Extremes of Gaussian processes over an infinite horizon

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Abstract

Consider a centered separable Gaussian process Y with a variance function that is regularly varying at infinity with index $2H \in (0, 2)$. Let ϕ be a 'drift' function that is strictly increasing, regularly varying at infinity with index $\beta > H$, and vanishing at the origin. Motivated by queueing and risk models, we investigate the asymptotics for $u \to \infty$ of the probability $P(\sup_{t\geq 0} Y_t - \phi(t) > u)$ as $u \to \infty$.

To obtain the asymptotics, we tailor the celebrated double sum method to our general framework. Two different families of correlation structures are studied, leading to four qualitatively different types of asymptotic behavior. A generalized Pickands' constant appears in one of these cases.

Our results cover both processes with stationary increments (including Gaussian integrated processes) and self-similar processes.

1 Introduction

Let Y be a centered separable Gaussian process, and let ϕ be a strictly increasing 'drift' function with $\phi(0) = 0$. Motivated by applications in telecommunications engineering and insurance mathematics, the probability

$$P\left(\sup_{t\geq 0} Y_t - \phi(t) > u\right) \tag{1}$$

has been analyzed under different levels of generality as $u \to \infty$. In these applications, Y_0 is supposed to be degenerate, i.e., $Y_0 = 0$. Letting u tend to infinity is known as investigating the *large buffer regime*, since u can be interpreted as a buffer level of a queue. Notice that (1) can be rewritten as

$$P\left(\sup_{t\geq 0} \frac{Y_{\mu(ut)}}{1+t} > u\right), \tag{2}$$

where μ is the inverse of ϕ. Special attention has been paid to the case that Y has stationary increments (e.g., [5, 6, 9, 10, 11, 13, 16, 18, 20, 24, 25, 29, 30]), and to the case that Y is self-similar or 'almost' self-similar [23].

*The research was supported by the Netherlands Organization for Scientific Research (NWO) under grant 631.000.002.

From a practical point of view, Gaussian processes lead to parsimonious yet flexible models, since a broad range of correlation structures can be described by few parameters. The study of Gaussian processes can also be justified by an approximation argument; they can appear as stochastic process limits, often as a result of a second-order scaling as in the central limit theorem. However, a warning is in place here: Wischik [40] argues that it is extremely important to check the appropriateness of this scaling before resorting to Gaussian models.

The main contribution of the present paper is that we extend the known results on the asymptotics of (1). For this, we introduce a wide class of local correlation structures, covering both processes with stationary increments and 'almost' self-similar processes. A motivation for studying the problem in this generality is to gain insight into the case that Y is the sum of a number of independent Gaussian processes, e.g., of a Gaussian integrated process and a number of fractional Brownian motions with different Hurst parameters. We study this case in somewhat more detail in forthcoming work.

Some words for the technical aspects of this paper. We use the *double sum method* to find the asymptotics of (2), see Piterbarg [34] or Piterbarg and Fatalov [35]. This method has been applied sucessfully to find the asymptotics of $P(\sup_{t\in[0,T]} X(t) > u)$, where X is either a stationary Gaussian process [33, 37] or a Gaussian process with a unique point of maximum variance [36]. These results are also available for fields, see [34, Section 8]. However, they cannot be applied to find the asymptotics of (1).

In this paper, we approach the double sum method differently. The idea in [36] is to first establish the asymptotics of a certain stationary Gaussian process on a subinterval of $[0, T]$. Then a comparison inequality is applied to see that the asymptotics of $P(\sup_{t\in[0,T]} X(t) > u)$ equal the asymptotics of this stationary field. Here, we do not make a comparison to stationary processes, but we apply the ideas underlying the double sum method directly to the processes $Y_{\mu(ut)}/(1 + t)$. Given our results, it can be seen immediately that the comparison approach cannot work in the generality of this paper: a so-called *generalized* Pickands' constant appears, which is not present in the stationary case. It is also obtained in the analysis of suprema of Gaussian integrated processes, see Dębicki [11]. The appearance of this constant in the present study is not surprising, since our results also cover Gaussian integrated processes.

Several related problems appear in the vast body of literature on asymptotics for Gaussian processes. For instance, Dębicki and Rolski [17] study the asymptotics of (1) over a finite horizon, i.e., the supremum is taken over $[0, T]$ for some $T > 0$. We remark that the asymptotics found in [17] differ qualitatively from the asymptotics established in the present paper. Another problem closely related to the present setting is where Y has the form $Z/\sqrt{n}$ for some Gaussian process Z independent of n. One then fixes u and studies the probability (1) as $n \to \infty$. The resulting asymptotics were studied by Dębicki and Mandjes [12]; these asymptotics are often called *many sources asymptotics*, since convolution of identical Gaussian measures amounts to scaling a single measure.

It is worthwhile to compare our results with those of Berman [2] on extremes of Gaussian processes with stationary increments. Berman studies the probability $P(\sup_{t\in B} \overline{Y}_t > u)$ for $u \to \infty$, where $\overline{Y}$ is constructed from Y by standardization (so that its variance is constant) and B is some fixed compact interval. The problem of finding the asymptotics of (2) does not fit into Berman's framework: our assumptions will imply that $Y_{\mu(ut)}/(1 + t)$ has a point of maximum variance, which is asymptotically unique. Another difference is that this point depends (asymptotically) linearly on u, so that it cannot belong to B for large u.

The paper is organized as follows. The main result and its assumptions are described in Section 2. In Section 3, we work out two cases of special interest: processes with stationary increments and self-similar processes. Furthermore, we relate our formulas with the literature by giving some examples.

Sections 4–7 are devoted to proofs. In Section 4, the classical Pickands' lemma is generalized into an appropriate direction. Section 5 distinguishes four instances of this lemma. The resulting observations are key to the derivation of the upper bounds, which is the topic of Section 6. Lower bounds are given in Section 7, where we use a double sum-type argument to see that the upper and lower bounds coincide asymptotically.

To slightly reduce the length of the proofs and make them more readable, details are often omitted when a similar argument has already been given, or when the argument is standard. We then use curly brackets (e.g., {**T1**}) to indicate which assumptions are needed to make the claim precise.

