF. We first study $(\nabla V(y))' b(y)$. From (5.24), we have for all $y \in \mathbb{R}^K$,

$$(\nabla V(y))' = 2(e'y)e' + 2\kappa\big(y' - p'\phi(e'y)\big)\widetilde{Q}[I - pe'\dot{\phi}(e'y)]. \tag{5.27}$$

We discuss the cases $e'y \ge 0$, $e'y \le -\varepsilon$ and $-\varepsilon < e'y < 0$ separately.

Case 1. $e'y \ge 0$. In this case, let $x = e'y$ and $z = y - px = (I - pe')y$, then we have

$$\begin{aligned}
&(\nabla V(y))' b(y) \\
&\quad = [2(e'y)e' + 2\kappa y'(I - ep')\tilde{Q}(I - pe')]\big(-R(I - pe')y - \alpha pe'y - \beta p\big) \\
&\quad = -2\alpha x^2 - \kappa z'[\widetilde{Q}(I - pe')R + R'(I - ep')\widetilde{Q}]z - 2x\beta - 2xe'Rz.
\end{aligned}$$

Suppose we have shown that there exists $C > 0$ such that

$$z'[\widetilde{Q}(I - pe')R + R'(I - ep')\widetilde{Q}]z \ge C|z|^2, \tag{5.28}$$

we then obtain that

$$(\nabla V(y))' b(y) \le -2\alpha x^2 - \kappa C|z|^2 - 2x\beta - 2xe'Rz.$$

Since $\alpha > 0$, we can select $\kappa > 0$ large so that $\frac{1}{2}(2\alpha x^2 + \kappa C|z|^2) > 2|xe'Rz|$ for any (x, z), where κ is independent of (x, z) or y. Then we have,

$$(\nabla V(y))' b(y) \le -\alpha x^2 - \tfrac{1}{2}\kappa C|z|^2 - 2x\beta.$$

Note that $|y| = |px + z| \le C|(x, z)|$, so that $|(x, z)|$ is large whenever $|y|$ is large. We conclude that for $|y|$ large,

$$\begin{aligned}
(\nabla V(y))' b(y) &\le -C|(x, z)|^2 \\
&\le -C|y|^2.
\end{aligned}$$

It remains to prove (5.28). We use a similar argument as for (5.19). Observe that

$$(R^{-1}p)'[\widetilde{Q}(I - pe')R + R'(I - ep')\widetilde{Q}](R^{-1}p) = 0,$$

which implies that $\widetilde{Q}R^{-1}p = be$ for some $b \in \mathbb{R}$. Thus, we obtain

$$
\begin{aligned}
&z'[\widetilde{Q}(I - pe')R + R'(I - ep')\widetilde{Q}]z \\
&\qquad = z'R'(I - ep')[(R^{-1})'\widetilde{Q} + \widetilde{Q}R^{-1}](I - pe')Rz.
\end{aligned}
\tag{5.29}
$$

Since R is a nonsingular M-matrix, R^{-1} is a nonnegative matrix Berman and Plemmons [(1994), Chapter 6], and we deduce that

$$
e'R^{-1}p > 0. \tag{5.30}
$$

This implies that $(I - pe')Rz \neq 0$ since $e'z = e'(I - pe')y = 0$ in this case. From (5.22) we know that $(R^{-1})'\widetilde{Q} + \widetilde{Q}R^{-1}$ is a positive definite matrix. Now (5.28) follows from (5.29).

Case 2. $e'y < -\varepsilon$. In this case, we have $\phi(e'y) = -\frac{1}{2}\varepsilon$ and $\dot{\phi}(e'y) = 0$. From (5.22), there exists $C > 0$ such that

