Nonnegativity of solutions to the basic adjoint relationship for some diffusion processes

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Version: Februari 10, 2011

1 Introduction

This note describes an open problem for two classes of diffusion processes. The first class is semimartingale reflecting Brownian motions (SRBMs) and the second class is piecewise Ornstein-Uhlenbeck (OU) processes. The open problem asks for conditions under which a solution to a basic adjoint relationship (BAR) associated with such a diffusion process does not change sign. The state space of a d-dimensional SRBM is a d-dimensional orthant, and thus the SRBM is a constrained diffusion process. SRBMs arise as diffusion approximations for open queueing networks in conventional heavy traffic when the number of servers at each station is fixed to be small; see, for example, [13]. A d-dimensional piecewise OU process is an unconstrained diffusion process that lives in $\mathbb{R}^d$. These diffusion processes serve as diffusion approximations for many-server queues when the service time distributions are of phase-type [7, 17].

The open problem was first stated as a conjecture in [4] for SRBMs in a two-dimensional rectangle and in [5] for SRBMs in a d-dimensional orthant. The conjecture is a critical ingredient in determining stationary distributions of SRBMs, both numerically and analytically. For instance, numerical algorithms have been developed in [4, 5] to compute the stationary distribution of an SRBM. The convergence of these algorithms critically relies on the validity of the conjecture, and the open problem plays the same role in the context of piecewise OU processes [6]. Moreover, in some known cases where a stationary distribution of an SRBM has been determined analytically, an essential and challenging step is to verify that the measure is unsigned, see [11, 16]. The validity of the conjecture would remove the need for this verification.

The open problem could be discussed in a general setting such as general diffusion processes or even general Markov processes. Restricting attention to SRBMS and piecewise OU processes allows us to convey the essence of the problem in concrete settings with minimal notation and assumptions, while the distinct features of the two classes of diffusion processes allow us to discuss a variety of tools that could be relevant in the context of the open problem. Also, resolving the open problem for these two specific classes of processes is important in itself, because of the applications mentioned above.

Before we end this introduction, we introduce some notation that will be used in the rest of this note. For a vector v, we write $v > 0$ to mean that each component of v is positive, and we write $v \geq 0$ to mean that each component of v is non-negative. For an $x \in \mathbb{R}$, $x^+ = \max(x, 0)$. For a vector $v = (v_i)$, $v^+ = (v_i^+)$ and $\|v\|$ denotes its Euclidean norm. Let $C_0(\mathbb{R}^d)$ be the set of continuous functions on $\mathbb{R}^d$ vanishing at infinity, equipped with the topology generated by the norm $\|f\| = \sup_{x\in\mathbb{R}^d} |f(x)|$. Let $C_b^2(\mathbb{R}_+^d)$ be the set of twice continuously differentiable functions f on $\mathbb{R}_+^d$ such that f and its first- and second-order derivatives are bounded. Also let $C_c^\infty(\mathbb{R}^d)$ be the set of infinitely differentiable functions that have compact supports in $\mathbb{R}^d$. For a signed measure π on $\mathbb{R}^d$, we use $\pi = \pi_+ - \pi_-$ to denote its Jordan decomposition and set $|\pi| = \pi_+ + \pi_-$.

2 SRBMs

Multidimensional SRBMs are first introduced in the pioneering paper of [14]; see [9, 18] for surveys on SRBMs. The state space for a d-dimensional SRBM $Z = \{Z(t), t \geq 0\}$ is $\mathbb{R}_+^d$. The data of the SRBM are a drift vector μ, a non-singular covariance matrix Σ, and a $d \times d$ "reflection matrix" R that specifies boundary behavior. Roughly speaking, in the interior of the orthant, Z behaves as an ordinary Brownian motion with parameters μ and Σ; Z is pushed in direction R^j whenever the boundary surface $\{z \in \mathbb{R}_+^d : z_j = 0\}$ is hit, where R^j is the jth column of R, for $j = 1, \ldots, d$. To make this description more precise, one represents Z in the form

$$Z(t) = X(t) + RY(t), \quad t \geq 0, \tag{2.1}$$

where X is an unconstrained Brownian motion with drift vector μ, covariance matrix Σ, and $Z(0) = X(0) \in \mathbb{R}_+^d$, and Y is a d-dimensional process with components $Y_1, \ldots, Y_d$ such that