We frequently apply standard results for regularly varying functions, for which the main reference is Bingham, Goldie and Teugels [4]. Recall that a positive function f is regularly varying at infinity with index ρ if for all $t > 0$,

$$\lim_{\alpha\to\infty} \frac{f(\alpha t)}{f(\alpha)} = t^\rho.$$

Implicitly, this convergence is uniform on intervals of the form $[a, b]$ with $b > a > 0$ by the Uniform Convergence Theorem (Theorem 1.5.2 in [4]). Often one can obtain uniformity on a wider class of intervals, although additional conditions may be required (see Theorem 1.5.2 and Theorem 1.5.3 in [4]). The Uniform Convergence Theorem is used extensively, and therefore abbreviated as UCT. It is applied without reference to the specific version that is used.

2 Description of the results and assumptions

This section presents our main theorem. Since many (yet natural and weak) assumptions underly our result, we defer a detailed description of these assumptions to Section 2.2.

2.1 Main theorem

The supremum in (2) is asymptotically 'most likely' attained at a point where the variance is close to its maximum value. Let t_u^* denote a point that maximizes the variance $\sigma^2(\mu(ut))/(1+t)^2$ (existence will be ensured by continuity conditions). Our main assumptions are that σ^2 (defined by $\sigma^2(t) := \mathbb{V}\mathrm{ar} Y_t$) and μ (defined as the inverse of ϕ in (1)) are regularly varying at infinity with indices $2H \in (0, 2)$ and $1/\beta < 1/H$ respectively. Note that the UCT implies that t_u^* converges to $t^* := H/(\beta - H)$. In that sense, t_u^* is asymptotically unique.

For an appropriately chosen δ with $\delta(u)/u \to 0$ and $\sigma(\mu(u))/\delta(u) \to 0$, (1) and (2) are asymptotically equivalent to

$$P\left(\sup_{t\in[t_u^*\pm\delta(u)/u]} \frac{Y_{\mu(ut)}}{1+t} > u\right),$$

see Lemma 7. Hence, in some sense, the variance $\sigma^2(\mu(ut))$ of $Y_{\mu(ut)}$ determines the *length* of the 'most probable' hitting interval by the requirement that $\sigma(\mu(u))/\delta(u) \to 0$.

Not only the length of this interval plays a role in the asymptotics of (2). There is one other important element: the *local correlation structure* of the process on $[t_u^* \pm \delta(u)/u]$. Traditionally, it was assumed that $\mathbb{V}\mathrm{ar}\left(Y_{\mu(us)}/\sigma(\mu(us)) - Y_{\mu(ut)}/\sigma(\mu(ut))\right)$ behaves locally like $|s - t|^\alpha$ for some $\alpha \in (0, 2]$ [32]. It was soon realized that $|s - t|^\alpha$ can be replaced by a regularly varying function (at zero) with minimal additional effort [37]; see also [3, 11, 23], to mention a few recent contributions.

However, by imposing such a correlation structure, it is impossible to find the asymptotics of (1) for a general Gaussian process with stationary increments, for instance. We solve this problem by introducing two wide classes of correlation structures, resulting in qualitatively different asymptotics in four cases. These specific structures must be imposed to be able to perform explicit calculations. The main novelty of this paper is that the local behavior may depend on u. Our framework is specific enough to derive generalities, yet general enough to include many interesting processes as special cases (to our best knowledge, all processes are covered for which the asymptotics of (1) appear in the literature; see the examples in Section 3.3).

Often there is a third element playing a role in the asymptotics: the *local variance structure* of $Y_{\mu(ut)}/(1+t)$ near $t = t_u^*$. By the structure of the problem and the differentiability assumptions that we will impose on σ and μ, this third element is only implicitly present in our analysis. However, if one is interested in the asymptotics of some probability different from (1), it may play a role. In that case, the reasoning of the present paper is readily adapted.

We now introduce the first family of correlation structures, leading to three different types of asymptotics. Suppose that the following holds:

$$\sup_{\substack{s,t\in[t_u^*\pm\delta(u)/u]\\ s\neq t}} \left| \frac{\mathbb{V}\mathrm{ar}\left(\frac{Y_{\mu(us)}}{\sigma(\mu(us))} - \frac{Y_{\mu(ut)}}{\sigma(\mu(ut))}\right)}{\mathcal{D}\tau^2(|\nu(us)-\nu(ut)|)/\tau^2(\nu(u))} - 1\right| \to 0, \tag{3}$$

as $u \to \infty$, where $\mathcal{D}$ is some constant and τ and ν are suitable functions. It is assumed that τ and ν are regularly varying at infinity with indices $\iota_\tau \in (0, 1)$ and $\iota_\nu > 0$ respectively. To gain some intuition, suppose that ν is the identity, and write $\tau(t) = \ell(t)t^{\iota_\tau}$ for some slowly varying function at infinity ℓ. The denominator in (3) then equals $\mathcal{D}|s-t|^{2\iota_\tau}\ell^2(u|s-t|)/\ell^2(u)$. From the analysis of the problem it follows that one must consider $|s - t| \le \Delta(u)/u$, where Δ is some function satisfying $\Delta(u) = o(\delta(u))$. As a result, the denominator is of the order $[\Delta(u)/u]^{2\iota_\tau}\ell^2(\Delta(u))/\ell^2(u)$; due to the term $\ell^2(\Delta(u))$, three cases can now be distinguished: Δ tends to infinity, to a constant, or to zero. Interestingly, the Pickands' constant appearing in the asymptotics is determined by the behavior of τ at infinity in the first case, and at zero in the last case (one needs an additional assumption on the behavior of τ at zero). The second 'intermediate' case is special, resulting in the appearance of a so-called generalized Pickands' constant.

