Conditional limit theorems for queues with Gaussian input
a weak convergence approach

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ABSTRACT

We consider a buffered queueing system that is fed by a Gaussian source and drained at a constant rate. The fluid offered to the system in a time interval $(0, t]$ is given by a separable continuous Gaussian process Y with stationary increments. The variance function $\sigma^2 : t \mapsto \mathbb{V}\mathrm{ar} Y_t$ of Y is assumed to be regularly varying with index $2H$, for some $0 < H < 1$.

By proving conditional limit theorems, we investigate how a high buffer level is typically achieved. The underlying large deviation analysis also enables us to establish the logarithmic asymptotics for the probability that the buffer content exceeds u as $u \to \infty$. In addition, we study how a busy period longer than T typically occurs as $T \to \infty$, and we find the logarithmic asymptotics for the probability of such a long busy period.

The study relies on the weak convergence in an appropriate space of $\{Y_{\alpha t}/\sigma(\alpha) : t \in \mathbb{R}\}$ to a fractional Brownian motion with Hurst parameter H as $\alpha \to \infty$. We prove this weak convergence under a fairly general condition on σ^2, sharpening recent results of Kozachenko $et\ al.$ [22]. The core of the proof consists of a new type of uniform convergence theorem for regularly varying functions with positive index.

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1. Introduction

When studying a buffered queueing system, one is often interested in the following two questions:

Q1: How is a high buffer level achieved?

Q2: If the buffer is nonempty for a long time, how does this event occur?

This paper considers these questions for a single server queue fed by a Gaussian process with stationary increments. The buffer is drained at a constant rate.

There are good reasons to investigate the above questions in a Gaussian framework. Firstly, Gaussian processes can model both short-range dependence (as in, e.g., an Ornstein-Uhlenbeck process) and long-range dependence (as in, e.g., a fractional Brownian motion with Hurst parameter exceeding 1/2). Moreover, in several situations (e.g., [23, 31]) the normality assumption is motivated by a central limit-type result. From a theoretical point of view, Gaussian processes are easier to analyze due to the vast body of literature on these processes.

The behavior of a queue conditioned on the occurrence of a rare event has been studied in different contexts. Hooghiemstra [21] studies the behavior of the waiting times if a long busy period occurs, and obtains in the (weak) limit a Brownian excursion. A different type of limit theorem is found by Anantharam [4]. He studies how a queue must have evolved when the waiting time has become large. By underlying independence assumptions, this occurs by 'staying close' to a piecewise linear path. We also mention Bertoin and Doney [7], who focus on the initial behavior of a random walk conditioned to stay nonnegative.

The typical behavior of a queue has also been studied in connection to the first question Q1. Due to the close relationship between queueing processes and risk processes (see, e.g., [6, Ch. II.3]), it is equivalent to ask how ruin occurs in the corresponding risk model. In a compound Poisson setting, this is addressed in Chapter IV.7 of [6] (see also Asmussen [5]). It turns out that the path is linear. However, as a result of possible correlations in the system input, this need not be the case in the Gaussian setting.

Although the second question Q2 has not been investigated explicitly in the literature, there is some related work. In queueing language, the question deals with the length of the steady-state busy period. Norros [28] considers a queue with fractional Brownian motion input, and studies the probability that the length of the busy period exceeds T as $T \to \infty$. He formulates a variational problem for which the solution determines the logarithmic asymptotics. This result is generalized by Kozachenko *et al.* [22], who also allow for other Gaussian input processes. In the present paper, we considerably widen the class of Gaussian processes for which the logarithmic asymptotics hold.

As answers to the above questions, we provide two conditional limit theorems. As for Q1, we identify a path x^* such that, under a certain condition, the (scaled) distribution of the Gaussian process Y given that the buffer reaches a high level u converges (in a sense that will be made precise) to a Dirac mass δ_{x^*} at x^*. That is, for every regular set of paths A, as $u \to \infty$,

$$P\left(\left\{\frac{1}{u}Y_{ut} : t \in \mathbb{R}\right\} \in A \,\middle|\, \sup_{t\geq 0} Y_t - t \geq u\right) \to \begin{cases} 1 & \text{if } x^* \in A; \\ 0 & \text{otherwise.} \end{cases}$$

A similar conditional limit theorem is given for the busy-period problem.

A weak convergence approach

In order to precisely explain the contributions of our work, we need to formalize our framework.

Let Y denote a centered separable Gaussian process with stationary increments. The central assumption is that the *variance function* $\sigma^2 : t \in \mathbb{R} \to \mathbb{V}\mathrm{ar} Y_t \in [0,\infty)$ is continuous and regularly varying with index $2H$ for some $0 < H < 1$, i.e.,

$$\lim_{\alpha\to\infty} \frac{\sigma^2(\alpha t)}{\sigma^2(\alpha)} = |t|^{2H}. \tag{1.1}$$

Notice that the function σ^2 characterizes the finite-dimensional distributions of Y.

Interestingly, by a powerful theorem of Chevet [11], the proofs of the conditional limit theorems rely only on a (sufficiently strong) type of weak convergence of the processes Y^α as $\alpha \to \infty$, with

$$Y_t^\alpha := \frac{Y_{\alpha t}}{\sigma(\alpha)}. \tag{1.2}$$

We now precisely describe the type of weak convergence that we show; it is explained in Section 3.4 that other types of weak convergence are not strong enough to provide satisfactory answers to the above questions. For $\gamma \geq 0$, set

$$\Omega^\gamma := \left\{ x : \mathbb{R} \to \mathbb{R} \text{ such that } x \text{ continuous}, x(0) = 0, \lim_{t\to\pm\infty} \frac{x(t)}{1+|t|^\gamma} = 0 \right\},$$

and equip Ω^γ with the topology generated by the norm

$$\|x\|_{\Omega^\gamma} := \sup_{t\in\mathbb{R}} \frac{|x(t)|}{1+|t|^\gamma}, \tag{1.3}$$

under which Ω^γ is a separable Banach space. Endow Ω^γ with the Borel σ-field induced by this topology, denoted by $\mathcal{B}(\Omega^\gamma)$. As pointed out in Section 4.2, under the condition $\lim_{t\to 0} \sigma^2(t) |\log|t||^{1+\epsilon} < \infty$ for some $\epsilon > 0$, Y^α takes almost surely values in Ω^γ for $\gamma > H$ and the law of Y^α in $(\Omega^\gamma, \mathcal{B}(\Omega^\gamma))$ exists; it is denoted by ν_α^γ. Hence, it is legitimate to ask whether ν_α^γ has a weak limit for $\alpha \to \infty$.

By considering the finite-dimensional distributions, it is readily seen that the only candidate weak limit is the law $\mathcal{L}(B_H)$ in Ω^γ of a fractional Brownian motion B_H with Hurst parameter H. Recall that a fractional Brownian motion B_H is a continuous centered Gaussian process with stationary increments and variance function $\mathbb{V}\mathrm{ar} B_H(t) = |t|^{2H}$; for $H = 1/2$, it reduces to ordinary Brownian motion. We write $Y^\alpha \Rightarrow B_H$ and $\nu_\alpha^\gamma \Rightarrow \mathcal{L}(B_H)$ for convergence in distribution and weak convergence respectively; when this notation is used, we also specify the space (and topology) in which this convergence takes place.

Comparison with previous results

Conditions for the weak convergence of Y^α in Ω^1 have been derived by Kozachenko *et al.* [22]. Their conditions are based on the *majorizing variance*

$$\overline{\sigma}^2(t) = \sup_{0<s<t} \sup_{\alpha\geq 1} \frac{\sigma^2(\alpha s)}{\sigma^2(\alpha)}. \tag{1.4}$$

Unfortunately, apart from some special cases, $\overline{\sigma}^2(t)$ is difficult to bound or compute.

By taking a different approach than Kozachenko *et al.* [22], we show that Y^α converges weakly to B_H in Ω^γ for $\gamma > H$ under the *same* condition that we use to guarantee the existence of ν_α^γ: $\lim_{t\to 0}\sigma^2(t)|\log|t||^{1+\epsilon} < \infty$ for some $\epsilon > 0$. This not only relaxes the condition in Proposition 2.9 of [22], but is also easier to check. As in [22], we rely on metric entropy techniques. However, we exploit the regular variation of the variance function before applying these techniques. Specifically, we present a new type of uniform convergence theorem for regularly varying functions with positive index.

To illustrate the advantage of the condition developed in this paper, consider the situation that the process Y is the superposition of a finite number m of independent Gaussian process with stationary increments. The variance functions of the m individual Gaussian processes are denoted by $\sigma_1^2, \dots, \sigma_m^2$, and σ_i is assumed to be regularly varying with index $H_i \in (0,1)$. The variance function $\sigma^2 = \sum_i \sigma_i^2$ of Y is then regularly varying with index $2\max_i H_i$, but it is in general impossible to compute the majorizing variance (1.4). In contrast, $\lim_{t\to 0}\sigma^2(t)|\log|t||^{1+\epsilon} < \infty$ for some $\epsilon > 0$ if and only if the same is true for the *individual* variance functions $\sigma_1^2, \dots, \sigma_m^2$.

The outline of the paper is as follows. In Section 2, we introduce the two queueing problems in more detail, and we state the theorems that provides answers to the above questions. Section 3 provides background material on notions that are crucial in the proofs of these theorems, including the new uniform convergence result for regularly varying functions (proven in Section 6). The convergence in distribution of Y^α to a fractional Brownian motion is the subject of Section 4; we present both a necessary and a sufficient condition. With these weak convergence results at our disposal, the proofs of the claims in Section 2 are given in Section 5.