$$
\begin{aligned}
(\nabla V(y))'b(y) &= \bigl(2(e'y)e' + 2\kappa y'\widetilde{Q} + \kappa\varepsilon p'\widetilde{Q}\bigr)(-Ry - \beta p) \\
&= -2\kappa\bigl[y'(\widetilde{Q}R + R'\widetilde{Q})y + \tfrac{1}{2}(\varepsilon p'\widetilde{Q}R + \beta p'\widetilde{Q})y + \tfrac{1}{2}\varepsilon\beta p'\widetilde{Q}p\bigr] \\
&\qquad - 2e'y \cdot (e'Ry + \beta) \\
&\le -2\kappa\bigl[C|y|^2 + \tfrac{1}{2}(\varepsilon p'\widetilde{Q}R + \beta p'\widetilde{Q})y + \tfrac{1}{2}\varepsilon\beta p'\widetilde{Q}p\bigr] \\
&\qquad - 2e'y \cdot (e'Ry + \beta) \\
&\le -2\kappa\bigl[C|y|^2 + \tfrac{1}{2}(\varepsilon p'\widetilde{Q}R + \beta p'\widetilde{Q})y + \tfrac{1}{2}\varepsilon\beta p'\widetilde{Q}p\bigr] \\
&\qquad + \kappa C(|y|^2 + |y|) \\
&\le -\kappa(C|y|^2 - C|y| - C),
\end{aligned}
$$

where κ is again chosen to be independent of y, but large enough such that $|2e'y \cdot (e'Ry + \beta)| < \kappa C(|y|^2 + |y|)$. Thus for $|y|$ large and $e'y < -\varepsilon$, we have

$$
(\nabla V(y))'b(y) \le -C|y|^2.
$$

Case 3. $-\varepsilon \le e'y \le 0$. In this case we use the property that $0 \le \dot{\phi}(e'y) \le 1$. Note that we have

$$
\begin{aligned}
&(\nabla V(y))'b(y) \\
&\quad = \bigl(2(e'y)e' + 2\kappa\bigl(y' - p'\phi(e'y)\bigr)\widetilde{Q}\bigl(I - pe'\dot{\phi}(e'y)\bigr)\bigr)(-Ry - \beta p) \\
&\quad = 2e'ye'(-Ry - \beta p) \\
&\qquad + 2\kappa\dot{\phi}(e'y)\bigl(y' - p'\phi(e'y)\bigr)\widetilde{Q}(I - pe')(-Ry - \beta p) \\
&\qquad + 2\kappa\bigl(1 - \dot{\phi}(e'y)\bigr)\bigl(y' - p'\phi(e'y)\bigr)\widetilde{Q}(-Ry - \beta p).
\end{aligned}
$$

We write

$$
y = aR^{-1}p + \xi,
$$

where ξ is orthogonal to $R^{-1}p$ and $a \in \mathbb{R}$, so that

$$|y|^2 = ca^2 + |\xi|^2 \qquad \text{for some } c > 0. \tag{5.31}$$

From (5.30), we have $e'R^{-1}p > 0$. Without loss of generality we assume that $e'R^{-1}p = 1$. Then $e'y = a + e'\xi$ and we get

$$\begin{aligned}
&(\nabla V(y))'b(y) \\
&\quad = -2(a + e'\xi)(\beta + e'R\xi + a) \\
&\qquad + \kappa\dot{\phi}(e'y)\big(\xi'[\widetilde{Q}(-(I - pe')R) + (-(I - pe')R)'\widetilde{Q}]\xi \\
&\qquad\qquad - 2p'\widetilde{Q}(I - pe')R\xi\phi(e'y)\big) \\
&\qquad + \kappa\big(1 - \dot{\phi}(e'y)\big) \\
&\qquad\quad \times \big(y'[-\widetilde{Q}R - R'\widetilde{Q}]y + \beta y'\widetilde{Q}p - \phi(e'y)p'\widetilde{Q}Ry - p'\widetilde{Q}p\beta\big).
\end{aligned} \tag{5.32}$$

Since $\xi'R^{-1}p = 0$, one checks, as for (5.28), that there exists a constant $C > 0$ such that

$$\xi'[\widetilde{Q}(-(I - pe')R) + (-(I - pe')R)'\widetilde{Q}]\xi \le -C|\xi|^2. \tag{5.33}$$

Moreover, from (5.22) and (5.31), we deduce that

$$y'[-\widetilde{Q}R - R'\widetilde{Q}]y \le -C|y|^2 = -Ca^2 - C|\xi|^2. \tag{5.34}$$