The second family of correlation structures, resulting in the fourth type of asymptotics, is given by

$$\sup_{\substack{s,t\in[t_u^*\pm\delta(u)/u]\\ s\neq t}} \left| \frac{\mathbb{V}\mathrm{ar}\left(\frac{Y_{\mu(us)}}{\sigma(\mu(us))} - \frac{Y_{\mu(ut)}}{\sigma(\mu(ut))}\right)}{\tau^2(|\nu(us)-\nu(ut)|/\nu(u))} - 1\right| \to 0, \tag{4}$$

where ν is regularly varying at infinity with index $\iota_\nu > 0$ and τ is regularly varying at zero with index $\tilde{\iota}_\tau \in (0, 1)$ (the tilde emphasizes that we consider regular variation at zero). A detailed description of the assumptions on each of the functions are given in Section 2.2. Here, if ν is the identity, the denominator equals $\ell^2(|s-t|)|s-t|^{2\tilde{\iota}_\tau}$ for some slowly varying function at zero ℓ. Therefore, it cannot be written in the form (3) unless ℓ is constant.

Having introduced the four cases intuitively, we now present them in somewhat more detail. The cases are referred to as case A, B, C, and D. We set

$$\mathcal{G} := \lim_{u\to\infty} \frac{\sigma(\mu(u))\tau(\nu(u))}{u}, \tag{5}$$

assuming the limit exists.

A. Case A applies when (3) holds and $\mathcal{G} = \infty$.

B. Case B applies when (3) holds and $\mathcal{G} \in (0, \infty)$.

C. Case C applies when (3) holds and $\mathcal{G} = 0$. We then also suppose that τ be regularly varying at zero with index $\tilde{\iota}_\tau \in (0, 1)$.

D. Case D applies when (4) holds.

In order to state the main result, we first introduce some further notation. For a centered separable Gaussian process η with stationary increments and variance function σ_η^2, we define

$$\mathcal{H}_\eta := \lim_{T\to\infty} \frac{1}{T} \mathcal{H}_\eta(T) := \lim_{T\to\infty} \frac{1}{T} \mathbb{E} \exp \left(\sup_{t\in[0,T]} \left[\sqrt{2}\eta_t - \sigma_\eta^2(t) \right] \right), \tag{6}$$

provided both the expectation and the limit exist. Depending on the context, we also write $\mathcal{H}_{\sigma_\eta^2}$ for $\mathcal{H}_\eta$. If η is a *fractional Brownian motion* with Hurst parameter $H \in (0, 1)$, it is denoted as B_H throughout this paper. Recall that a fractional Brownian motion is defined by setting $\sigma_\eta^2(t) = t^{2H}$, and that these constants are strictly positive (in particular, they exist). These constants appear in Pickands' classical analysis of stationary Gaussian processes [32, 33]. In the present generality, they have been introduced by Dębicki [11], and the field analogue shows up in the study of Gaussian fields; see Piterbarg [34].

Given a stochastic process Y, we use both $Y(t)$ and Y_t for the value of Y at time epoch t. Moreover, we write

$$\Psi(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{1}{2}w^2} dw,$$

and it is standard that, for $x \to \infty$,

$$\sqrt{2\pi} x \Psi(x) \sim e^{-x^2/2}, \tag{7}$$

where asymptotic equivalence $f \sim g$ as $x \to X \in [-\infty, \infty]$ means $f(x) = g(x)(1 + o(1))$ as $x \to X$.

Provided it exists, we denote an asymptotic inverse of τ by $\overleftarrow{\tau}$; recall that it is (asymptotically uniquely) defined by

$$\overleftarrow{\tau}(\tau(t)) \sim \tau(\overleftarrow{\tau}(t)) \sim t. \tag{8}$$

It depends on the context whether $\overleftarrow{\tau}$ is an asymptotic inverse near zero or infinity, i.e., whether (8) holds for $t \to 0$ or $t \to \infty$ respectively. Unless stated otherwise, regular variation should always be understood as regular variation at infinity, and measurability of such functions is implicit (it is often ensured by continuity assumptions).

It is convenient to introduce the notation

$$C_{H,\beta,\iota_\nu,\iota_\tau} := \sqrt{2^{1-1/\iota_\tau} \pi \iota_\nu} \left(\frac{\beta}{H} \right)^{1/\iota_\tau} \left(\frac{H}{\beta - H} \right)^{\iota_\nu + \frac{H}{\beta} - \frac{1}{2} + \frac{1}{\iota_\tau}\left(1 - \frac{H}{\beta}\right)}$$

and, for case B,

$$\mathcal{M} := \frac{\beta^2}{2\mathcal{G}^2 H^{2H/\beta} (\beta - H)^{2-2H/\beta}},$$

where $\mathcal{G} \in (0, \infty)$ is defined as in (5). Here is our main result. The assumptions are detailed in Section 2.2.

Theorem 1 *Let μ and σ satisfy assumptions* **M1–M4** *and* **S1–S4** *below for some $\beta > H$.*

In case A, i.e., when **A1**, **A2**, **T1**, **T2**, **N1**, **N2** *below hold, we have*

$$P\left(\sup_{t\geq 0} Y_{\mu(t)} - t > u \right) \sim \mathcal{H}_{B_{\iota_\tau}} C_{H,\beta,\iota_\nu,\iota_\tau} \sqrt{\mathcal{D}^{1/\iota_\tau}} \frac{\sigma(\mu(u))\nu(u)}{u \overleftarrow{\tau}\left(\frac{\sigma(\mu(u))\tau(\nu(u))}{u} \right)} \Psi\left(\inf_{t\geq 0} \frac{u(1+t)}{\sigma(\mu(ut))} \right).$$

In case B, i.e., when **B1**, **B2**, **T1**, **T2**, **N1**, **N2** *below hold, then $\mathcal{H}_{\mathcal{D}\mathcal{M}\tau^2}$ exists and we have*

$$P\left(\sup_{t\geq 0} Y_{\mu(t)} - t > u \right) \sim \mathcal{H}_{\mathcal{D}\mathcal{M}\tau^2} \sqrt{2\pi} \iota_\nu \left(\frac{H}{\beta - H} \right)^{\iota_\nu + \frac{H}{\beta} - \frac{1}{2}} \frac{\sigma(\mu(u))\nu(u)}{u} \Psi\left(\inf_{t\geq 0} \frac{u(1+t)}{\sigma(\mu(ut))} \right).$$