2. Queueing results

In this section, we present the two conditional limit theorems that serve as answers to the two questions raised in the Introduction. As indicated there, a key role in the proofs of the results is played by the convergence in distribution of Y^α to B_H in Ω^γ. Since this convergence is the subject of Section 4, we defer all proofs for the present section to Section 5.

2.1 Conditional limit theorem for high buffer level

Before presenting the announced conditional limit theorem, it is insightful to first have a closer look at the probability

$$P\left(\sup_{t\geq 0} Y_t - ct^\beta \geq u\right) \tag{2.1}$$

for $\beta > H$ and $c > 0$, as $u \to \infty$. In case $\beta = 1$, this probability equals the steady-state probability that the buffer content exceeds u when the input is Y the drain rate is c; this situation is described in the Introduction. Since we allow $\beta > H$, we analyze the problem in slightly more generality. We note that studying $u \to \infty$ is known as considering the *large buffer* asymptotic regime.

There exists a vast body of literature that deals with the logarithmic asymptotics of (2.1), under different levels of generality (Duffield and O'Connell [19], Dębicki *et al.* [14], and Kozachenko *et al.* [22]). An important contribution in this setting was made by Dębicki [13], who establishes the logarithmic asymptotics for $\beta = 1$ under the technical requirement that $\lim_{u\to\infty} P(\sup_{t\geq 0} Y_t - \epsilon t > u) = 0$ for $\epsilon > 0$. However, this condition is automatically satisfied in case Y has stationary increments, since $Y_t/t \to 0$ almost surely (see Lemma 3).

In [13] it is also assumed that σ^2 increases, but this assumption can be avoided by invoking the Uniform Convergence Theorem for regularly varying functions (Theorem 1.5.2 of [9]) in Lemma 3.1. Hence, only assuming continuity of the sample paths of Y and regular variation of the variance function suffices to establish the logarithmic asymptotics of (2.1).

We remark that the *exact* asymptotics of (2.1) have been studied extensively in the past few years. Quite general expressions have recently been found [18], and we refer to that article for background and references.

We now turn to logarithmic asymptotics again. It was already noted that these are known to hold under the the condition that Y has continuous sample paths. However, the proof given in this paper relies on the weak convergence of the processes Y^α, and we therefore require the (stronger) condition $\lim_{t\to 0}\sigma^2(t)|\log|t||^{1+\epsilon}<\infty$ for some $\epsilon>0$; see Section 5.

Proposition 1 *If $\lim_{t\to 0}\sigma^2(t)|\log|t||^{1+\epsilon}<\infty$ for some $\epsilon>0$, then for $\beta>H$,*

$$\lim_{u\to\infty}\frac{\sigma^2(u^{1/\beta})}{u^2}\log P\left(\sup_{t\geq 0}Y_t-ct^\beta\geq u\right)=-\frac{1}{2}c^{2H/\beta}\left(\frac{H}{\beta-H}\right)^{-2H/\beta}\left(\frac{\beta}{\beta-H}\right)^2.$$

One of the advantages of using the weak convergence approach is that one can analyze the large deviations on a path level. This large deviation study yields a path x^* that can be interpreted as the 'most probable' path. We believe that it is impossible to do such a large deviation analysis in case one only requires continuity of Y instead of the stronger assumption $\lim_{t\to 0}\sigma^2(t)|\log|t||^{1+\epsilon}<\infty$. In the setting of this subsection, the path is given by

$$\begin{aligned} x^*(t) &= \frac{1+(t^*)^\beta}{2(t^*)^{2H}}\mathbb{C}\mathrm{ov}(B_H(t),B_H(t^*)) \\ &= \frac{\beta}{2(\beta-H)}\left(1+\left|\frac{t}{t^*}\right|^{2H}-\left|1-\frac{t}{t^*}\right|^{2H}\right), \end{aligned} \tag{2.2}$$

for $t\in\mathbb{R}$, where $t^*=(H/[\beta-H])^{1/\beta}$. We formalize the intuition that x^* (suitably scaled) is the 'most likely' trajectory of Y when Y_t-ct^β reaches u.

Theorem 1 *If $\lim_{t\to 0}\sigma^2(t)|\log|t||^{1+\epsilon}<\infty$ for some $\epsilon>0$, then for any $\beta>H$, the law of*

$$\frac{\sigma([u/c]^{1/\beta})}{u}Y^{(u/c)^{1/\beta}}\Bigg|\sup_{t\geq 0}Y_t-ct^\beta\geq u$$

converges weakly in Ω^β to the Dirac measure δ_{x^*} at x^* as $u\to\infty$.

Notice that the weak convergence stated in the theorem implies, for instance, that for any $\eta>0$, $\beta>H$,

$$\lim_{u\to\infty}P\left(\sup_{t\in\mathbb{R}}\frac{\left|\frac{1}{u}Y_{u^{1/\beta}t}-x^*(c^{1/\beta}t)\right|}{1+c|t|^\beta}\geq\eta\,\Bigg|\sup_{t\geq 0}Y_t-ct^\beta\geq u\right)=0.$$

For $\beta=1$, the most likely time epoch for Y_t-ct to hit u is ut^* and hence linear in u. Interestingly, according to Theorem 1, if Y_t-ct^β reaches u, $\frac{\sigma([u/c]^{1/\beta})}{u}Y^{(u/c)^{1/\beta}}$ is typically 'close' to x^*, which is only a straight line when $H = 1/2$. See Addie *et al.* [1] for 'most likely paths' in the *many sources* asymptotic regime.

Recently, there has been some interest in the probability (2.1) where the supremum is taken over $[0, T]$ instead of the entire positive halfline [15, 20]. It is legitimate to ask whether the weak convergence approach can also be taken in this setting to obtain logarithmic asymptotics and a conditional limit theorem. This is not the case; the probability then equals

$$P\left(\sup_{t\in[0,T]} Y_t - ct^\beta \geq u\right) = P\left(\sup_{t\in[0,[u/c]^{-1/\beta}T]} \frac{\frac{1}{u}Y_{[u/c]^{1/\beta}t}}{1+t^\beta} \geq 1\right),$$

so that knowledge of the large deviations of $\frac{1}{u}Y_{[u/c]^{1/\beta}t}$ is useless due to the presence of u in the interval. Of course, this is readily solved if Y is self-similar, and in that case we can indeed obtain logarithmic asymptotics and a conditional limit theorem. However, for a self-similar process Y, weak convergence in the scaling (1.2) is trivial and the large deviations are given by Schilder's theorem; see Section 3.3.

2.2 Conditional limit theorem for the length of a busy period

In this subsection, we gain some insights in the steady-state distribution of the length of a so-called busy period. We refer to Norros [28] for an introduction to the problem; here, we only review the notation.

For $x \in \Omega^1$, define the function $s : \Omega^1 \to \mathbb{R}^\mathbb{R}$ by

$$s(x)(t) := \sup_{s\leq t}(x(t) - x(s) - (t - s)). \tag{2.3}$$

While $x(t)$ $(-x(-t))$ represents the amount of work arriving in the interval $[0, t]$ $([-t, 0])$ for $t \geq 0$, $s(x)(t)$ can be thought of as the amount of work in the buffer at time t when the system 'input' is x. We set

$$t_-(x) := \sup\{t \leq 0 : s(x)(t) = 0\}, \quad t_+(x) := \inf\{t \geq 0 : s(x)(t) = 0\},$$

i.e., $t_-(x)$ $(t_+(x))$ is the last (first) time $s(x)$ hits zero before (after) zero. We say that zero is contained in a *busy period*, since an imaginary server is constantly draining the buffer during the time interval $[t_-(x), t_+(x)]$.

The set of paths in Ω^1 for which the busy period containing zero is strictly longer than T is denoted by K_T, i.e.,

$$K_T := \{x \in \Omega^1 : t_-(x) < 0 < t_+(x), t_+(x) - t_-(x) > T\}.$$

It is our aim to find the logarithmic asymptotics of $P(Y \in K_T)$ as $T \to \infty$.

Norros [28] considers this setting for $Y = B_H$, and his results are generalized by Kozachenko *et al.* [22] to allow for more general input processes. The next proposition generalizes their findings. A key role in the result is played by a separable Hilbert space $\mathcal{H}_H$: the Cameron-Martin space associated with the law of B_H. More details on this space can be found in [28]. The norm induced by the inner product on $\mathcal{H}_H$ is denoted by $\|\cdot\|_{\mathcal{H}_H}$.

Proposition 2 *If* $\lim_{t\to 0} \sigma^2(t)|\log|t||^{1+\epsilon} < \infty$ *for some* $\epsilon > 0$*, then*

$$\lim_{T\to\infty} \frac{\sigma^2(T)}{T^2} \log P(Y \in K_T) = -\frac{1}{2} \inf_{x\in K_1\cap\mathcal{H}_H} \|x\|^2_{\mathcal{H}_H}. \tag{2.4}$$

Let us stress the fact that $\mathcal{H}_H$ is the Cameron-Martin space associated with the law of B_H; the right hand side of (2.4) does not depend on the specific form of σ^2, but only on its index of variation.

Proposition 2 can also be used to derive the logarithmic asymptotics in case $c(t-s)$ is substracted from $x(t)-x(s)$ in the definition of $s(x)$ in (2.3). Equation (2.3) shows that we essentially replace the distribution of Y by the distribution of $\tilde{Y}=Y/c$. Evidently, the variance function $\tilde{\sigma}^2$ of $\tilde{Y}$ then equals $\tilde{\sigma}^2=\sigma^2/c^2$. We conclude that the following logarithmic asymptotics apply:

$$\lim_{T\to\infty}\frac{\sigma^2(T)}{T^2}\log P(\tilde{Y}\in K_T)=-\frac{c^2}{2}\inf_{x\in K_1\cap\mathcal{H}_H}\|x\|^2_{\mathcal{H}_H}.$$

The constant $\inf_{x\in K_1\cap\mathcal{H}_H}\|x\|^2_{\mathcal{H}_H}$ is generally difficult to identify, except for the case $H=1/2$; in that case, it equals 1. An expression for the path $\overline{x}\in\overline{K}_1\cap\mathcal{H}_H$ with $\inf_{x\in K_1\cap\mathcal{H}_H}\|x\|^2_{\mathcal{H}_H}=\|\overline{x}\|^2_{\mathcal{H}_H}$ in the complementary case $H\neq 1/2$ has been found recently [27]. Even without this knowledge, it is possible to formulate the analog of Theorem 1.