Substituting (5.33) and (5.34) into (5.32), and using $0 \le \dot{\phi}(e'y) \le 1$ as well as $|\phi(e'y)| \le \varepsilon$, we obtain

$$\begin{aligned}
&(\nabla V(y))'b(y) \\
&\quad \le -2(a^2 + C|a||\xi| + C|a|) + \kappa(-C|\xi|^2 + C|\xi| + C|a| + C).
\end{aligned} \tag{5.35}$$

Since $e'y = a + e'\xi \in [-\varepsilon, 0]$, we must have $|a| \le C + |\xi|$ and consequently $|y| \le C|a| + |\xi| \le C|\xi| + C$. Thus for $|y|$ large, we can choose κ large so that the dominating term in (5.35) is $-\kappa C|\xi|^2$. Using the fact that $|y|^2 \le C|\xi|^2$ when $|y|$ is large, we then deduce that there exists a constant $C > 0$ such that for $|y|$ large,

$$(\nabla V(y))'b(y) \le -C|y|^2.$$

This concludes the proof for the third case.

On combining the above three cases we obtain that, for $|y|$ large,

$$(\nabla V(y))'b(y) \le -C|y|^2,$$

as claimed in the proposition.

We now proceed to study the second derivative of V, which is denoted by $\ddot{V}$. We also write $\ddot{\phi}$ for the second derivative of ϕ. From (5.27), we find

$$\begin{aligned}
\ddot{V}(y) = 2ee' + 2\kappa\big[&\widetilde{Q} + ee' \cdot p'\widetilde{Q}p\big(\ddot{\phi}(e'y)\phi(e'y) + \dot{\phi}(e'y)^2\big) \\
&- (\widetilde{Q}pe' + ep'\widetilde{Q})\dot{\phi}(e'y) - ee' \cdot y'\widetilde{Q}p\ddot{\phi}(e'y)\big].
\end{aligned} \tag{5.36}$$

If $e'y \notin [-\varepsilon, 0]$, we obtain $0 \le \dot{\phi}(e'y) \le 1$ and $\ddot{\phi}(e'y) = 0$. Therefore, for any i, j, there exists some $C > 0$ such that

$$\left|\frac{\partial^2 V}{\partial y_i\,\partial y_j}(y)\right| \le C.$$

If $e'y \in [-\varepsilon, 0]$, then $|\ddot{\phi}(e'y)| \le C$ for some $C > 0$ since $\phi \in C^2(\mathbb{R})$ and $[-\varepsilon, 0]$ is compact. Moreover, since $0 \le \dot{\phi}(e'y) \le 1$, the dominating term in (5.36) is $-2\kappa ee' \cdot y'\widetilde{Q}p\ddot{\phi}(e'y)$ for $|y|$ large. This implies that if $e'y \in [-\varepsilon, 0]$ and $|y|$ is large, then there exists a constant $C > 0$ such that for any i, j,

$$\left|\frac{\partial^2 V}{\partial y_i\,\partial y_j}(y)\right| \le C|y|,$$

where C is independent of y. This concludes the proof of (5.25). Now for $|y|$ large, we deduce from (5.25) that

$$GV(y) = (\nabla V(y))'b(y) + \frac{1}{2}\sum_{i,j}(\sigma\sigma')_{ij}\frac{\partial^2 V}{\partial y_i\,\partial y_j}(y) \le -C|y|^2 \le -1.$$

The proof of Proposition 4 is complete. $\square$

PROOF OF THEOREM 3. In order to show that Y is positive recurrent and has a unique stationary distribution, we only have to check that $V(y) \to \infty$ as $|y| \to \infty$ in view of Proposition 1 and (5.26).

Let $x = e'y$ and $z = y - px^+$, then $|y|^2 \le C(x^2 + |z|^2)$. We can rewrite (5.24) as follows:

$$\begin{aligned}
V(y) &= x^2 + \kappa\big(y' - p'\phi(x)\big)\widetilde{Q}\big(y - p\phi(x)\big) \\
&\ge x^2 + C|y - p\phi(x)|^2 \\
&= x^2 + C\big|z + p\big(x^+ - \phi(x)\big)\big|^2 \\
&\ge x^2 + C|z|^2 - C\varepsilon^2 \\
&\ge C|y|^2 - C\varepsilon^2,
\end{aligned}$$

where the second last inequality uses the fact $0 \le x^+ - \phi(x) \le \frac{1}{2}\varepsilon$. Therefore, $V(y) \to \infty$ as $|y| \to \infty$ and we conclude that Y has a unique stationary distribution.