In case C, i.e., when **C1–C3**, **T1**, **N1**, **N2** *below hold, we have*

$$P\left(\sup_{t\geq 0} Y_{\mu(t)} - t > u \right) \sim \mathcal{H}_{B_{\tilde{\iota}_\tau}} C_{H,\beta,\iota_\nu,\tilde{\iota}_\tau} \sqrt{\mathcal{D}^{1/\tilde{\iota}_\tau}} \frac{\sigma(\mu(u))\nu(u)}{u \overleftarrow{\tau}\left(\frac{\sigma(\mu(u))\tau(\nu(u))}{u} \right)} \Psi\left(\inf_{t\geq 0} \frac{u(1+t)}{\sigma(\mu(ut))} \right).$$

In case D, i.e., when **D1**, **D2**, **N1**, **N2** *below hold, we have*

$$P\left(\sup_{t\geq 0} Y_{\mu(t)} - t > u \right) \sim \mathcal{H}_{B_{\tilde{\iota}_\tau}} C_{H,\beta,\iota_\nu,\tilde{\iota}_\tau} \frac{\sigma(\mu(u))}{u \overleftarrow{\tau}\left(\frac{\sigma(\mu(u))}{u} \right)} \Psi\left(\inf_{t\geq 0} \frac{u(1+t)}{\sigma(\mu(ut))} \right).$$

Observe that $\overleftarrow{\tau}$ is an asymptotic inverse of τ at infinity in case A, and at zero in case C and D. Hence, the factors preceding the function Ψ are regularly varying with index $(H/\beta + \iota_\nu \iota_\tau - 1)(1 - 1/\iota_\tau) + (1 - \iota_\tau)\iota_\nu$ in case A, with index $H/\beta + \iota_\nu - 1$ in case B, with index $H/\beta + \iota_\nu - 1 - (H/\beta + \iota_\tau \iota_\nu - 1)/\tilde{\iota}_\tau$ in case C, and with index $(H/\beta - 1)(1 - 1/\tilde{\iota}_\tau)$ in case D. Note that case B is special in a number of ways: a non-classical Pickands' constant is present and no inverse appears in the formula.

We now formally state the underlying assumptions.

2.2 Assumptions

Two types of assumptions are distinguished: general assumptions and case-specific assumptions. The general assumptions involve the variance σ^2 of Y, the time change μ, and the functions ν and τ appearing in (3) and (4). The case-specific assumptions formalize the four regimes introduced in the previous subsection.

2.2.1 General assumptions

We start by stating the assumptions on μ.

M1 μ is regularly varying at infinity with index $1/\beta$,

M2 μ is strictly increasing, $\mu(0) = 0$,

M3 μ is ultimately continuously differentiable and its derivative $\dot{\mu}$ is ultimately monotone.

M4 μ is twice continuously differentiable and its second derivative $\ddot{\mu}$ is ultimately monotone.

Assumption **M2** is needed to ensure that the probabilities (1) and (2) be equal. The remaining conditions imply that $\beta u \dot{\mu}(u) \sim \mu(u)$ and $\beta^2 u^2 \ddot{\mu}(u) \sim (1 - \beta)\mu(u)$, see Exercise 1.11.13 of [4]. In particular, $\dot{\mu}$ and $\ddot{\mu}$ are regularly varying with index $1/\beta - 1$ and $1/\beta - 2$ respectively.

Now we formulate the assumptions on σ and one assumption on both μ and σ.

S1 σ is continuous and regularly varying at infinity with index H for some $H \in (0,1)$,

S2 σ^2 is ultimately continuously differentiable and its first derivative $\dot{\sigma}^2$ is ultimately monotone,

S3 σ^2 is ultimately twice continuously differentiable and its second derivative $\ddot{\sigma}^2$ is ultimately monotone,

S4 There exist some $T, \epsilon > 0$, $\gamma \in (0,2]$ such that

1. $\limsup_{u\to\infty} \sup_{s,t\in(0,(1+\epsilon)T^{1/\beta}]} \mathbb{V}\mathrm{ar}(Y_{us} - Y_{ut})\sigma^{-2}(u)|s-t|^{-\gamma} < \infty$, and
2. $\limsup_{u\to\infty} \frac{\sigma^2(\mu(u))}{u^2} \log P\left(\sup_{t\geq T} \frac{Y_{\mu(ut)}}{1+t} > u\right) < -\frac{1}{2}\frac{(1+t^*)^2}{(t^*)^{H/\beta}}$.

We emphasize that $\dot{\sigma}^2$ denotes the derivative of σ^2, and not the square derivative of σ. As earlier, conditions **S1**–**S3** imply that $u\dot{\sigma}^2(u) \sim 2H\sigma^2(u)$ and $u^2\ddot{\sigma}^2(u) \sim 2H(2H-1)\sigma^2(u)$. The first point of **S4**, which is Kolmogorov's weak convergence criterion, ensures the existence of a modification with continuous sample paths; we always assume to work with this modification. The second point of **S4** ensures that the probability $P(\sup_{t\geq uT} Y_{\mu(t)} - t > u)$ cannot dominate the asymptotics. We choose to formulate this as an assumption, although it is possible to give sharp conditions for **S4**.2 to hold. However, these conditions look relatively complicated, while the second point is in general easier to verify on a case by case basis. In the next section, we show that it holds for processes with stationary increments and self-similar processes.

Note that if **M1**–**M4** and **S1**–**S4** hold, the first and second derivative of $\sigma^2(\mu(\cdot))$ are also regularly varying, with indices $2H/\beta - 1$ and $2H/\beta - 2$ respectively. It is this fact that guarantees the existence of the limits that are implicitly present in the notation '$\sim$' in Theorem 1.

The function ν appearing in (3) and (4) also has to satisfy certain assumptions, which are similar to the assumptions imposed on μ:

N1 ν is regularly varying at infinity with index $\iota_\nu > 0$,

N2 ν is ultimately continuously differentiable and its derivative $\dot{\nu}$ is ultimately monotone.

Finally, we formulate the assumptions on τ in (3) or (4).

T1 τ is continuous and regularly varying at infinity with index ι_τ for some $\iota_\tau \in (0,1)$,

T2 $\tau(t) \leq Ct^{\gamma'}$ on a neighborhood of zero for some $C, \gamma' > 0$.