Theorem 2 *If* $\lim_{t\to 0}\sigma^2(t)|\log|t||^{1+\epsilon}<\infty$ *for some* $\epsilon>0$*, then the law of*

$$\left.\frac{\sigma(T)}{T}Y^T\right| Y\in K_T$$

converges for $\gamma>H$ *weakly in* Ω^γ *to the Dirac measure* $\delta_{\overline{x}}$ *at* $\overline{x}$ *as* $T\to\infty$

As in the preceding subsection, this theorem implies for instance that for any $\eta,\zeta>0$,

$$\lim_{T\to\infty}P\left(\sup_{t\in\mathbb{R}}\frac{\left|\frac{1}{T}Y_{Tt}-\overline{x}(t)\right|}{1+|t|^{H+\zeta}}\geq\eta\,\middle|\,Y\in K_T\right)=0.$$

3. Background

In this section, we discuss some background on three concepts that we use extensively in the remainder of the paper: regular variation, metric entropy and large deviations. Moreover, we address some topological issues that we raised in the Introduction.

Before we start, we introduce the notation

$$C([-T,T]):=\{x:[-T,T]\to\mathbb{R}\text{ such that }x\text{ continuous},x(0)=0\},$$

and equip $C([-T,T])$ with the topology of uniform convergence, i.e., the topology generated by the norm $\|x\|_T:=\sup_{t\in[-T,T]}|x(t)|$. Note that $C([-T,T])$ equipped with this topology is a separable Banach space. We write $\mathcal{B}(C([-T,T]))$ for the Borel σ-field on $C([-T,T])$ generated by the topology of uniform convergence.

3.1 Regular variation

We first give the definition of regular variation, cf. Equation (1.1).

Definition 1 *A nonnegative measurable function* f *on* $[0,\infty)$ *is said to be regularly varying at infinity with index* $\rho\in\mathbb{R}$ *if for all* $t>0$,

$$\lim_{\alpha\to\infty}\frac{f(\alpha t)}{f(\alpha)}=t^\rho.$$

Unless otherwise stated, we only consider regular variation at infinity. For more details and extensions of regular variation, the reader is referred to Bingham, Goldie and Teugels [9].

We now present a property of regularly varying functions that is crucial in this paper, particularly in Section 4. Define $L_\epsilon : \mathbb{R} \to [0, \infty)$ by

$$L_\epsilon(t) := \begin{cases} |\log|t||^{1+\epsilon} & \text{if } |t| \le 1/e; \\ 1 & \text{otherwise.} \end{cases} \tag{3.1}$$

Proposition 3 *Let f be regularly varying with index $\rho > 0$. If fL_ϵ is bounded on each interval $(0, \cdot]$ for some $\epsilon > 0$, then we have*

$$\lim_{\alpha\to\infty} \frac{f(\alpha t)}{f(\alpha)} L_\epsilon(t) = t^\rho L_\epsilon(t),$$

uniformly in t on each $(0, \cdot]$.

Proof. The proof is given in Section 6.1. □

Notice that the requirement that fL_ϵ be bounded on intervals of the form $(0, \cdot]$ is equivalent to local boundedness of f and $\limsup_{t\downarrow 0} f(t)|\log t|^{1+\epsilon} < \infty$. Alternatively, one can replace the L_ϵ by other continuous positive functions with the following two properties: on compact subsets of $(0, \infty)$, it is bounded away from zero and bounded from above, and near zero it is equivalent to L_ϵ. An example of such a function is $(\log(1 + 1/t))^{1+\epsilon}$.

3.2 Metric entropy

Metric entropy is an important tool in studying continuity and boundedness of trajectories of Gaussian processes. In order to introduce the main ideas of the concept, let Z be a centered Gaussian process on a set $\mathbb{T} \subset \mathbb{R}$ and define the semimetric

$$d(s, t) := \sqrt{\mathbb{E}|Z_s - Z_t|^2}, \quad s, t \in \mathbb{T}.$$

For simplicity, we suppose that d is continuous on $\mathbb{T} \times \mathbb{T}$; in the context of the present paper, this is guaranteed by the fact that σ^2 is continuous. We say that $S \subset \mathbb{T}$ is a ϑ-net in $\mathbb{T}$ with respect to the semimetric d, if for any $t \in \mathbb{T}$ there exists an $s \in S$ such that $d(s, t) \le \vartheta$.

Definition 2 *The metric entropy $\mathbb{H}_d(\mathbb{T}, \vartheta)$ is defined as $\log N_d(\mathbb{T}, \vartheta)$, where $N_d(\mathbb{T}, \vartheta)$ denotes the minimal number of points in a ϑ-net in $\mathbb{T}$ with respect to d.*

The quantity $\int_0^\infty \sqrt{\mathbb{H}_d(\mathbb{T}, \vartheta)}d\vartheta$ is called the Dudley integral.

Obviously, if $\mathbb{T}$ is completely bounded with respect to d, $\mathbb{H}_d(\mathbb{T}, \vartheta) = 0$ for ϑ large enough, so that the convergence of the Dudley integral is equivalent to its convergence at zero.

A useful fact is that Z has an almost surely continuous modification if the Dudley integral converges, see, e.g., Lemma 2.1.1 and Theorem 2.1.5 of Adler and Taylor [3], or Corollary 4.15 of Adler [2]. A simple sufficient condition for this is given in the next lemma; it is satisfied by many processes.

Lemma 1 *Let* $\mathbb{T} = [-T, T]$ *for some* $T > 0$*. If there exist* $\epsilon, \kappa, C > 0$ *such that for any* $s, t \in \mathbb{T}$ *with* $|s - t| < \kappa$*,*

$$\mathbb{E}|Z_s - Z_t|^2 \leq \frac{C}{L_\epsilon(s-t)}, \tag{3.2}$$

then there exists a probability measure ν *on* $(C([-T, T]), \mathcal{B}(C([-T, T])))$ *such that for any finite sequence* $\{t_1, \ldots, t_n\} \subset [-T, T]$ *and sets* $A_i \in \mathcal{B}(\mathbb{R})$*,*

$$P(Z_{t_1} \in A_1, \ldots, Z_{t_n} \in A_n) = \nu \left(x \in C([-T, T]) : x(t_1) \in A_1, \ldots, x(t_n) \in A_n \right).$$

Proof. We adopt the (mostly standard) terminology on stochastic processes (see, e.g., Revuz and Yor [30]). Without loss of generality, we may suppose that Y^α is the coordinate mapping on the canonical probability space $(\mathbb{R}^{[-T,T]}, \mathcal{B}(\mathbb{R})^{[-T,T]}, \tilde{P})$, where $\mathcal{B}(\mathbb{R})$ denotes the usual Borel σ-field on $\mathbb{R}$, and the superscripts indicate that we deal with product spaces and product σ-fields. We refer to Section I.3 of [30] for more details.

Since the metric entropy $\mathbb{H}_d([-T, T], \vartheta)$ is upper bounded by $C_0 \vartheta^{-\frac{2}{1+\epsilon}}$ for some constant $C_0 > 0$ and for $\vartheta > 0$ small, the Dudley integral converges and there exists a continuous modification of Z. One can now construct a probability space $(C([-T, T]), \mathcal{B}(\mathbb{R})^{[-T,T]} \cap C([-T, T]), \nu)$ with the required property. The claim follows by noting that $\mathcal{B}(\mathbb{R})^{[-T,T]} \cap C([-T, T]) = \mathcal{B}(C([-T, T]))$ (see, e.g., Theorem VII.2.1 of Parthasarathy [29]). □

In Section 4.1, we establish tightness of a sequence of probability measures on $C([-T, T])$ in a similar way. However, instead of using the finiteness of the Dudley integral, we then carefully derive upper bounds in order to obtain the desired uniformity. A key tool in this analysis is Proposition 3.

3.3 Large deviations on separable Banach spaces

In this subsection, we present some background on large deviations. After discussing some standard notions of large deviation theory, we give some useful facts on large deviations for Gaussian measures. We then relate these to weak convergence.

Let $\mathcal{X}$ denote a separable Banach space, and endow $\mathcal{X}$ with the topology induced by the norm on $\mathcal{X}$. The Borel σ-field generated by this topology is called $\mathcal{B}$. A function $I : \mathcal{X} \to [0, \infty]$ is said to be a *rate function* if it is lower semicontinuous, i.e., $\{x \in \mathcal{X} : I(x) \leq \gamma\}$ is a closed subset of $\mathcal{X}$ for any $\gamma \in [0, \infty)$. A rate function is *good* if $\{x \in \mathcal{X} : I(x) \leq \gamma\}$ is also compact in $\mathcal{X}$. For $B \subset \mathcal{X}$, we denote the interior and closure of B by B^o and $\overline{B}$ respectively.

The central notion in large deviation theory is known as the *large deviation principle.* More background on large deviation techniques can be found in the book by Dembo and Zeitouni [16].