To prove that Y is exponentially ergodic, we observe from (5.24) that there exists some $C > 0$ such that $V(y) \le C|y|^2 + C$ for all $y \in \mathbb{R}^K$. Moreover, (5.26) implies that for $|y|$ large,

$$GV(y) \le -CV(y) + C.$$

Putting this together with the fact that $V \in C^2(\mathbb{R}^K)$, we know that there exist some $c > 0$ and $d < \infty$ such that

$$GV(y) \le -cV(y) + d \qquad \text{for any } y \in \mathbb{R}^K.$$

Since $V \geq 0$, Proposition 2 implies that Y is f-exponentially ergodic, where $f = V + 1$. In particular, Y is exponentially ergodic since $f \geq 1$. □

APPENDIX A: PROOF OF PROPOSITION 3

We first outline the key idea behind the proof. Suppose that (B, g) is not controllable or that (B, h) is not observable in the CQLF existence problem. Then we can "reduce" them to suitable subspaces such that (B_1, g_1) is controllable and (B_1, h_1) is observable, where B_1 is a new matrix of lower dimension than B and similarly for g_1, h_1. In the process of "reduction," two desired properties are preserved: (a) $B(B - gh')$ has no real negative eigenvalues if and only if $B_1(B_1 - g_1h_1')$ has no real negative eigenvalues; (b) $(B, B - gh')$ has a CQLF if and only if $(B_1, B_1 - g_1h_1')$ has a CQLF. Therefore, applying Theorem 3.1 in Shorten et al. (2009) to $(B_1, B_1 - g_1h_1')$ yields the result.

To make the ideas concrete, we now introduce a lemma giving an equivalent formulation of the CQLF existence problem, which makes the "reduction" possible. The lemma is an analog of Proposition 2 in King and Nathanson (2006). In King and Nathanson (2006), each matrix of the pair is nonsingular while in our case one of the matrices is singular.

LEMMA 1. *Suppose that all eigenvalues of the matrix B have negative real part and all eigenvalues of $B - gh'$ have negative real part, except for a simple zero eigenvalue. Then the following statements are equivalent*:

(a) *The pair $(B, B - gh')$ does not have a CQLF.*
(b) *There are positive semidefinite matrices X and Z such that*

$$BX + XB' + (B - gh')Z + Z(B' - hg') = 0,$$
$$BX + XB' \neq 0 \quad \textit{and} \quad (B - gh')Z + Z(B' - hg') \neq 0.$$

(c) *There are nonzero, positive semidefinite matrices X and Z such that*

$$BX + XB' + (B - gh')Z + Z(B' - hg') = 0, \tag{A.1}$$

where $Z \neq cB^{-1}gg'(B^{-1})'$ for any $c \in \mathbb{R}$.

PROOF. We first prove the equivalence of (a) and (b). To set up the notation, let $S^{K\times K}$ be the space of real symmetric $K \times K$ matrices. For an arbitrary matrix $A \in \mathbb{R}^{K\times K}$, define the linear operator L_A on $S^{K\times K}$ by

$$L_A : S^{K\times K} \to S^{K\times K}, \qquad L_A(H) = AH + HA'. \tag{A.2}$$

It is well known that if A has eigenvalues $\{\lambda_i\}$ with eigenvectors $\{v_i\}$, then L_A has eigenvalues $\{\lambda_i + \lambda_j\}$ with eigenvectors $\{v_iv_j' + v_jv_i'\}$ for all $i \leq j$. Since all eigenvalues of the matrix B have negative real part, L_B is invertible.