$$Y \text{ is continuous and non-decreasing with } Y(0) = 0, \tag{2.2}$$

$$Y_j \text{ only increases at times } t \text{ for which } Z_j(t) = 0, \quad j = 1, \ldots, d, \quad \text{and} \tag{2.3}$$

$$Z(t) \in \mathbb{R}_+^d, \quad t \geq 0. \tag{2.4}$$

An SRBM Z is defined to be a (weak) solution to (2.1)–(2.4); namely, there exists (X, Y, Z), defined on some filtered probability space $(\Omega, \{\mathcal{F}_t\}, \mathbb{P})$, such that (X, Y, Z) is adapted to $\{\mathcal{F}_t\}$, X is an $\{\mathcal{F}_t\}$-Brownian motion, and (X, Y, Z) satisfies (2.1)–(2.4) almost surely; see, for example, Appendix A of [2] for a complete definition.

A $d \times d$ matrix R is said to be an $\mathcal{S}$*-matrix* if there exists a d-vector $w \geq 0$ such that $Rw > 0$, and R is said to be *completely-*$\mathcal{S}$ if each of its principal sub-matrices is an $\mathcal{S}$-matrix. Given a drift vector μ and a non-singular covariance matrix Σ, there exists an (Σ, μ, R)-SRBM satisfying (2.1)–(2.4) for each initial point $Z(0) \in \mathbb{R}_+^d$ if and only if R is a completely-$\mathcal{S}$ matrix; moreover, Z is unique in distribution when R is completely-$\mathcal{S}$. References for these fundamental results can be found in the survey papers [9, 18].

Let (Σ, μ, R) be fixed. Hereafter we assume that Σ is positive definite and R is completely-$\mathcal{S}$. Let Z be the SRBM associated with the data $(\mathbb{R}_+^d, \Sigma, \mu, R)$. Assume that the SRBM has a stationary distribution. A necessary condition of the existence of the stationary distribution is

$$R \text{ is non-singular and } R^{-1}\mu < 0; \tag{2.5}$$

see, for example, [2] for a proof. It follows from the proof in [15] that the stationary distribution π is unique, and each component of $\mathbb{E}_\pi(Y(1))$ is finite and $\mathbb{E}_\pi(Y(1)) = -R^{-1}\mu > 0$, $\mathbb{E}_\pi$ is the expectation operator conditioned on $Z(0)$ having stationary distribution π. (Harrison and Williams [15] consider an SRBM with reflection matrix R being an $\mathcal{M}$-matrix, a class of matrices defined in Chapter 6 of [1]; but its proof can be straightforwardly carried over to a general reflection matrix.) For a Borel set $A \subset \mathbb{R}^d_+$, define

$$\nu_i(A) = \mathbb{E}_\pi \int_0^1 1_{\{Z(u)\in A\}} dY_i(u), \quad i = 1, \ldots, d.$$

Clearly, ν_i defines a finite measure on $\mathbb{R}^d_+$, which has a support on $F_i = \{x \in \mathbb{R}^d_+ : x_i = 0\}$.

Using Itô formula, one can immediately obtain the following relationship governing the stationary distribution π and boundary measures $\nu_1, \ldots, \nu_d$. For each $f \in C^2_b(\mathbb{R}^d_+)$,

$$\int_{\mathbb{R}^d_+} Lf(x)\pi(dx) + \sum_{i=1}^d \int_{\mathbb{R}^d_+} D_i f(x)\nu_i(dx) = 0, \tag{2.6}$$

where

$$Lf(x) = \frac{1}{2}\sum_{i,j=1}^d \Sigma_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{i=1}^d \mu_i \frac{\partial f}{\partial x_i}(x),$$

$$D_i f(x) = \sum_{j=1}^d R_{ji}\frac{\partial f}{\partial x_j}(x), \quad i = 1, \ldots, d.$$

Equation (2.6) is known as the basic adjoint relationship (BAR). Clearly, (2.6) holds for all $f \in C^2_b(\mathbb{R}^d)$ is equivalent to that (2.6) holds for all $f \in C^\infty_c(\mathbb{R}^d)$. The fact that a stationary distribution of an SRBM in $\mathbb{R}^d_+$ must satisfy BAR (2.6) was first proved in [15] when R is an $\mathcal{M}$-matrix. Extension to an SRBM with a general reflection matrix, when its stationary distribution exists, can be proved without any modification; see, for example, Proposition 3 of [5] for this fact. The following proposition establishes the converse.