Definition 3 *We say that a family of probability measures* $\{\mu_n : n \in \mathbb{N}\}$ *on* $(\mathcal{X}, \mathcal{B})$ *satisfies a large deviation principle (LDP) with rate function* $I : \mathcal{X} \to [0, \infty]$ *and scale sequence* $\{\lambda_n : n \in \mathbb{N}\}$ *if for all* $B \in \mathcal{B}$*,*

$$\begin{aligned} \liminf_{n\to\infty} \frac{1}{\lambda_n} \log \mu_n(B) &\geq -\inf_{x \in B^o} I(x), \\ \limsup_{n\to\infty} \frac{1}{\lambda_n} \log \mu_n(B) &\leq -\inf_{x \in \overline{B}} I(x). \end{aligned}$$

Of particular interest is the case that the μ_n are centered Gaussian measures, i.e., the image of μ_n under any continuous linear map $\xi : \mathcal{X} \to \mathbb{R}$ is a centered Gaussian distribution. The most well-known theorem in this framework is Schilder's theorem, which states that the large deviation principle holds for the empirical mean of i.i.d. copies of a Gaussian measure μ. Let $\mathcal{H}_\mu$ denote the Cameron-Martin space associated with μ. Given a sequence $\{a_n\}$ tending to ∞, Schilder's theorem states that the family $\{\mu(a_n \cdot)\}$ satisfies an LDP with scale sequence $\{a_n^2\}$ and good rate function

$$I(x) = \begin{cases} \frac{1}{2}\|x\|^2_{\mathcal{H}_\mu} & \text{if } x \in \mathcal{H}_\mu; \\ \infty & \text{otherwise.} \end{cases} \tag{3.3}$$

We refer the reader to Deuschel and Stroock [17] or Lifshits [25] for more details.

Schilder's theorem is a special case of the following theorem, which is Theorem 2 of Chevet [11].

Theorem 3 *Let μ, μ_n be centered Gaussian measures on $\mathcal{X}$, and let $\{a_n\}$ be a sequence of positive real numbers tending to ∞. If $\mu_n \Rightarrow \mu$ in $\mathcal{X}$, then $\{\mu_n(a_n \cdot)\}$ satisfies an LDP with scale sequence $\{a_n^2\}$ and good rate function I, where I is the good rate function associated with Schilder's theorem for $\{\mu(a_n \cdot)\}$.*

Informally, Theorem 3 states that the families $\{\mu(a_n \cdot)\}$ and $\{\mu_n(a_n \cdot)\}$ have the same large deviation behavior if μ_n converges weakly to μ.

3.4 Topological issues

In this subsection, we motivate the choice for the space Ω^γ and its topology. As pointed out in the introduction, convergence in distribution of Y^α to B_H in Ω^γ is only useful in applications if the topology on Ω^γ is strong enough. For explanatory reasons, we suppose in this subsection that Y has continuous sample paths.

The most natural path space to work with is the space $C(\mathbb{R})$ of continuous functions on $\mathbb{R}$; it is usually equipped with the topology induced by the metric

$$d_p(x, y) = \sum_{n=1}^{\infty} 2^{-n} \sup_{t \in [-n,n]} \min(|x(t) - y(t)|, 1). \tag{3.4}$$

This (product) topology is also referred to as the topology of uniform convergence on compacts. Note that convergence of a sequence in $(C(\mathbb{R}), d_p)$ is equivalent to uniform convergence in $C([-T, T])$ for any $T > 0$. A similar statement holds for weak convergence of measures on $C(\mathbb{R})$: a sequence of measures converges weakly in $(C(\mathbb{R}), d_p)$ if and only if the image measure under the projection map $p_T : C(\mathbb{R}) \to C([-T, T])$ converges in $C([-T, T])$ for any $T > 0$.

For many applications, however, the product topology is not strong enough; the weaker the topology, the less information is contained by stating that measures converge weakly. In fact, the topology of uniform convergence on compacts cannot be used in either of the applications studied in Section 2. To illustrate this, we introduce a set A^β that is used in the first application. For $\beta > 0$, we set

$$A^\beta \quad := \quad \{x \in C(\mathbb{R}) : \sup_{t \geq 0} x(t) - t^\beta \geq 1\}, \tag{3.5}$$

$$A^\beta_o \quad := \quad \{x \in C(\mathbb{R}) : \sup_{t \geq 0} x(t) - t^\beta > 1\} \tag{3.6}$$

Suppose we have an LDP in the space $(C(\mathbb{R}), d_p)$. As this provides an upper bound for closed sets, it is desirable that A^β is closed in $(C(\mathbb{R}), d_p)$. However, this is not the case; construct a sequence $\{x_n\} \subset A^\beta$ as follows:

$$x_n(t) = \begin{cases} (1+n^\beta)(t-n+1) & t \in [n-1, n] \\ -(1+n^\beta)(t-n-1) & t \in (n, n+1] \\ 0 & \text{otherwise.} \end{cases}$$

It is readily seen that x_n converges in $(C(\mathbb{R}), d_p)$ to zero, but $0 \notin A^\beta$.

This example indicates that the tail of the sample paths in $(C(\mathbb{R}), d_p)$ may cause problems. In Ω^γ, however, the topology is sufficiently strong to make A^β closed, as the following lemma shows. The lemma is used in the proof of Proposition 1 and Theorem 1.

Lemma 2 *For $\beta \geq \gamma$, $A^\beta \cap \Omega^\gamma$ is closed in $(\Omega^\gamma, \|\cdot\|_{\Omega^\gamma})$, and $A_o^\beta \cap \Omega^\gamma$ is open in $(\Omega^\gamma, \|\cdot\|_{\Omega^\gamma})$.*

Proof. To prove the first claim, we consider an arbitrary sequence $\{x_n\} \subset A^\beta \cap \Omega^\gamma$ converging in Ω^γ to some $x \in \Omega^\gamma$; we show that $x \in A^\beta \cap \Omega^\gamma$. We derive a contradiction by supposing that this is not the case. Notice that A^β can be written as

$$A^\beta = \left\{ x \in C(\mathbb{R}) : \sup_{t \geq 0} \frac{x(t)}{1+t^\beta} \geq 1 \right\}.$$

Define

$$\eta := 1 - \sup_{t \geq 0} \frac{x(t)}{1+t^\beta} > 0. \tag{3.7}$$

As $x_n \to x$ in Ω^γ, we can select an n_0 such that for $n \geq n_0$,

$$\sup_{t \geq 0} \frac{|x_n(t) - x(t)|}{1+t^\beta} < \eta/2. \tag{3.8}$$

We combine (3.7) with (3.8) to see that

$$\sup_{t \geq 0} \frac{x_n(t)}{1+t^\beta} \leq \sup_{t \geq 0} \frac{x_n(t) - x(t)}{1+t^\beta} + \sup_{t \geq 0} \frac{x(t)}{1+t^\beta} < \eta/2 + 1 - \eta < 1,$$

implying $x_n \notin A^\beta$; a contradiction.

A similar argument can be given to see that $A_o^\beta \cap \Omega^\gamma$ is open in Ω^γ. □

4. Weak convergence results

This section is devoted to necessary and sufficient conditions for the convergence in distribution of Y^α to a fractional Brownian motion. For background on weak convergence, we refer to Billingsley [8]. After dealing with weak convergence on compact intervals (Section 4.1), we extend the results to weak convergence in Ω^γ for $\gamma > H$ (Section 4.2).

Throughout, σ_α^2 denotes the variance function of Y^α, i.e.,

$$\sigma_\alpha^2(t) := \frac{\sigma^2(\alpha t)}{\sigma^2(\alpha)}.$$

It is no coindidence that the candidate weak limit is self-similar (i.e., the finite-dimensional distributions of $a^H B_H(\cdot)$ equal those of $B_H(a\cdot)$); see Lamperti [24].

4.1 Weak convergence on compacts
Fix some time horizon $T > 0$ throughout this subsection and consider the compact interval $[-T, T]$. We slightly abuse notation by restricting Y^α to $[-T, T]$ while keeping the notation Y^α. Under the condition that $\lim_{t\to 0} \sigma^2(t)|\log|t||^{1+\epsilon} < \infty$ for some $\epsilon > 0$, Lemma 1 implies that the distribution of Y^α is equivalent to a probability measure ν_α on the measurable space $(C([-T, T]), \mathcal{B}(C([-T, T])))$.

To get some feeling for the necessity of this condition, we note that continuity of the sample paths of Y imply that $\lim_{t\to 0} \sigma^2(t)|\log|t|| < \infty$ under an extremely weak condition on σ. Indeed, the following theorem, based on Sudakov's inequality, provides a simple necessary condition for continuity of the sample paths. We omit a proof, since one can repeat the arguments in van der Vaart and van Zanten [32, Corollary 2.7].

Theorem 4 (necessity) *Suppose that σ is strictly increasing on some neighborhood of zero. If Y has continuous sample paths, then $\lim_{t\to 0} \sigma^2(t)|\log|t|| < \infty$.*

We now turn to a sufficient condition for the weak convergence of Y^α. Since $C([-T, T])$ is a separable and complete metric space (i.e., a Polish space), by Prohorov's theorem [8, Th. 5.1 and 5.2], weak convergence in $C([-T, T])$ is equivalent to convergence of finite-dimensional distributions and tightness of $\{\nu_\alpha\}$.

It is easy to see that the finite-dimensional distributions of Y^α converge in distribution to B_H, since

$$\mathbb{C}\mathrm{ov}(Y_s, Y_t) = \frac{1}{2}\left[\sigma^2(s) + \sigma^2(t) - \sigma^2(|s-t|)\right]$$

and σ^2 is regularly varying. Therefore, weak convergence of Y^α in $C([-T, T])$ is equivalent to tightness of $\{\nu_\alpha\}$ in $C([-T, T])$. By Theorem 7.3 of [8], this is in turn equivalent to

$$\lim_{\delta\to 0} \limsup_{\alpha\to\infty} P\left(\sup_{\substack{|s-t|\le\delta \\ s,t\in[-T,T]}} |Y_s^\alpha - Y_t^\alpha| \ge \zeta\right) = 0, \tag{4.1}$$

for any $\zeta > 0$. For notational convenience, we leave out the requirement $s, t \in [-T, T]$ explicitly in the remainder.