Assumption **T2** is essential to prove uniform tightness at some point in the proof, which yields the existence of the Pickands' constants.

2.2.2 Case-specific assumptions

We now formulate the case-specific assumptions in each of the cases A, B, C, and D. These assumptions are also mentioned in the Introduction, but it is convenient to label them for reference purposes. If we write that the correlation structure is determined by (3) or (4), the function δ is supposed to satisfy $\delta(u) = o(u)$ and $\sigma(\mu(u)) = o(\delta(u))$ as $u \to \infty$.

After recalling the definition of $\mathcal{G}$ in (5), we start with case A.

A1 the correlation structure is determined by (3),

A2 $\mathcal{G} = \infty$.

Similar conditions are imposed in case B.

B1 the correlation structure is determined by (3),

B2 $\mathcal{G} \in (0,\infty)$.

In case C, we need an additional condition (**C3**). Note that the index of variation in **C3** appears at several places in the asymptotics, cf. Theorem 1. It also implies the existence of an asymptotic inverse $\overleftarrow{\tau}$ at zero, cf. Theorem 1.5.12 of [4].

C1 the correlation structure is determined by (3),

C2 $\mathcal{G} = 0$,

C3 τ is regularly varying at zero with index $\tilde{\iota}_\tau \in (0,1)$.

Case D is slightly different from the previous three cases, although here the regular variation of τ at zero also plays a role. In fact, the index of variation appears in exactly the same way in the asymptotics as in case C.

D1 the correlation structure is determined by (4),

D2 τ is regularly varying at zero with index $\tilde{\iota}_\tau \in (0,1)$.

3 Special cases: stationary increments and self-similarity

In this section, we apply Theorem 1 to calculate the asymptotics of (2) for two specific cases: (i) Y has stationary increments and (ii) Y is self-similar. In both examples, the imposed assumptions imply that $\sigma^2(0) = 0$, so that $Y_0 = 0$ almost surely.

In case Y has stationary increments, the finite-dimensional distributions are completely determined by the variance function σ^2. For self-similar processes, (2) has been studied by Hüsler and Piterbarg [23]. We show that their results are reproduced and even slightly generalized by Theorem 1.

We conclude this section with some examples that have been studied in the literature.

3.1 Stationary increments

Since σ determines the finite-dimensional distributions of Y, it also fixes the local correlation structure; we record this in the next proposition. To get some feeling for the result, observe that for $s, t \in [t_u^* \pm \delta(u)/u]$,

$$\mathbb{V}\mathrm{ar}\left(\frac{Y_{\mu(us)}}{\sigma(\mu(us))} - \frac{Y_{\mu(ut)}}{\sigma(\mu(ut))}\right) \approx \frac{\mathbb{V}\mathrm{ar}\left(Y_{\mu(us)} - Y_{\mu(ut)}\right)}{\sigma^2(\mu(ut^*))} = \frac{\sigma^2(|\mu(us) - \mu(ut)|)}{\sigma^2(\mu(ut^*))}.$$

This intuitive reasoning is now made precise. Note the proposition also entails that case D does *not* occur in this setting.

Proposition 1 *Let* **S1**–**S2**, **M1**–**M3** *hold for some $\beta > H$. Let δ be regularly varying with index $\iota_\delta \in (1 - 1/\beta, 1)$. Then (3) holds with $\tau = \sigma$, $\nu = \mu$ and $\mathcal{D} = (t^*)^{-2H/\beta}$.*

Proof. Since $s, t \in [t_u^* \pm \delta(u)/u]$, we have by the UCT {**S1**, **M1**},

$$\lim_{u\to\infty} \sup_{\substack{s,t\in[t_u^*\pm\delta(u)/u]\\ s\neq t}} \left| \frac{\sigma^2(\mu(u))}{\mathcal{D}\sigma(\mu(us))\sigma(\mu(ut))} - 1 \right| = 0.$$

Moreover, the stationarity of the increments implies that

$$2\left[\sigma(\mu(us))\sigma(\mu(ut)) - \mathbb{C}\mathrm{ov}\left(Y_{\mu(us)}, Y_{\mu(ut)}\right)\right] = \sigma^2(|\mu(us) - \mu(ut)|) - [\sigma(\mu(us)) - \sigma(\mu(ut))]^2.$$

Hence, it suffices to prove that

$$\lim_{u\to\infty} \sup_{\substack{s,t\in[t_u^*\pm\delta(u)/u]\\ s\neq t}} \frac{[\sigma(\mu(us)) - \sigma(\mu(ut))]^2}{\sigma^2(|\mu(us) - \mu(ut)|)} = 0. \tag{9}$$

For this, observe that the left hand side of (9) is majorized by $t_1(u)t_2(u)$, where

$$t_1(u) := \sup_{\substack{s,t\in[t_u^*\pm\delta(u)/u]\\ s\neq t}} \frac{[\sigma(\mu(us)) - \sigma(\mu(ut))]^2}{[\mu(us) - \mu(ut)]^2}; \quad t_2(u) := \sup_{\substack{s,t\in[t_u^*\pm\delta(u)/u]\\ s\neq t}} \frac{[\mu(us) - \mu(ut)]^2}{\sigma^2(|\mu(us) - \mu(ut)|)}.$$

As for $t_1(u)$, by the Mean Value Theorem {**S2**, **M3**} there exist $t^\wedge(u,s,t), t^\vee(u,s,t)$ such that, for u large enough,

$$t_1(u) = \sup_{\substack{s,t\in[t_u^*\pm\delta(u)/u]\\ s\neq t}} \frac{[\tilde{\sigma}_\mu(ut^\wedge(u,s,t))]^2}{[\dot{\mu}(ut^\vee(u,s,t))]^2} \le \left(\frac{\sup_{t\in[t_u^*\pm\delta(u)/u]} \tilde{\sigma}_\mu(ut)}{\inf_{t\in[t_u^*\pm\delta(u)/u]} \dot{\mu}(ut)} \right)^2,$$

where $\tilde{\sigma}_\mu(\cdot)$ denotes the derivative of $\sigma(\mu(\cdot))$. As a consequence of the UCT {**M1**, **M3**, **S1**, **S2**}, $t_1(u)$ can therefore be upper bounded by $C'\sigma^2(\mu(u))/\mu^2(u)$ for some constant $C' < \infty$.