Proposition 1. *Assume that $\pi, \nu_1, \ldots, \nu_d$ are finite measures on $\mathbb{R}^d_+$ and that ν_i is supported on F_i, $i = 1, \ldots, d$. If $(\pi, \nu_1, \ldots, \nu_d)$ jointly satisfies BAR (2.6), then π equals the stationary distribution of Z up to a multiplicative constant.*

Proposition 1 was first proved in [8] for SRBMs in polyhedron domains and more recently in [3] for a class of reflecting Brownian motions that do not necessarily have the semimartingale representation as in (2.1). We are now in a position to formulate our first open problem.

Open problem 1. *Prove Proposition 1 with '$\pi, \nu_1, \ldots, \nu_d$ are finite measures' replaced by '$\pi, \nu_1, \ldots, \nu_d$ are finite signed measures'.*

To resolve this open problem it suffices to show that solutions $(\pi, \nu_1, \ldots, \nu_d)$ of (2.6) cannot change sign, since one can then invoke Proposition 1 to conclude that π must be equal to the stationary distribution up to a multiplicative constant.

3 Piecewise OU process

A d-dimensional piecewise Ornstein-Uhlenbeck (OU) process Z is defined to be a (strong) solution to

$$Z(t) = X(t) + \int_0^t b(Z(s))ds \quad t \geq 0, \tag{3.1}$$

where X is a d-dimensional driftless Brownian motion with covariance matrix Σ and the drift function $b : x \in \mathbb{R}^d \to b(x) \in \mathbb{R}^d$ is a piecewise linear function that is Lipschitz continuous in x. We are particularly interested in the case when the drift function b is of the form

$$b(x) = a + Bx + C(Dx)^+,$$

where B, C and B are $d \times d$ matrices and a is a d-dimensional vector. The diffusion limits in [7, 17] have such a form for their drift functions.

Assume that π is a stationary distribution of Z. By applying Ito's formula, one can show that π satisfies the following basic adjoint relationship (BAR)

$$\int_{\mathbb{R}^d} Lf(x)\pi(dx) = 0 \quad \text{for each } f \in C_b^2(\mathbb{R}^d), \tag{3.2}$$

where

$$Lf = \frac{1}{2}\sum_{i,j=1}^d \Sigma_{ij}\frac{\partial f}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(x)\frac{\partial f}{\partial x_i}.$$

The following lemma says that the converse is also true. The proof simply verifies all the conditions in Echeverría's theorem, see Theorem 4.9.17 of [12].

Lemma 1. *If π is a probability measure that satisfies (3.2), then it is a stationary distribution.*

Proof. Clearly b is locally bounded. Because b is assumed to be Lipschitz, there exists some $K > 0$ such that

$$\langle x, b(x)\rangle \leq K(1 + \|x\|^2), \quad x \in \mathbb{R}^d.$$

By Proposition 4.9.17 of [12], there exists a stationary solution of the martingale problem for $(\mathcal{L}, \pi)$, where $\mathcal{L} = \{(f, Lf) : f \in C_c^\infty(\mathbb{R}^d)\}$. Let Z be the piecewise OU process satisfying (3.1) with $Z(0) = X(0)$ following distribution π. Applying Ito's formula, one can check that Z is a solution to the martingale problem $(\mathcal{L}, \pi)$. By Theorem 8.1.7 of [12], the solution to the martingale problem is unique. Therefore, the piecewise OU process Z is stationary with martingale distribution π. The claim follows from Echeverría's theorem. $\square$

Note that Lemma 1 is the analog of Proposition 1 in the piecewise OU setting. We now formulate our second open problem.

Open problem 2. *Assume that Σ and $b(x)$ are such that the piecewise OU process has a stationary distribution, and assume that π is a finite signed measure on $\mathbb{R}^d$ that satisfies (3.2). Decide if any additional conditions on b should be imposed in order to prove that π is a linear combination of stationary probability measures.*

The conditions, if any, should be mild so that they are satisfied for the piecewise OU process arising from the many-server diffusion approximations introduced in [7]. It has recently been shown in [10] that these piecewise OU processes have a unique stationary distribution. For these diffusions, the open problem thus reduces to showing that any π satisfying (3.2) is proportional to the stationary distribution. In this sense, Open problem 2 is the analog of Open problem 1 in the setting of [7].