Following King and Nathanson (2006), we formulate the CQLF existence problem in terms of separating convex cones in $S^{K\times K}$. Define $\mathrm{Cone}(B) = \{L_B(X)|X \geq 0\}$ and $\mathrm{Cone}(B-gh') = \{L_{(B-gh')}(Z)|Z \geq 0\}$. Both are closed convex cones in $S^{K\times K}$. Let $S^{K\times K}$ be equipped with the usual Hilbert–Schmidt inner product $\langle X, Z\rangle = \mathrm{tr}(XZ)$. We obtain that for any $Q \in S^{K\times K}$,

$$\langle X, QB + B'Q\rangle = \langle Q, BX + XB'\rangle = \langle Q, L_B(X)\rangle.$$

Note that for a nonzero positive semidefinite matrix X, we have $QB + B'Q < 0$ if and only if $\langle X, QB + B'Q\rangle < 0$, where the "if" part can be checked by taking $X = xx'$ for any nonzero $x \in \mathbb{R}^K$, and the "only if" part follows from the spectral decomposition of the positive semidefinite matrix X. Therefore, we have $QB + B'Q < 0$ if and only if $\langle Q, M\rangle < 0$ for all nonzero $M \in \mathrm{Cone}(B)$. Using a similar argument one finds that $Q(B - gh') + (B - hg')Q \leq 0$ if and only if $\langle Q, T\rangle \leq 0$ for all nonzero $T \in \mathrm{Cone}(B - gh')$. Moreover, since B only has eigenvalues with negative real part, we deduce that $QB + B'Q < 0$ for $Q \in S^{K\times K}$ implies that Q is positive definite by Theorem 2.2.3 in Horn and Johnson (1994). By definition of CQLF, we thus obtain that $(B, B - gh')$ has a CQLF if and only if there exists a $Q \in S^{K\times K}$ such that $QB + B'Q < 0$ and $Q(B-gh')+(B-hg')Q \leq 0$. Equivalently, $(B, B-gh')$ has a CQLF if and only if there exists a $Q \in S^{K\times K}$ such that $\langle Q, M\rangle > 0$ for all nonzero $M \in \mathrm{Cone}(-B)$ and $\langle Q, T\rangle \leq 0$ for all nonzero $T \in \mathrm{Cone}(B - gh')$. Therefore, finding a CQLF for the pair $(B, B - gh')$ is the same as finding a separating hyperplane in $S^{K\times K}$ for $\mathrm{Cone}(-B)$ and $\mathrm{Cone}(B - gh')$. By the separating hyperplane theorem, we conclude that $(B, B - gh')$ not having a CQLF is equivalent to $\mathrm{Cone}(-B)$ and $\mathrm{Cone}(B-gh')$ having nonzero intersection. This completes the proof of the equivalence of (a) and (b).

We now turn to the equivalence of (b) and (c), for which we use the aforementioned spectral properties of the linear operator (A.2). Since L_B is invertible, we deduce that $L_B(X) = 0$ is equivalent to $X = 0$. We know that all eigenvalues of $(B - gh')$ have negative real part except for a simple zero eigenvalue, hence, $L_{(B-gh')}$ also has a simple zero eigenvalue with eigenvector $cB^{-1}gg'(B^{-1})'$ for some nonzero $c \in \mathbb{R}$ while all of its other eigenvalues have negative real part. Consequently, $(B - gh')Z + Z(B - gh')' \neq 0$ is equivalent to $Z \neq cB^{-1}gg'(B^{-1})'$ for any $c \in \mathbb{R}$. The proof of the lemma is complete. □

PROOF OF PROPOSITION 3. In view of Theorem 3.1 of Shorten et al. (2009), we need to check that controllability of (B, g) and observability of (B, h) need not be verified in the CQLF existence problem. Recall that controllability of (B, g) means that the vectors $g, Bg, B^2g, \ldots$ span $\mathbb{R}^K$, and observability of (B, h) means that the vectors $h, B'h, (B')^2h, \ldots$ span $\mathbb{R}^K$. To simplify the notation, let $\widetilde{B} = B - gh'$.