We now turn to $t_2(u)$. A substitution {**M2**} shows that

$$t_2(u) = \sup_{\substack{s,t\in[\mu(ut_u^*-\delta(u)),\mu(ut_u^*+\delta(u))]\\ s>t}} \frac{(s-t)^2}{\sigma^2(s-t)} = \sup_{0<t\le\mu(ut_u^*+\delta(u))-\mu(ut_u^*-\delta(u))} \frac{t^2}{\sigma^2(t)}.$$

Observe that, again by the Mean Value Theorem and the UCT {**M1**, **M3**},

$$\mu(ut_u^* + \delta(u)) - \mu(ut_u^* - \delta(u)) \le 2 \sup_{t\in[t_u^*\pm\delta(u)/u]} \dot{\mu}(ut)\delta(u) \sim \frac{2}{\beta}(t^*)^{1/\beta-1}\mu(u)\delta(u)/u,$$

which tends to infinity by the assumption on ι_δ.

Suppose for the moment that the map $x \mapsto x^2/\sigma^2(x)$ is bounded on sets of the form $(0,\cdot]$. Since it is regularly varying with index $2 - 2H > 0$ {**S1**}, we have by the UCT and the assumption that $\iota_\delta > 1 - 1/\beta$, for u large enough,

$$t_2(u) \le \sup_{0<t\le 3/\beta(t^*)^{1/\beta-1}} \frac{[\mu(u)\delta(u)/u]^2 t^2}{\sigma^2(\mu(u)\delta(u)/ut)} \sim \left(\frac{3}{\beta}(t^*)^{1/\beta-1}\right)^{2-2H} \frac{[\mu(u)\delta(u)/u]^2}{\sigma^2(\mu(u)\delta(u)/u)}.$$

In conclusion, there exists a constant $\mathcal{K} < \infty$ such that

$$\sup_{\substack{s,t\in[t_u^*\pm\delta(u)/u]\\ s\neq t}} \frac{[\sigma(\mu(us)) - \sigma(\mu(ut))]^2}{\sigma^2(|\mu(us) - \mu(ut)|)} \le \mathcal{K}\frac{\sigma^2(\mu(u))\delta^2(u)/u^2}{\sigma^2(\mu(u)\delta(u)/u)},$$

which is regularly varying with index $2(1-H)(\iota_\delta - 1) < 0$, so that (9) follows.

It remains to show that $x \mapsto x^2/\sigma^2(x)$ is locally bounded. To see this, we use an argument introduced by Dębicki [11, Lemma 2.1]. By **S2**, one can select some (large) $s \ge 0$ such that σ^2 is continuously differentiable at s. Then, for some small $x > 0$,

$$\sigma^2(s) - \sigma^2(s-x) \le \sigma^2(s) + \sigma^2(x) - \sigma^2(s-x) = 2\mathbb{C}\mathrm{ov}(Y_s, Y_x) \le 2\sigma(s)\sigma(x),$$

and by the Mean Value Theorem there exists some $\rho_x \in [s-x, s]$ such that $\sigma^2(s) - \sigma^2(s-x) = \dot{\sigma}^2(\rho_x)x$. By continuity of $\dot{\sigma}^2$ at s,

$$\limsup_{x\downarrow 0} \frac{x}{\sigma(x)} \le \limsup_{x\downarrow 0} 2\frac{\sigma(s)}{\dot{\sigma}^2(\rho_x)} = 2\frac{\sigma(s)}{\dot{\sigma}^2(s)} < \infty.$$

The claim follows upon combining this observation with **S1**. □

Lemma 1 *Let Y have stationary increments, and suppose that* **S1** *and* **M1** *hold. If $\sigma^2(t) \le Ct^\gamma$ on a neighborhood of zero for some $C, \gamma > 0$, then* **S4** *holds.*

Proof. By the stationarity of the increments, the first point of **S4** follows immediately from the UCT for $t \mapsto \sigma^2(t)t^{-\gamma}$ (this map is locally bounded by the condition in the lemma). In fact, it holds for all $T, \epsilon > 0$.

To check the second requirement of **S4**, select some ω such that $H/\beta < \omega < 1$. By the UCT {**M1**}

$$\lim_{T\to\infty} \lim_{u\to\infty} \sup_{t\ge T} \frac{ut^\omega}{\overleftarrow{\mu}(\mu(u)t^{1/\beta})} = \lim_{T\to\infty} T^{\omega-1} = 0.$$

Hence, we may suppose without loss of generality that T is such that $\overleftarrow{\mu}(\mu(u)t^{1/\beta})/u \ge t^\omega$ for every $t \ge T$ and large u. This implies that

$$P\left(\sup_{t\ge T} \frac{Y_{\mu(ut)}}{1+t} > u\right) \le P\left(\sup_{t\ge[\mu(uT)/\mu(u)]^\beta} \frac{Y_{\mu(u)t^{1/\beta}}}{1+t^\omega} > u\right) \le P\left(\sup_{t\ge T/2} \frac{Y_{\mu(u)t^{1/\beta}}}{1+t^\omega} > u\right).$$

We now apply some results from earlier work [18]. By Corollary 3 and the arguments in the proof of Proposition 1 of [18], we have

$$\limsup_{u\to\infty} \frac{\sigma^2(\mu(u))}{u^2} \log P\left(\sup_{t\ge T} \frac{Y_{\mu(ut)}}{1+t} > u\right) \le -\frac{1}{2} \inf_{t\ge T/2} \frac{(1+t^\omega)^2}{t^{2H/\beta}}.$$

Note that we have used the continuity of the functional $x \mapsto \sup_{t\ge(T/2)^{1/\beta}} x(t)/(1+t^{\omega\beta})$ in a certain topology, cf. Lemma 2 of [18]. The claim is obtained by choosing T large enough, which is possible since $t^{2\omega}/t^{2H/\beta} \to \infty$ as $t \to \infty$. □

With Proposition 1 and Lemma 1 at our disposal, we readily find the asymptotics of (1) when Y has stationary increments.