The remainder of this section shows that, under the additional assumption that the drift function b is bounded, Open problem 2 can be resolved using existing theory of Markov processes. The boundedness assumption prevents us from applying the result to piecewise OU processes arising from many-server diffusion approximations. Our proof hints at the difficulties to resolve the open problem in general.

Proposition 2. *Suppose that Z has a unique stationary distribution. Assume that the drift rate $b : \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz continuous and bounded. Let π be a finite signed measure that satisfies (3.2). Then π must be proportional to the stationary distribution.*

Proof. By Theorem 8.1.6 of [12], the minimal closed linear extension of the operator $\mathcal{L} = \{(f, Lf) : f \in C_c^\infty(\mathbb{R}^d)\}$ on $C_0(\mathbb{R}^d) \times C_0(\mathbb{R}^d)$ is single-valued and generates a Feller semigroup $\{T(t)\}$ on $C_0(\mathbb{R}^d)$. This semigroup must coincide with the transition semigroup generated by our piecewise OU diffusion since the generator of the diffusion equals $\mathcal{L}$ on $C_c^\infty(\mathbb{R}^d)$ by Ito's formula. Write $\bar{\mathcal{L}} = \{(f, \bar{L}f) : f \in \mathcal{D}(\bar{L})\}$ for the extended generator with corresponding domain $\mathcal{D}(\bar{L})$.

Let π satisfy (3.2). Then we have $\int \bar{L}f(x)\pi(dx) = 0$ for all $f \in \mathcal{D}(\bar{L})$, since

$$\left|\int \bar{L}f(x)\pi(dx)\right| = \left|\int (\bar{L}f(x) - Lf_n(x))\pi(dx)\right| \leq |\pi|(\mathbb{R}^d) \cdot \|\bar{L}f - Lf_n\| \to 0$$

whenever $(f_n, Lf_n) \to (f, \bar{L}f)$ for some $f \in \mathcal{D}(\bar{L})$ and $\{f_n\} \subset C_c^\infty(\mathbb{R}^d)$.

Let $h \in C_0(\mathbb{R}^d)$. By Proposition 1.1.5 of [12], for any $t \geq 0$, $\int_0^t T(s)hds \in \mathcal{D}(\bar{L})$ and

$$T(t)h - h = \bar{L}\int_0^t T(s)hds.$$

In conjunction with $\int \bar{L}f(x)\pi(dx) = 0$ for $f = \int_0^t T(s)hds$, this yields $\int (T(t)h)(x)\pi(dx) = \int h(x)\pi(dx)$ for $h \in C_0(\mathbb{R}^d)$. Application of Lemma 2 in Section 4 below, with $P = T(t)$, completes the proof in conjunction with the assumption of a unique stationary distribution. □

4 Analogs in discrete time and discrete space

In this section, we show that, in the discrete state space setting or in the discrete time setting, the problem analogous to Open problems 1 and 2 has been resolved completely. Thus, the difficulty of Open problems 1 and 2 stems from the continuity of time coupled with the continuity of space of a Markov process.

When Z is a continuous time Markov chain (CTMC) on a discrete space S and it is irreducible and positive recurrent, its stationary distribution π exists and is unique. Furthermore, π satisfies

$$\langle \pi, Lf \rangle = 0 \quad \text{for any bounded function } f \text{ on } S. \tag{4.1}$$

Here $\langle \cdot, \cdot \rangle$ denotes the inner product of two vectors and L is the generator (a matrix) of the CTMC. Equation (4.1) is a CTMC analog of a basic adjoint relationship. By the irreducibility of L, any signed vector π satisfying (4.1) and $\sum_{i \in S} |\pi_i| < \infty$ must be proportional to the unique stationary distribution. Therefore, the result analogous to the ones proposed in Open problems 1 and 2 is well known in the CTMC setting.