Theorem 5 (sufficiency) *If $\lim_{t\to 0} \sigma^2(t)|\log|t||^{1+\epsilon} < \infty$ for some $\epsilon > 0$, then $Y^\alpha \Rightarrow B_H$ in $C([-T, T])$.*

Proof. Our objective is to prove (4.1). Since σ^2 is assumed to be continuous, the condition in the theorem implies the boundedness of $\sigma^2 L_\epsilon$ on intervals of the form $(0, \cdot]$ (recall the definition of L_ϵ in (3.1)). Therefore, as a consequence of Proposition 3, we have for any $\delta > 0$, and α large enough, uniformly in $t \in [-\delta, \delta]\backslash\{0\}$ (obviously, $\sigma_\alpha^2(0) = 0$):

$$\sigma_\alpha^2(t) \le 2\delta^{2H} \frac{L_\epsilon(\delta)}{L_\epsilon(t)}. \tag{4.2}$$

Use the fact that L_ϵ is non-increasing and (4.2) to see that, for any $\zeta > 0$ and α sufficiently large,

$$\begin{aligned} P\left(\sup_{|s-t|\le\delta} |Y_s^\alpha - Y_t^\alpha| \ge \zeta\right) &= P\left(\sup_{\{(s,t):2\delta^{2H}L_\epsilon(\delta)/L_\epsilon(s-t)\le 2\delta^{2H}\}} |Y_s^\alpha - Y_t^\alpha| \ge \zeta\right) \\ &\le P\left(\sup_{\sigma_\alpha^2(|s-t|)\le 2\delta^{2H}} |Y_s^\alpha - Y_t^\alpha| \ge \zeta\right) \\ &\le \frac{1}{\zeta}\mathbb{E}\left(\sup_{\sigma_\alpha^2(|s-t|)\le 2\delta^{2H}} |Y_s^\alpha - Y_t^\alpha|\right). \end{aligned}$$

Define $\mathbb{H}_\alpha(\mathbb{T},\cdot)$ as the metric entropy of $\mathbb{T} \subset \mathbb{R}$ under the semimetric induced by σ_α^2; see Subsection 3.2 for definitions. Motivated by the proof of Lemma 1, we set $\mathbb{H}(\mathbb{T},\vartheta) = C_0\vartheta^{-2/(1+\epsilon)}$ for some constant C_0 depending on the Lebesgue measure of $\mathbb{T}$; it can be regarded as the metric entropy under the semimetric induced by L_ϵ, being only valid for small $\vartheta > 0$.

We use Corollary 2.1.4 of [3] to see that there exists a constant $C > 0$ such that

$$\mathbb{E}\left(\sup_{\sigma_\alpha^2(|s-t|)\le 2\delta^{2H}} |Y_s^\alpha - Y_t^\alpha|\right) \le C\int_0^{2\delta^{2H}} \sqrt{\mathbb{H}_\alpha([-T,T],\vartheta)}d\vartheta.$$

Another application of (4.2) shows that for $\vartheta > 0$,

$$\mathbb{H}_\alpha([-T,T],\vartheta) \le \mathbb{H}\left([-T,T], \frac{\vartheta}{\sqrt{2\delta^{2H}L_\epsilon(\delta)}}\right),$$

so that

$$\int_0^{2\delta^{2H}} \sqrt{\mathbb{H}_\alpha([-T,T],\vartheta)}d\vartheta \le \sqrt{2\delta^{2H}L_\epsilon(\delta)}\int_0^{2\delta^{2H}/\sqrt{2\delta^{2H}L_\epsilon(\delta)}} \sqrt{\mathbb{H}([-T,T],\vartheta)}d\vartheta.$$

To summarize, we have

$$\limsup_{\alpha\to\infty} P\left(\sup_{|s-t|\le\delta} |Y_s^\alpha - Y_t^\alpha| \ge \zeta\right) \le \frac{C\sqrt{2\delta^{2H}L_\epsilon(\delta)}}{\zeta}\int_0^{\delta^H/\sqrt{L_\epsilon(\delta)/2}} \sqrt{\mathbb{H}([-T,T],\vartheta)}d\vartheta.$$

As $\int \sqrt{\mathbb{H}([-T,T],\vartheta)}d\vartheta < \infty$, we obtain (4.1) by letting $\delta \to 0$. □

The remainder of this subsection is devoted to easy corollaries of the sufficient condition in Theorem 5. We first show the relation with Lemma 4.2 of [12].

Corollary 1 *Suppose that σ^2 is regularly varying at zero with index $\lambda \in (0,2]$, and that σ^2 is continuous. Then we have $Y^\alpha \Rightarrow B_H$ in $C([-T,T])$.*

Proof. Since σ^2 is regularly varying at zero with index λ, $t \mapsto \sigma^2(1/t)$ is regularly varying at infinity with index $-\lambda$. Apply Proposition 1.5.1 of [9] to conclude that $\sigma^2(1/t)|t|^{\lambda/2} \to 0$ as $t \to \infty$. Equivalently, $\sigma^2(t)|t|^{-\lambda/2} \to 0$ as $t \to 0$, implying the condition in Theorem 5. □

Similarly, one proves the following Kolmogorov-type criterion for tightness.

Corollary 2 *If $\lim_{t\to 0}\sigma^2(t)|t|^{-\lambda} < \infty$ for some $\lambda \in (0,2]$ and σ^2 is continuous, then $Y^\alpha \Rightarrow B_H$ in $C([-T,T])$.*

4.2 Weak convergence on Ω^γ
In this subsection, we focus on the weak convergence of Y^α to B_H in Ω^γ for $\gamma > H$. Obviously, this convergence can only take place when the laws ν_α^γ of Y^α in Ω^γ exist.

Lemma 3 *If $\lim_{t\to 0} \sigma^2(t)|\log|t||^{1+\epsilon} < \infty$ for some $\epsilon > 0$, then the probability measures ν_α^γ on $(\Omega^\gamma, \mathcal{B}(\Omega^\gamma))$ exist for $\gamma > H$.*

Proof. We first note that, by the assumption on $\sigma^2 L_\epsilon$, Y has almost surely continuous trajectories as detailed in the proof of Lemma 1. Therefore, in order to show that $Y^\alpha \in \Omega^\gamma$ almost surely, it suffices to prove that $\lim_{t\to\pm\infty} Y_t^\alpha/t^\gamma = 0$ almost surely. We use the reasoning of Addie *et al.* [1], to which we add an essential argument.

Since σ^2 is supposed to be regularly varying with index $2H$, we have $\sigma^2(t)/t^{\gamma+H} \to 0$, which can be exploited to see that for $\epsilon > 0$, $\sum_k \mathbb{P}(Y_k/k^\gamma > \epsilon) < \infty$. By the Borel-Cantelli lemma, $Y_k/k^\gamma \to 0$ almost surely. Note that, for $Z_k := \sup_{s\in[k,k+1]} |Y_s - Y_k|$,

$$|Y_t| \le |Y_{\lfloor t \rfloor}| + Z_{\lfloor t \rfloor},$$

so that it suffices to show that $Z_k/k^\gamma \to 0$ almost surely. For this, we first remark that $\mathbb{E}\exp(\alpha Z_k^2) = \mathbb{E}\exp(\alpha Z_1^2) < \infty$ for $\alpha > 0$ small enough, as a consequence of Borell's inequality [3, Thm. 2.3.1]. Notice that we used the continuity to ensure that this inequality can be applied. By Chernoff's bound, we have for any $\epsilon > 0$,

$$\sum_k P\left(Z_k/k^\gamma > \epsilon\right) \le \sum_k P\left(Z_k^2 > \epsilon^2 k^{2\gamma}\right) \le \sum_k \exp(-\alpha\epsilon^2 k^{2\gamma})\mathbb{E}\exp(\alpha Z_1^2) < \infty.$$

Another application of the Borel-Cantelli lemma now proves that $Z_k/k^\gamma \to 0$ and therefore $Y_t^\alpha/t^\gamma \to 0$ almost surely.

The measure ν_α^γ can now be constructed as in Lemma 1; it only remains to show that $\mathcal{B}(\mathbb{R})^{\mathbb{R}} \cap \Omega^\gamma = \mathcal{B}(\Omega^\gamma)$. It is easy to see that this holds for $\gamma = 0$, and Ω^γ is isometrically isomorphic to Ω^0. □

We now investigate the probabilistic meaning of weak convergence in Ω^γ. While the weak convergence in the uniform topology on compacts is obtained by an application of Theorem 5, the convergence in Ω^γ is substantially stronger (see Section 3.4). Therefore, an additional condition is needed to strengthen the convergence. Such a condition is given in Lemma 3 of Buldygin and Zaiats [10, cited according to [22]]. Lemma 4 below is closely related to this key result; only 'sup' has been replaced by 'lim sup'. See also Majewski [26] for a related result in a large deviation setting.