Proposition 2 *Let Y have stationary increments. Suppose that* **S1**–**S3** *hold, and that $\sigma^2(t) \le Ct^\gamma$ on a neighborhood of zero for some $C, \gamma > 0$. Moreover, suppose that* **M1**–**M4** *hold for some $\beta > H$.*

If $\sigma^2(\mu(u))/u \to \infty$, then

$$P\left(\sup_{t\ge 0} Y_{\mu(t)} - t > u\right) \sim \mathcal{H}_{B_H} C_{H,\beta,1/\beta,H} \left(\frac{\beta - H}{H}\right)^{1/\beta} \frac{\sigma(\mu(u))\mu(u)}{u\overleftarrow{\sigma}\left(\frac{\sigma^2(\mu(u))}{u}\right)} \Psi\left(\inf_{t\ge 0} \frac{u(1+t)}{\sigma(\mu(ut))}\right).$$

If $\sigma^2(\mu(u))/u \to \mathcal{G} \in (0,\infty)$, then

$$P\left(\sup_{t\geq 0} Y_{\mu(t)} - t > u\right) \sim \mathcal{H}_{(2/\mathcal{G}^2)\sigma^2} \frac{\sqrt{\pi/2}}{H} \frac{\sigma(\mu(u))\mu(u)}{u} \Psi\left(\inf_{t\geq 0} \frac{u(1+t)}{\sigma(\mu(ut))}\right).$$

If $\sigma^2(\mu(u))/u \to 0$ and σ is regularly varying at zero with index $\lambda \in (0,1)$, then

$$P\left(\sup_{t\geq 0} Y_{\mu(t)} - t > u\right) \sim \mathcal{H}_{B_\lambda} C_{H,\beta,1/\beta,\lambda} \left(\frac{\beta - H}{H}\right)^{H/(\beta\lambda)} \frac{\sigma(\mu(u))\mu(u)}{u \overleftarrow{\sigma}\left(\frac{\sigma^2(\mu(u))}{u}\right)} \Psi\left(\inf_{t\geq 0} \frac{u(1+t)}{\sigma(\mu(ut))}\right).$$

Proof. Directly from Theorem 1. For the case $\sigma^2(\mu(u))/u \to \mathcal{G} \in (0,\infty)$, observe that necessarily $2H = \beta$. □

3.2 Self-similar processes

We now suppose that Y is a self-similar process with Hurst parameter H, i.e., $\mathbb{V}\text{ar}(Y_t) = t^{2H}$ and for any $\alpha > 0$ and $s, t \geq 0$,

$$\mathbb{C}\text{ov}\left(Y_{\alpha t}, Y_{\alpha s}\right) = \alpha^{2H} \mathbb{C}\text{ov}\left(Y_t, Y_s\right). \tag{10}$$

The self-similarity property has been observed statistically in several types of data traffic, see, e.g., [31]. Two examples of self-similar Gaussian processes are the fractional Brownian motion and the Riemann-Liouville process.

Another (undoubtedly related) reason why self-similar processes are interesting is that the weak limit obtained by scaling a process both in time and space must be self-similar (if it exists); see Lamperti [27]. In the setting of Gaussian processes with stationary increments, a strong type of weak convergence is studied in [18]. We also mention the interesting fact that self-similar processes are closely related to stationary processes by the so-called Lamperti-transformation; see [1] for more details.

We make the following assumption about the behavior of the (standardized) variance of Y near $t = t^*$: for some function τ which is regularly varying at zero with index $\tilde{\iota}_\tau \in (0,1)$,

$$\lim_{s,t\to t^*} \frac{\mathbb{V}\text{ar}\left(\frac{Y_{s^{1/\beta}}}{s^{H/\beta}} - \frac{Y_{t^{1/\beta}}}{t^{H/\beta}}\right)}{\tau^2(|s-t|)} = 1. \tag{11}$$

By the self-similarity, one may equivalently require that a similar condition holds for s, t tending to an arbitrary strictly positive number; see [23]. In the proof of Proposition 3 below we show that (11) implies that self-similar processes are covered by case D.

We also need the following assumption on the variance structure of Y: for some $\gamma > 0$,

$$\sup_{s,t\in(0,1]} \mathbb{V}\text{ar}(Y_s - Y_t)|s-t|^{-\gamma} < \infty. \tag{12}$$

This Kolmogorov criterion ensures that there exists a continuous modification of Y. Notice that without loss of generality it suffices to take the supremum over any interval $(0,\cdot]$ by the self-similarity.

The following proposition generalizes Theorem 1 of Hüsler and Piterbarg [23]; it is left to the reader to check that the formulas indeed coincide when $\phi(t) = ct^\beta$ for some $c > 0$. Although no condition of the type (12) appears in [23], it is implicitly present; the process $\tilde{Z}$ in [23] is claimed to satisfy condition (E3) on page 19 of [34].

Proposition 3 *Let Y be self-similar with Hurst parameter H, and let μ satisfy* **M1–M4** *for some $\beta > H$. If (11) and (12) hold, then,*

$$P\left(\sup_{t\geq 0} Y_{\mu(t)} - t > u\right) \sim \mathcal{H}_{B_{\tilde{\iota}_\tau}} C_{H,\beta,1,\tilde{\iota}_\tau} \frac{\mu(u)^H}{u \overleftarrow{\tau}\left(\frac{\mu(u)^H}{u}\right)} \Psi\left(\inf_{t\geq 0} \frac{u(1+t)}{\mu(ut)^H}\right)$$

Proof. Note that by (11), for δ with $\delta(u) = o(u)$,

$$\lim_{u\to\infty} \sup_{s,t\in[t_u^*\pm\delta(u)/u]} \left| \frac{\mathbb{V}\text{ar}\left(\frac{Y_{\mu(us)/\mu(u)}}{(\mu(us)/\mu(u))^H} - \frac{Y_{\mu(ut)/\mu(u)}}{(\mu(ut)/\mu(u))^H}\right)}{\tau^2(|\mu(us)^\beta - \mu(ut)^\beta|/\mu(u)^\beta)} - 1 \right| = 0.$$