We first show that in the CQLF existence problem for the pair $(B, B - gh')$, we can assume without loss of generality that (B, g) is controllable. The proof relies on Lemma 1. Let U be the span of vectors $g, Bg, B^2g \dots$. Suppose U is a proper subspace of $\mathbb{R}^K$ with $\dim(U) < K$, and note that $\mathbb{R}^K = U \oplus U^\perp$ where $U^\perp$ is the orthogonal complement of U. In view of this decomposition, we perform a change of basis and rewrite B, $\tilde{B}$, X and Y in the block form

$$B = \begin{pmatrix} B_1 & B_2 \\ 0 & B_3 \end{pmatrix}, \qquad \widetilde{B} = \begin{pmatrix} \widetilde{B}_1 & \widetilde{B}_2 \\ 0 & B_3 \end{pmatrix},$$
$$X = \begin{pmatrix} X_1 & X_2 \\ X_2' & X_3 \end{pmatrix}, \qquad Z = \begin{pmatrix} Z_1 & Z_2 \\ Z_2' & Z_3 \end{pmatrix}, \tag{A.3}$$

where $B - \widetilde{B} = gh'$ and g, h are represented in the new basis. We use the same notation for the matrices and vectors after the change of basis to save space, and we remark that the orthogonal transformation does not affect the existence of a CQLF for the pair $(B, \tilde{B})$ or the existence of real negative eigenvalues of $B\tilde{B}$. Namely, for any orthonormal matrix $O \in \mathbb{R}^K$, one readily checks that the pair $(B, \tilde{B})$ has a CQLF if and only if the pair $(OBO', O\tilde{B}O')$ has a CQLF. Furthermore, $B\tilde{B}$ has no real negative eigenvalues if and only if $(OBO')(O\tilde{B}O')$ has no real negative eigenvalues. Let g_1, h_1 be the orthogonal projection of g, h on the subspace U, so that $B_1 - \widetilde{B}_1 = g_1 h_1'$. Since U is the span of the vectors $g, Bg, B^2g \dots$, we deduce that $g_1, B_1 g_1, B_1^2 g_1 \dots$ span U by (A.3), that is, (B_1, g_1) is controllable. We now use Lemma 1 to argue that there exists a CQLF for $(B, \widetilde{B})$ if and only if there exists a CQLF for $(B_1, \widetilde{B}_1)$, where (B_1, g_1) is controllable. Note that (A.3) implies, using (A.1) in Lemma 1,

$$B_3(X_3 + Z_3) + (X_3 + Z_3)B_3' = 0.$$

Equivalently,

$$L_{B_3}(X_3 + Z_3) = 0,$$

where the linear operator L_{B_3} is defined in (A.2). Since B has only eigenvalues with negative real part, B_3 also has this property. This implies the linear operator L_{B_3} is invertible. We thus obtain $X_3 + Z_3 = 0$. Using the fact that X and Z are positive semidefinite, we deduce that $X_3 = Z_3 = 0$, and consequently $X_2 = Z_2 = 0$. This leads to

$$B_1 X_1 + X_1 B_1' + \widetilde{B}_1 Z_1 + Z_1 \widetilde{B}_1' = 0. \tag{A.4}$$

Thus, for the pair $(B, B - gh')$, the existence of nonzero $X, Z \geq 0$ such that (A.1) holds implies the existence of nonzero $X_1, Z_1 \geq 0$ such that (A.4) holds. Conversely, if there exists nonzero $X_1, Z_1 \geq 0$ such that (A.4) holds, setting $X_2 = X_3 = Z_2 = Z_3 = 0$, we then obtain that there exists nonzero $X, Z \geq 0$ such that (A.1) holds. Since $B - gh'$ has only eigenvalues with negative real part except for a simple zero eigenvalue, so does $B_1 - g_1 h_1'$. For $c \in \mathbb{R}$, since $g \in U$, one finds that $g'(B^{-1})' = (g_1'(B_1^{-1})', 0')$ by (A.3). Thus $Z \neq cB^{-1}gg'(B^{-1})'$

is equivalent to $Z_1 \neq cB_1^{-1}g_1g_1'(B_1^{-1})'$. Putting these facts together, we apply Lemma 1 to conclude that $(B, \widetilde{B})$ has no CQLF if and only if $(B_1, \widetilde{B}_1)$ has no CQLF, where (B_1, g_1) is controllable. Therefore, without loss of generality, we can assume that (B, g) is controllable in the CQLF existence problem for the pair $(B, B - gh')$.