Lemma 4 *Let a family of probability measures $\{\mu_n\}$ on Ω^γ be given. Suppose that the image of $\{\mu_n\}$ under the projection map $p_T : \Omega^\gamma \to C([-T,T])$ is tight in $C([-T,T])$ for all $T > 0$. Then $\{\mu_n\}$ is tight in Ω^γ if and only if for any $\zeta > 0$,*

$$\lim_{T\to\infty} \limsup_{n\to\infty} \mu_n \left(x \in \Omega^\gamma : \sup_{|t|\ge T} \frac{|x(t)|}{1+|t|^\gamma} \ge \zeta \right) = 0. \tag{4.3}$$

Proof. To prove necessity, let $\{\mu_n\}$ be tight in Ω^γ and fix $\zeta > 0$. Given $\eta > 0$, choose an Ω^γ-compact set K such that $\mu_n(K) > 1 - \eta$ for all n. We denote an Ω^γ-ball centered at x with radius ζ by $B_\zeta(x)$, so that $\{B_{\zeta/2}(x) : x \in K\}$ is an Ω^γ-open cover of K. Since K is Ω^γ-compact, one can select $m < \infty$ and $x_1, \ldots, x_m$ such that $\{B_{\zeta/2}(x_i) : i = 1, \ldots, m\}$ also covers K. Let T_i be such that $\sup_{|t|\geq T_i} |x_i(t)|/(1 + |t|^\gamma) < \zeta/2$, and set

$$A_{\zeta,T} := \left\{ x \in \Omega^\gamma : \sup_{|t|\geq T} \frac{|x(t)|}{1 + |t|^\gamma} < \zeta \right\}.$$

Note that for $T := \max_i T_i$, $K \subset A_{\zeta,T}$. We have now shown that for any $\eta > 0$ one can find $T > 0$ such that

$$\sup_{n\geq 1} \mu_n(A^c_{\zeta,T}) \leq \eta. \tag{4.4}$$

Obviously, this implies (4.3).

For sufficiency, instead of supposing (4.3), we may suppose without loss of generality that for any $\eta > 0$ there exists a $T > 0$ such that (4.4) holds. Indeed, since Ω^γ is separable and complete, any probability measure on Ω^γ is tight. In particular, the above reasoning used to prove necessity implies that for any $\eta > 0$ and $n \geq 1$, one can find $T_n > 0$ such that $\mu_n(A^c_{\zeta,T_n}) \leq \eta$. As a consequence of (4.3), there exists a $T' > 0$ and n_0 such that $\sup_{n\geq n_0} \mu_n(A^c_{\zeta,T'}) \leq \eta$. Hence we have (4.4) for $T := \max(T', \max_{n\leq n_0} T_n)$.

Suppose the image of $\{\mu_n\}$ under the projection map is tight in $C([-T, T])$ for all $T > 0$. We can then choose a set K that is compact in the topology of uniform convergence on compact intervals such that $\sup_n \mu_n(K^c) \leq \eta/2$. For brevity, we call K $\mathcal{U}$-compact. Using (4.4), we can select for any $m \in \mathbb{N}$ a $T_m > 0$ such that $\sup_{n\geq 1} \mu_n(A^c_{1/m,T_m}) \leq \eta/2^{m+1}$. Set $K' := K \cap \bigcap_{m\in\mathbb{N}} A_{1/m,T_m}$ and note that $\inf_{n\geq 1} \mu_n(K') \geq 1 - \eta$. Therefore, we have established the claim once we have shown that the Ω^γ-closure of K' is Ω^γ-compact. For this, let $\{x_\ell\}$ be a sequence in K', and let $\delta > 0$ be arbitrary. Since K is $\mathcal{U}$-compact, we can find a subsequence $\{x_{\ell_k}\}$ of $\{x_\ell\}$ that converges uniformly on compact intervals, say, to x. Moreover, $\{x_{\ell_k}\} \subset \bigcap_{m\in\mathbb{N}} A_{1/m,T_m}$ implies that we can find a $T > 0$ such that $\sup_{k\geq 1} \sup_{|t|\geq T} |x_{\ell_k}(t)|/(1 + |t|) < \delta/2$. As $x \in \Omega^\gamma$, we can also choose $T' > 0$ such that $\sup_{|t|\geq T'} |x(t)|/(1 + |t|^\gamma) \leq \delta/2$. From the convergence of x_{ℓ_k} to x on compacts we deduce that $\sup_{|t|\leq \max(T,T')} |x_{\ell_k}(t) - x(t)|/(1 + |t|^\gamma) \leq \delta$. We now readily infer that $\|x_{\ell_k} - x\|_\Omega \leq \delta$, i.e., K' is Ω^γ-compact. □

Having a characterization of weak convergence in Ω^γ at our disposal, we now specialize to the framework of the present paper. The main result of this section is that the family $\{\nu^\gamma_\alpha\}$ is tight in Ω^γ under the conditions of Theorem 5. It may seem rather surprising that it is possible to establish this tightness in Ω^γ without an additional condition on large time scale behavior. Apparently, the fact that the variance function varies regularly with an index $2H < 2\gamma$ and the Gaussian nature suffice to control the process over large time scales.

Theorem 6 (sufficiency) *If $\lim_{t\to 0} \sigma^2(t)|\log|t||^{1+\epsilon} < \infty$ for some $\epsilon > 0$, then $Y^\alpha \Rightarrow B_H$ in Ω^γ for any $\gamma > H$.*

Proof. Having Lemma 4 at our disposal, we need to establish (4.3) for the family $\{\nu_\alpha^\gamma\}$, or, equivalently, for any $\zeta > 0$,

$$\lim_{T\to\infty} \limsup_{\alpha\to\infty} P\left(\sup_{t\geq T} \frac{|Y_t^\alpha|}{1+t^\gamma} \geq \zeta\right) = 0. \tag{4.5}$$

An upper bound for the probability in the preceding display is based on Markov's inequality: for $\zeta > 0$, $\alpha, k \geq 1$,

$$\begin{aligned} P\left(\sup_{t\geq e^k} \frac{|Y_t^\alpha|}{1+t^\gamma} \geq \zeta\right) &\leq 2P\left(\sup_{t\geq e^k} \frac{Y_t^\alpha}{1+t^\gamma} \geq \zeta\right) \leq 2\sum_{j=k}^\infty P\left(\sup_{t\in[e^j,e^{j+1}]} \frac{Y_t^\alpha}{1+t^\gamma} \geq \zeta\right) \\ &\leq \frac{2}{\zeta}\sum_{j=k}^\infty \frac{\mathbb{E}\sup_{t\in[e^j,e^{j+1}]} Y_t^\alpha}{1+e^{j\gamma}}. \end{aligned} \tag{4.6}$$

As in the proof of Theorem 5, we use metric entropy techniques to find a further upper bound. Recall the notation $\mathbb{H}_\alpha(\mathbb{T},\cdot)$ and $\mathbb{H}(\mathbb{T},\cdot)$ that we used in the proof of Theorem 5. By Theorem 14.1 of Lifshits [25], there exists a constant $C > 0$ such that $\mathbb{E}\sup_{t\in[e^j,e^{j+1}]} Y_t^\alpha \leq C\int \sqrt{\mathbb{H}_\alpha([e^j,e^{j+1}],\vartheta)}d\vartheta$.

We now derive a bound on $\int \sqrt{\mathbb{H}_\alpha([e^k,e^{k+1}],\vartheta)}d\vartheta$ for k large, uniformly in α. The first step is to bound the variance σ_α^2. As a consequence of Proposition 3, we have for $\alpha\to\infty$,

$$\sup_{|t|\leq 1/e} \sigma_\alpha^2(t)L_\epsilon(t) \to \sup_{|t|\leq 1/e} t^{2H}L_\epsilon(t) = e^{-2H}.$$

Moreover, by the Uniform Convergence Theorem (Theorem 1.5.2 of [9]), for large α, we have $\sigma_\alpha^2(t) \leq 2e^{\gamma-H}t^{H+\gamma}$ for all $|t|\geq 1/e$. Therefore, the function M_ϵ given by

$$M_\epsilon(t) := \begin{cases} 2e^{-2H}/L_\epsilon(t) & \text{if } |t|\leq 1/e; \\ 2e^{\gamma-H}t^{H+\gamma} & \text{otherwise.} \end{cases}$$

majorizes σ_α^2 uniformly in (large) α. It is important to notice that M_ϵ is continuous and strictly increasing for $t\in\mathbb{R}_+$.

Using the stationarity of the increments and the fact that the inverse of $1/\sqrt{L_\epsilon(\cdot)}$ is given by $\vartheta\mapsto\exp\left(-\vartheta^{-2/(1+\epsilon)}\right)$ for $\vartheta\in[0,\sqrt{2}e^{-H}]$, we see that for large j,

$$\begin{aligned} \int_0^{\sqrt{2}e^{-H}} \sqrt{\mathbb{H}_\alpha([e^j,e^{j+1}],\vartheta)}d\vartheta &= \int_0^{\sqrt{2}e^{-H}} \sqrt{\mathbb{H}_\alpha([0,e^j(e-1)],\vartheta)}d\vartheta \\ &\leq \sqrt{2}e^{-H}\int_0^1 \sqrt{\log\left(\frac{e^j(e-1)}{2\exp\left(-\vartheta^{-2/(1+\epsilon)}\right)}+1\right)}d\vartheta \\ &\leq e^{-H}\log[e^j(e-1)]+1+\frac{1}{\epsilon}, \end{aligned}$$

implying

$$\lim_{k\to\infty}\sum_{j=k}^\infty \frac{\int_0^{\sqrt{2}e^{-H}} \sqrt{\mathbb{H}_\alpha([e^j,e^{j+1}],\vartheta)}d\vartheta}{1+e^{j\gamma}} = 0,$$

so that it remains to show a similar statement for the integration interval $[\sqrt{2}e^{-H}, \infty)$. For this, observe that, for some constant $\mathcal{C} > 0$,