Let $Z = \{Z_n : n = 0, 1, \ldots\}$ be a discrete time Markov chain with probability transition function $P(x, A)$ for $x \in \mathbb{R}^d$ and Borel set $A \subset \mathbb{R}^d$. Here, for each $x \in \mathbb{R}^d$, $P(x, \cdot)$ is a probability measure on $\mathbb{R}^d$, and for each Borel set A, the function $P(\cdot, A)$ is a Borel measurable function on $\mathbb{R}^d$. A distribution π on $\mathbb{R}^d$ is a stationary distribution for Z if and only if $\pi(A) = \int_{\mathbb{R}^d} P(x, A)\pi(dx)$ for each Borel set A. The latter condition is equivalent to $\int_{\mathbb{R}^d} f(x)\pi(dx) = \int_{\mathbb{R}^d} Pf(x)\pi(dx)$ for each $f \in C_0(\mathbb{R}^d)$, which is further equivalent to

$$\int_{\mathbb{R}^d} Lf(x)\pi(dx) = 0 \quad \text{for all} \quad f \in C_0(\mathbb{R}^d), \tag{4.2}$$

where $Pf(x) = \int_{\mathbb{R}^d} P(x, dy)f(y)$ and $Lf(x) = Pf(x) - f(x)$. We have the following lemma.

Lemma 2. *Let P be a probability transition function on $\mathbb{R}^d$. Assume π is a finite signed measure on $\mathbb{R}^d$ that satisfies (4.2). Then both π_+ and π_- satisfy (4.2), where $\pi = \pi_+ - \pi_-$ is the Jordan decomposition of π.*

Remark. As a consequence of Lemma 2, if the corresponding Markov chain has a unique stationary distribution, any finite signed measure π that satisfies (4.2) and $\pi(\mathbb{R}^d) = 1$ must be a probability measure, which is equal to the stationary distribution. Therefore, the lemma says that the problem analogous Open problems 1 and 2 is also completely resolved in the discrete time setting.

Proof of Lemma 2. It readily seen that (4.2) holds for all $f \in C_0(\mathbb{R}^d)$ implies that it holds for all $f \in B(\mathbb{R}^d)$, the set of bounded functions on $\mathbb{R}^d$. Indeed, for every $f \in B(\mathbb{R}^d)$ there exists a sequence $\{f_n\} \subset C_0(\mathbb{R}^d)$ such that $\sup_n \sup_x |f_n(x)| < \infty$ and $\lim_{n\to\infty} f_n(x) = f(x)$ for each $x \in \mathbb{R}^d$. Applying the bounded convergence theorem twice, once relying on the boundedness of the operator P, we obtain

$$\int_{\mathbb{R}^d} f(x)\pi(dx) = \lim_{n\to\infty} \int_{\mathbb{R}^d} f_n(x)\pi(dx) = \lim_{n\to\infty} \int_{\mathbb{R}^d} Pf_n(x)\pi(dx) = \int_{\mathbb{R}^d} Pf(x)\pi(x).$$

Taking f to be an indicator of a Borel set A, we have $\pi(A) = \int_{\mathbb{R}^d} P(x, A)\pi(dx)$. Write $\pi = \pi_+ - \pi_-$ for the Jordan decomposition of π, and let Γ be the support of π_+. It follows that

$$\pi_+(\Gamma) = \pi(\Gamma) = \int_{\mathbb{R}^d} P(x,\Gamma)\pi(dx) = \int_{\Gamma} P(x,\Gamma)\pi_+(dx) - \int_{\Gamma^c} P(x,\Gamma)\pi_-(dx),$$

where Γ^c denotes the complement of Γ. The right-hand side is bounded from above by $\pi_+(\Gamma)$. We must therefore have that $P(x,\Gamma) = 1$ for $x \in \Gamma$, and similarly $P(x,\Gamma^c) = 1$ for $x \in \Gamma^c$. It is readily seen that this implies that both π_+ and π_- satisfy (4.2). □

Acknowledgement

We thank Tom Kurtz for valuable discussions, in particular for outlining the proof of Lemma 2. We are grateful to the Newton Institute for their hospitality during the program Stochastic Processes in Communication Sciences. JD is supported in part by NSF grants CMMI-0727400, CMMI-0825840, CMMI-1030589, and by an IBM Faculty Award. ABD gratefully acknowledges NSF grant EEC-0926308.

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