We next show that without loss of generality we can assume that (B, h) is observable in the CQLF existence problem for the pair $(B, B - gh')$. Note that for $Q > 0$, we have $QB + B'Q < 0$ and $Q(B - gh') + (B' - hg')Q \leq 0$ if and only if $Q^{-1}B' + BQ^{-1} < 0$ and $Q^{-1}(B - hg') + (B' - gh')Q^{-1} \leq 0$. Hence, $(B, B - gh')$ has a CQLF if and only if $(B', B' - hg')$ has a CQLF. From the preceding paragraph, we know that in the CQLF existence problem for the pair $(B', B' - hg')$, we can assume that (B', h) is controllable without loss of generality. By definition, (B', h) being controllable is the same as (B, h) being observable. Therefore, we conclude that we can assume without loss of generality that (B, h) is observable.

Finally, we argue that the pair $(B, B - gh')$ has a CQLF if and only if the matrix product $B(B - gh')$ has no real negative eigenvalues. Assuming that (B, g) is controllable and that (B, h) is observable, Theorem 3.1 in Shorten et al. (2009) states that $(B, B - gh')$ has a CQLF if and only if the matrix product $B(B - gh')$ has no real negative eigenvalues. We have shown that we can always assume that (B, g) is controllable and that (B, h) is observable in the CQLF existence problem by reduction to proper subspaces. So it only remains to check that in the process of reduction, the spectral property of having no real negative eigenvalues of the matrix product is preserved. Specifically, in the above proof that controllability of (B, g) can be assumed without loss of generality, we obtain that $(B, B - gh')$ has a CQLF if and only if $(B_1, B_1 - g_1h_1')$ has a CQLF, where (B_1, g_1) is controllable. We next prove that $B(B - gh')$ has no real negative eigenvalues if and only if $B_1(B_1 - g_1h_1')$ has no real negative eigenvalues, that is, the desired spectral property of the matrix product is preserved in the process of reduction from $(B, B - gh')$ to $(B_1, B_1 - g_1h_1')$. Observe that the spectrum of $B(B - gh')$ is the union of the spectrum of $B_1(B_1 - g_1h_1')$ and B_3^2 by (A.3). Since all eigenvalues of B_3 have negative real part, we deduce that $B_1(B_1 - g_1h_1')$ having no real negative eigenvalues is equivalent to $B(B - gh')$ having no real negative eigenvalues. A similar argument applies for observability instead of controllability. We have therefore completed the proof of Proposition 3. $\square$

APPENDIX B: ANY QUADRATIC FUNCTION FAILS FOR $\alpha > 0$

In this section, we give a simple example showing that, in general, no quadratic function can serve as a Lyapunov function in the Foster–Lyapunov criterion to prove positive recurrence of the piecewise OU process Y for $\alpha > 0$. We first introduce a lemma which implies that the matrix $-R(I - pe') - \alpha pe'$ is nonsingular for $\alpha > 0$.

LEMMA 2. *If $\alpha > 0$, then all eigenvalues of the matrix $-R(I-pe')-\alpha pe'$ have negative real part.*

PROOF. It is clear that the matrix has an eigenvalue $-\alpha$ with right eigenvector p. Suppose $\lambda \neq -\alpha$ is an eigenvalue of the matrix with left eigenvector θ, that is,

$$\theta'\bigl(-R(I-pe')-\alpha pe'\bigr)=\lambda\theta', \tag{B.1}$$

then we obtain that $\theta' p = 0$. It follows from (B.1) that λ is an eigenvalue of the matrix $-R(I-pe')$. Moreover, λ cannot be zero since otherwise $\theta' = ce'R^{-1}$ for some nonzero $c \in \mathbb{R}$, which follows from the fact that $R(I-pe')$ has a simple zero eigenvalue. This contradicts the fact that $e'R^{-1}p > 0$ as seen in (5.30). From condition (b) in the proof of Theorem 1, we know that all nonzero eigenvalues of the matrix $-R(I-pe')$ have negative real part. This completes the proof of the lemma. $\square$

LEMMA 3. *Suppose that Q is a real $K \times K$ positive semidefinite matrix such that at least one of the matrices $Q(-R)+(-R')Q$ and $Q(-R(I-pe')-\alpha pe')+(-(I-ep')R'-\alpha ep')Q$ fails to be negative definite. Let the quadratic function L be given by $L(y)=y'Qy$ for $y \in \mathbb{R}^K$. Then there exists some $\beta \in \mathbb{R}$ and $v \in \mathbb{R}^K$ such that $GL(tv) \geq 0$ for any $t \geq 0$.*