$$\begin{aligned}\int_{\sqrt{2}e^{-H}}^{\infty} \sqrt{\mathbb{H}_\alpha([e^j, e^{j+1}], \vartheta)}d\vartheta &\leq \int_{\sqrt{2}e^{-H}}^{e^{H(j+1)}} \sqrt{\log\left(\frac{e^j(e-1)}{2^{1-1/(H+\gamma)}e^{(H-\gamma)/(H+\gamma)}\vartheta^{2/(H+\gamma)}}\right) + 1d\vartheta} \\ &\leq (e^{H(j+1)} - \sqrt{2}e^{-H})\sqrt{\log\left(\frac{\mathcal{C}e^j}{(2e^{-2H})^{1/(H+\gamma)}}\right)},\end{aligned}$$

from which the claim is readily obtained. □

We now relate Theorem 6 to Proposition 2.9 of Kozachenko *et al.* [22]. The criterion given in Proposition 2.9 of [22] states that

$$\sup_{t\in\mathbb{R}_+} \overline{\sigma}^2(t)(\log(1 + 1/t))^{1+\epsilon} < \infty \tag{4.7}$$

for some $\epsilon \in (0, 1)$ (recall the definition of $\overline{\sigma}^2$ in (1.4)). Since we already noted that $\lim_{t\downarrow 0}(\log(1 + 1/t))^{1+\epsilon}/L_\epsilon(t) = 1$, this condition implies that $\lim_{t\to 0} \sigma^2(t)|\log|t||^{1+\epsilon} < \infty$ for some $\epsilon > 0$. Although the continuity of σ is not stated explicitly in [22], it is necessary to obtain continuity of the sample paths of Y. Indeed, if Y has continuous sample paths and σ^2 is locally bounded (as implied by (4.7)), then the dominated convergence theorem implies the continuity of σ^2.

In conclusion, the condition in Theorem 6 improves this result of [22] in two ways: the condition is easier to check and weaker.

We get the following important corollary by combining Theorem 6 with Chevet's Theorem 3.

Corollary 3 *Let $\{a_\alpha\}$ be a sequence of positive real numbers tending to infinity as $\alpha \to \infty$. If $\lim_{t\to 0} \sigma^2(t)|\log|t||^{1+\epsilon} < \infty$ for some $\epsilon > 0$, then the distributions in Ω^γ ($\gamma > H$) of Y^α/a_α satisfy an LDP in Ω^γ with scale sequence $\{a_\alpha^2\}$ and rate function I given by (3.3), where $\mathcal{H}_H$ is the Cameron-Martin space associated with fractional Brownian motion on Ω^γ.*

5. Proofs for Section 2

5.1 Proof of Proposition 1

For A^β be given by (3.5), we note

$$\begin{aligned}P\left(\sup_{t\geq 0} Y_t - ct^\beta \geq u\right) &= P\left(\sup_{t\geq 0} Y_{(u/c)^{1/\beta}t} - ut^\beta \geq u\right) \\ &= P\left(\sup_{t\geq 0} \frac{1}{u}Y_{(u/c)^{1/\beta}t} - t^\beta \geq 1\right) \\ &= \nu^\beta_{(u/c)^{1/\beta}}\left(\frac{u}{\sigma([u/c]^{1/\beta})}A^\beta \cap \Omega^\beta\right).\end{aligned}$$

Since $A^\beta \cap \Omega^\beta$ is closed in Ω^β by Lemma 2, we have by Corollary 3,

$$\begin{aligned}
&\limsup_{u\to\infty} \frac{\sigma^2(u^{1/\beta})}{u^2} \log P\left(\sup_{t\geq 0} Y_t - ct^\beta \geq u\right) \\
=\ &\limsup_{u\to\infty} c^{2H/\beta} \frac{\sigma^2([u/c]^{1/\beta})}{u^2} \log P\left(\sup_{t\geq 0} Y_t - ct^\beta \geq u\right) \\
\leq\ & -c^{2H/\beta} \inf_{x\in A^\beta\cap\Omega^\beta} I(x),
\end{aligned}$$

where I is given by (3.3). It remains to calculate the quantity $\inf_{x\in A^\beta\cap\Omega^\beta} I(x)$, or equivalently $\inf_{t\geq 0}\inf_{\{x\in\Omega^\beta:x(t)-t^\beta\geq 1\}} I(x)$. It is left to the reader to repeat the argument in Addie *et al.* [1] to see that, also for $\beta\neq 1$, $2\inf_{\{x\in\Omega^\beta:x(t)-t^\beta\geq 1\}} I(x) = (1+t^\beta)^2/t^{2H}$. Straightforward calculus shows that the (unique) infimum over t is attained at $t = (H/[\beta - H])^{1/\beta}$. The analysis in [1] also shows that the minimizing argument x^* of $\inf_{x\in A^\beta\cap\Omega^\beta} I(x)$ is indeed given by (2.2).

For the lower bound, note that $A^\beta \supset A^\beta_o$, and that $A^\beta_o \cap \Omega^\beta$ is Ω^β-open by Lemma 2. Therefore,

$$\liminf_{u\to\infty} \frac{\sigma^2(u^{1/\beta})}{u^2} \log P\left(\sup_{t\geq 0} Y_t - ct^\beta \geq u\right) \geq -c^{2H/\beta} \inf_{x\in A^\beta_o\cap\Omega^\beta} I(x).$$

An elementary argument shows that $\inf_{x\in A^\beta_o\cap\Omega^\beta} I(x) = \inf_{x\in A^\beta\cap\Omega^\beta} I(x)$, so that the claim is proven.

5.2 Proof of Theorem 1

By the Portmanteau Theorem (e.g., Billingsley [8, Theorem 2.1]), it suffices to show that for all Ω^β-closed sets F,

$$\limsup_{u\to\infty} P\left(\frac{1}{u} Y_{(u/c)^{1/\beta}\cdot} \in F \,\middle|\, \sup_{t\geq 0} Y_t - ct^\beta \geq u\right) \leq \delta_{x^*}(F). \tag{5.1}$$

Since this assertion is trivial if $x^*\in F$, we suppose that $x^*\notin F$. Denote the probability on the left hand side of (5.1) by p_u, so that

$$\log p_u = \log \nu^\beta_{(u/c)^{1/\beta}}\left(\frac{u}{\sigma([u/c]^{1/\beta})}\left(A^\beta\cap F\right)\right) - \log \nu^\beta_{(u/c)^{1/\beta}}\left(\frac{u}{\sigma([u/c]^{1/\beta})} A^\beta\cap\Omega^\beta\right)$$

and by Corollary 3, as both F and $A^\beta\cap\Omega^\beta$ are closed,

$$\limsup_{u\to\infty} \frac{\sigma^2(u^{1/\beta})}{u^2} \log p_u \leq -c^{2H/\beta} \inf_{x\in A^\beta\cap F} I(x) + c^{2H/\beta} \inf_{x\in A^\beta_o\cap\Omega^\beta} I(x). \tag{5.2}$$

We proceed by showing that

$$\inf_{x\in A^\beta\cap F} I(x) > \inf_{x\in A^\beta\cap\Omega^\beta} I(x). \tag{5.3}$$

For this, we suppose that we have equality, so that we can find a sequence $\{x_n\}\subset A^\beta\cap F$ with $I(x_n) < I(x^*) + 1/n$. Without loss of generality, we may suppose that x_n is $(1+$

$t_n^\beta)/(2t_n^{2H})\mathbb{C}\mathrm{ov}(B_H(\cdot), B_H(t_n))$ for some $t_n \geq 0$, since the minimizer of the rate function over the set $\{x : x(t_n) - t_n^\beta \geq 1\}$ has this form, cf. (2.2). Moreover, by uniqueness of t^*, we must have $t_n \to t^*$ in order to ensure that $(1 + t_n^\beta)^2/(2t_n^{2H}) = I(x_n) < I(x^*) + 1/n$. An easy calculation shows that then x_n converges in Ω^β to $x^* \notin F$, which contradicts the fact that F is closed.

The claim follows by combining Equation (5.2) with Equation (5.3) and the observation that $\inf_{x\in A_o^\beta\cap\Omega^\beta} I(x) = \inf_{x\in A^\beta\cap\Omega^\beta} I(x)$.

5.3 Proof of Proposition 2

Corollary 3 implies that $Y_{\cdot T}/T$ satisfies a large deviation principle in Ω^1 with rate function I and scale sequence $\sigma^2(T)/T^2$. Having observed this, the remainder of the proof is a combination of the arguments contained in Kozachenko *et al.* [22] and Norros [28]; we do not repeat them. The idea is to use $P(Y \in K_T) = P(Y_{\cdot T}/T \in K_1)$, and then justify the limit in (2.4) by showing that K_1 is open and that $\inf_{x\in K_1} I(x) = \inf_{x\in \overline{K_1}} I(x)$.

5.4 Proof of Theorem 2

We first show the existence and uniqueness of $\overline{x}$. For this, note that the large deviation principle for $Y_{\cdot T}/T$ is governed by a strictly convex rate function. Both existence and uniqueness follow from Proposition 4.4 of [28], since $\inf_{x\in\overline{K_1}} I(x)$ can be written as an infimum of the rate function over a convex set. Notice that Proposition 4.3 and Proposition 4.4 of [28] together imply that $\inf_{x\in K_1} I(x) = I(\overline{x})$.

By a similar reasoning as in the proof of Theorem 1 and the fact that K_1 is open [28], the claim follows after showing that for Ω^γ-closed sets F with $\overline{x} \notin F$,

$$\inf_{x\in\overline{K_1}\cap F} I(x) > \inf_{x\in K_1} I(x) = I(\overline{x}),$$

cf. (5.3). Suppose we have equality in the preceding display. For every $n \in \mathbb{N}$, one can then select an $x_n \in \overline{K_1} \cap F$ such that $I(x_n) \leq I(\overline{x}) + 1/n$. Now define the sets

$$M_n := \{x \in \Omega : I(x) \leq I(\overline{x}) + 1/n\}.$$

By the goodness of the rate function, these sets are Ω^γ-compact. Since $\{x_n\} \subset M_1$, one can select a subsequence of $\{x_n\}$ that converges (in Ω^γ) to some $\underline{x}$. As the M_n decrease, one then has that $\underline{x} \in M_n$ for every $n \in \mathbb{N}$, implying that $I(\underline{x}) = I(\overline{x})$. By construction we also have $\underline{x} \in \overline{K_1} \cap F$ as the latter set is closed in Ω^γ. Uniqueness yields $\overline{x} = \underline{x}$, contradicting $\underline{x} \in F$.

6. Proofs for Section 3

6.1 Proof of Proposition 3

The proof is modeled after the proof of Theorem 1.5.2 in [9]. We start with some notation. Let $\eta > 0$ be arbitrary and let $\mathcal{T} = \mathcal{T}(\eta) < 1$ be such that

$$\sup_{t\in[0,\mathcal{T}]} t^\rho L_\epsilon(t) < \frac{1}{9}\eta. \tag{6.1}$$

Since f is regularly varying with index $\rho > 0$, we can find A_1 so that for $\alpha \geq A_1$,

$$\frac{f(\alpha)}{f(\alpha/\mathcal{T})} \leq 2\mathcal{T}^\rho. \tag{6.2}$$

Define

$$M := \sup_{0<\alpha\leq A_1} f(\alpha)L_\epsilon(\alpha),$$

which is finite by assumption. Using Proposition 1.5.1 of [9], we pick A_2 so that for $\alpha \geq A_2$,

$$\frac{(\log A_1)^{1+\epsilon} + (\log \alpha)^{1+\epsilon}}{f(\alpha)} \leq \frac{\eta}{2^{1+\epsilon}M}. \tag{6.3}$$

Without loss of generality, we may suppose that $A_2 \geq eA_1$ and $A_1 \geq e$.

The outline of the proof is as follows. In the first step, we show

$$\sup_{t\in(0,\mathcal{T}]} \sup_{\alpha\geq A_1/t} \left| \frac{f(\alpha t)}{f(\alpha)} L_\epsilon(t) - t^\rho L_\epsilon(t) \right| < \eta, \tag{6.4}$$

and then we show that

$$\sup_{t\in(0,A_1/A_2]} \sup_{A_2\leq\alpha\leq A_1/t} \left| \frac{f(\alpha t)}{f(\alpha)} L_\epsilon(t) - t^\rho L_\epsilon(t) \right| < \eta. \tag{6.5}$$

In the third and last step, we use (6.4) and (6.5) to establish the claim.

Step 1: Proof of (6.4) Apply Theorem 1.5.4 of [9] to the regularly varying function $f(\alpha)\alpha^{-\rho}$ to see that

$$f(\alpha) = c(\alpha)\alpha^{\rho/2}\phi(\alpha),$$

where $c(\alpha) \to 1$ as $\alpha \to \infty$, and ϕ is non-decreasing. Without loss of generality, we may assume that A_1 is such that $1/2 \leq c(\alpha) \leq 2$ for $\alpha \geq A_1$. For fixed $t \in (0, \mathcal{T}]$, we have $t \leq 1$ (since $\mathcal{T} < 1$), so that

$$\begin{aligned}
\sup_{\alpha\geq A_1/t} \frac{f(\alpha t)}{f(\alpha)} L_\epsilon(t) &= \sup_{\alpha\geq A_1} \frac{f(\alpha)}{f(\alpha/t)} L_\epsilon(t) \\
&= \sup_{\alpha\geq A_1} \frac{f(\alpha)}{c(\alpha/t)\alpha^{\rho/2}\phi(\alpha/t)} t^{\rho/2} L_\epsilon(t) \\
&\leq 2 \sup_{\alpha\geq A_1} \frac{f(\alpha)}{\alpha^{\rho/2}\phi(\alpha/t)} t^{\rho/2} L_\epsilon(t).
\end{aligned}$$

Since both $t^{\rho/2}L_\epsilon(t)$ and (for any α) $f(\alpha)\alpha^{-\rho/2}/\phi(\alpha/t)$ are non-decreasing in t on $(0, \mathcal{T}]$, we conclude with (6.2) that

$$\begin{aligned}
\sup_{t\in(0,\mathcal{T}]} \sup_{\alpha\geq A_1/t} \frac{f(\alpha t)}{f(\alpha)} L_\epsilon(t) &\leq 2 \sup_{\alpha\geq A_1} \frac{f(\alpha)}{\alpha^{\rho/2}\phi(\alpha/\mathcal{T})} \mathcal{T}^{\rho/2} L_\epsilon(\mathcal{T}) \\
&\leq 4 \sup_{\alpha\geq A_1} \frac{f(\alpha)}{c(\alpha/\mathcal{T})\alpha^{\rho/2}\phi(\alpha/\mathcal{T})} \mathcal{T}^{\rho/2} L_\epsilon(\mathcal{T}) \\
&= 4 \sup_{\alpha\geq A_1} \frac{f(\alpha)}{f(\alpha/\mathcal{T})} L_\epsilon(\mathcal{T}) \\
&\leq 8\mathcal{T}^\rho L_\epsilon(\mathcal{T}) \\
&\leq 8 \sup_{t\in[0,\mathcal{T}]} t^\rho L_\epsilon(t).
\end{aligned}$$

Inequality (6.4) is an easy consequence of combining this with (6.1).

Step 2: Proof of (6.5) Note that, since $t \in [0,\infty) \mapsto |t|^{1+\epsilon}$ is convex, we have for $t/\alpha \le 1/e$

$$\begin{aligned} L_\epsilon(t/\alpha) &= |\log t/\alpha|^{1+\epsilon} \le (|\log t| + \log\alpha)^{1+\epsilon} \le 2^{1+\epsilon}\left(\frac{1}{2}|\log t|^{1+\epsilon} + \frac{1}{2}(\log\alpha)^{1+\epsilon}\right) \\ &= 2^\epsilon(|\log t|^{1+\epsilon} + (\log\alpha)^{1+\epsilon}). \end{aligned}$$

Observe that $t/\alpha \le 1/e$ for $t \in [0, A_1]$ and $\alpha \ge A_2$, so that

$$\begin{aligned} \sup_{t\in(0,A_1/A_2]} \sup_{A_2\le\alpha\le A_1/t} \frac{f(\alpha t)}{f(\alpha)} L_\epsilon(t) &= \sup_{\alpha\ge A_2} \sup_{t\in(0,A_1/\alpha]} \frac{f(\alpha t)}{f(\alpha)} L_\epsilon(t) \\ &= \sup_{\alpha\ge A_2} \sup_{t\in(0,A_1]} \frac{f(t)}{f(\alpha)} L_\epsilon(t/\alpha) \\ &\le 2^\epsilon \sup_{\alpha\ge A_2} \sup_{t\in(0,A_1]} \frac{f(t)}{f(\alpha)} \left(|\log t|^{1+\epsilon} + (\log\alpha)^{1+\epsilon}\right). \end{aligned}$$

Since $A_1 \ge e$, we can bound $f(t)|\log t|^{1+\epsilon}$ as follows on $(0, A_1]$:

$$\begin{aligned} \sup_{t\in(0,A_1]} f(t)|\log t|^{1+\epsilon} &\le \max\left(\sup_{t\in(0,1/e]} f(t)L_\epsilon(t), \sup_{t\in(1/e,A_1]} f(t)|\log A_1|^{1+\epsilon}\right) \\ &\le \max(M, M(\log A_1)^{1+\epsilon}) \\ &= M(\log A_1)^{1+\epsilon}. \end{aligned}$$

By combining the two preceding displays, we obtain

$$\sup_{t\in(0,A_1/A_2]} \sup_{A_2\le\alpha\le A_1/t} \frac{f(\alpha t)}{f(\alpha)} L_\epsilon(t) \le 2^\epsilon M \sup_{\alpha\ge A_2} \frac{(\log A_1)^{1+\epsilon} + (\log\alpha)^{1+\epsilon}}{f(\alpha)} \le \frac{1}{2}\eta,$$

where the last inequality is (6.3). Inequality (6.5) readily follows by using (6.1).

Step 3: Proof of the claim It is readily checked that the first two steps imply that for any (small) $\eta > 0$ we can find a (small) κ and (large) A_2 such that

$$\sup_{\alpha\ge A_2} \sup_{t\in(0,\kappa]} \left|\frac{f(\alpha t)}{f(\alpha)} L_\epsilon(t) - t^\rho L_\epsilon(t)\right| < \eta. \tag{6.6}$$

In this last step, we establish the uniform convergence on each interval $(0, \cdot]$.

Let $T > 0$ be arbitrary, and set $M' := \sup_{t\in(0,T]} t^\rho L_\epsilon(t) < \infty$. By Theorem 1.2.1 of [9], it is possible to select A_3 so that for $\alpha \ge A_3$, uniformly in $t \in [\kappa, T]$,

$$\left|\frac{f(\alpha t)}{f(\alpha)t^\rho} - 1\right| < \frac{\eta}{M'}.$$

Now, for $\alpha \ge A_3$, we have

$$\sup_{t\in[\kappa,T]} \left|\frac{f(\alpha t)}{f(\alpha)} L_\epsilon(t) - t^\rho L_\epsilon(t)\right| = \sup_{t\in[\kappa,T]} t^\rho L_\epsilon(t) \left|\frac{f(\alpha t)}{f(\alpha)t^\rho} - 1\right| < \eta.$$

Combining this with (6.6) yields the claim.

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