PROOF. Suppose that $Q(-R)+(-R')Q$ fails to be negative definite, then there exists some $\lambda \geq 0$ and nonzero vector $v \in \mathbb{R}^K$ such that $[Q(-R)+(-R')Q]v = \lambda v$ and $e'v \leq 0$. By definition of generator of Y in (2.2), we thus obtain

$$\begin{aligned} GL(tv) &= \sum_{i,j} Q_{ij}(\sigma\sigma')_{ij} + (\nabla L(tv))'b(tv) \\ &= \sum_{i,j} Q_{ij}(\sigma\sigma')_{ij} + t^2 v'[Q(-R)+(-R')Q]v - 2t\beta p'Qv \\ &= \sum_{i,j} Q_{ij}(\sigma\sigma')_{ij} + \lambda v'vt^2 - 2t\beta p'Qv. \end{aligned} \tag{B.2}$$

Since Q is positive semidefinite, we infer that $\sum_{i,j} Q_{ij}(\sigma\sigma')_{ij} = \operatorname{tr}(Q\sigma\sigma') = \operatorname{tr}(\sigma'Q\sigma) \geq 0$. Set $\beta = 0$. We conclude from (B.2) that $GL(tv) \geq 0$ for any $t \geq 0$. A similar argument applies to the case where $Q(-R(I-pe')-\alpha pe')+(-(I-ep')R'-\alpha ep')Q$ fails to be negative definite. The proof of the lemma is complete. $\square$

In view of Lemmas 2 and 3, we give the following definition of strong CQLF which is slightly different than Definition 5 given in Section 3.1. For more details, refer to Shorten and Narendra (2003) and King and Nathanson (2006).

DEFINITION 6 (Strong CQLF). Let A and B be real $K \times K$ matrices having only eigenvalues with negative real part. For $Q \in \mathbb{R}^{K\times K}$, the quadratic form L given by $L(y) = y'Qy$ for $y \in \mathbb{R}^K$ is called a strong common quadratic Lyapunov function (strong CQLF) for the pair (A, B) if Q is positive definite and

$$QA + A'Q < 0,$$
$$QB + B'Q < 0.$$

We remark that it suffices to require Q to be a symmetric matrix in the above definition by Theorem 2.2.3 in Horn and Johnson (1994).

We now formulate an example showing that, in general, no quadratic function can serve as a Lyapunov function in the Foster–Lyapunov criterion to prove positive recurrence of the piecewise OU process Y for $\alpha > 0$. Let R be a matrix given by

$$R = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix},$$

so that R is a nonsingular M-matrix. Let $\alpha = 133$ and $p' = [0, 0, 1]$.

LEMMA 4. *For any quadratic function L given by $L(y) = y'Qy$ for some real $K \times K$ positive semidefinite matrix Q and all $y \in \mathbb{R}^K$, there exists some $\beta \in \mathbb{R}$ and $v \in \mathbb{R}^K$ such that $GL(tv) \geq 0$ for any $t \in \mathbb{R}$ in the above example.*

PROOF. In view of Lemma 3, it suffices to prove that there is no strong CQLF for the pair $(-R, -R(I - pe') - \alpha pe')$ for $\alpha > 0$. Equivalently, it suffices to show that the matrix product $R(R(I - pe') + \alpha pe')$ has real negative eigenvalues by Theorem 1 in King and Nathanson (2006). One readily checks that $R(R(I - pe') + \alpha pe')$ has three different eigenvalues: -7, $5 - \sqrt{82}$ and $5 + \sqrt{82}$. Thus, it has two real negative eigenvalues and we deduce that $(-R, -R(I - pe') - \alpha pe')$ has no strong CQLF in this example. Application of Lemma 3 completes the proof of the lemma. $\square$

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H. MILTON STEWART SCHOOL OF INDUSTRIAL
AND SYSTEMS ENGINEERING
GEORGIA INSTITUTE OF TECHNOLOGY
ATLANTA, GEORGIA 30332
USA
E-MAIL: ton.dieker@isye.gatech.edu
gxf1240@gatech.edu