The self-similarity implies

$$\mathbb{V}\text{ar}\left(\frac{Y_{\mu(us)/\mu(u)}}{(\mu(us)/\mu(u))^H} - \frac{Y_{\mu(ut)/\mu(u)}}{(\mu(ut)/\mu(u))^H}\right) = \mathbb{V}\text{ar}\left(\frac{Y_{\mu(us)}}{\mu(us)^H} - \frac{Y_{\mu(ut)}}{\mu(ut)^H}\right),$$

so that (4) holds for $\nu(t) = \mu(t)^\beta$ and the τ of (11); then we have **N1** and **N2** as a consequence of the assumption that **M1**–**M3** hold. Moreover, it is trivial that $\sigma^2(t) = t^{2H}$ satisfies **S1**–**S3**. We now show that **S4** holds. By the self-similarity, for any $T > 0$,

$$\sup_{s,t\in(0,T]} \frac{\mathbb{V}\text{ar}(Y_{us} - Y_{ut})}{u^{2H}|s-t|^\gamma} = T^{2H-\gamma} \sup_{s,t\in(0,1]} \frac{\mathbb{V}\text{ar}(Y_s - Y_t)}{|s-t|^\gamma},$$

so that the first condition of **S4** is satisfied due to (12). As for the second point, by the self-similarity and the reasoning in the proof of Lemma 1, it suffices to show that for large T

$$\limsup_{u\to\infty} \frac{\mu(u)^{2H}}{u^2} \log P\left(\sup_{t\geq T/2} \frac{Y_{t^{1/\beta}}}{1+t^\omega} > \frac{u}{\mu(u)^H}\right) < -\frac{1}{2} \frac{(1+t^*)^2}{(t^*)^{H/\beta}},$$

for some ω satisfying $H/\beta < \omega < 1$. This follows from Borell's inequality (e.g., Theorem D.1 of [34]) once it has been shown that $Y_t/t^{\omega\beta} \to 0$ as $t \to \infty$. We use a reasoning as in Lemma 3 of [18] to see that this is the case. First, one can exploit the fact that $\omega\beta > H$ to establish $\lim_{k\to\infty} Y_{2^k}/2^{k\omega\beta} = 0$ by the Borel-Cantelli lemma. It then suffices to show that also $Z_k/2^{k\omega\beta} \to 0$, where $Z_k := \sup_{s\in[2^k,2^{k+1}]} |Y_s - Y_{2^k}|$. Note that Z_k has the same distribution as $2^{kH} Z_0$ by the self-similarity of Y. The almost sure convergence follows again from the Borel-Cantelli lemma: for $\alpha, \epsilon > 0$,

$$\sum_k P(Z_k/2^{k\omega\beta} > \epsilon) \leq \sum_k P(Z_0 > \epsilon 2^{k(\omega\beta - H)}) \leq \sum_k \exp\left(-\alpha\epsilon^2 2^{2k(\omega\beta-H)}\right) \mathbb{E}\exp\left(\alpha Z_0^2\right).$$

If one chooses $\alpha > 0$ appropriately small, $\mathbb{E}\exp\left(\alpha Z_0^2\right)$ is finite as a consequence of Borell's inequality (which can be applied since Y is continuous).

In conclusion, case D applies and the asymptotics are given by Theorem 1. □

Hüsler and Piterbarg [23, Section 3] also consider a class of Gaussian processes that behave somewhat like a self-similar processes. Although we do not work this out, this class is also covered by (case D of) Theorem 1; note that their condition (18) is a special case of (4), for $\nu(t) = t$.

3.3 Examples

We now work out some examples that appear in the literature. In all examples, we obtain (modest) extensions of what is known already. For Gaussian integrated processes (Section 3.3.2), we also remove some technical conditions.

3.3.1 Fractional Brownian motion

In some sense, fractional Brownian motion (fBm) is the easiest instance of a process Y that fits into the framework of Proposition 2. Indeed, the variance function σ^2 of fBm is the canonical regularly varying function, $\sigma^2(t) = t^{2H}$ for some $H \in (0,1)$.

A fractional Brownian motion B_H is self-similar in the sense of (10). Therefore, it can appear as a weak limit of a time- and space-scaled process; for examples, see [18, 39]. The increments of a fractional Brownian motion are *long-range dependent* if and only if $H > 1/2$, i.e., the covariance function of the increments on an equispaced grid is then nonsummable. For more details on long-range dependence and an extensive list of references, see Doukhan *et al.* [19].

As fBm is both self-similar and has stationary increments, the asymptotics can be obtained by applying either Proposition 2 or Proposition 3. Interestingly, this implies that it should be possible to write the formulas in the three cases of Proposition 2 as a single formula for fBm. The proof given below is based on Proposition 2, but the reader easily verifies that Proposition 3 yields the same formula; one then uses

$$\left(\frac{\beta - H}{\beta}\right)^{1/\beta} C_{H,\beta,1/\beta,H} = \frac{\beta - H}{\beta H} C_{H,\beta,1,H}.$$

Note that fBm is the only process for which both Proposition 2 and 3 can be applied: it is the only Gaussian self-similar process with stationary increments.

Corollary 1 *Let B_H be a fractional Brownian motion with Hurst parameter $H \in (0,1)$. If μ satisfies conditions* **M1–M4** *for some $\beta > H$, then*

$$P\left(\sup_{t\geq 0} B_H(\mu(t)) - t > u\right) \sim \mathcal{H}_{B_H} C_{H,\beta,1/\beta,H} \left(\frac{\beta - H}{H}\right)^{1/\beta} \frac{u^{1/H-1}}{\mu(u)^{1-H}} \Psi\left(\inf_{t\geq 0} \frac{u(1+t)}{\mu(ut)^H}\right).$$

Proof. First note that $\mu(u)^{2H}/u$ has a limit in $[0,\infty]$ as a consequence of **M2**. If $\mu(u)^{2H}/u$ tends to either zero or infinity, the formula follows readily from Proposition 2 by setting $\sigma^2(t) = t^{2H}$ (so that $\lambda = H$ in case C). In case $\mu(u)^{2H}/u \to \mathcal{G} \in (0,\infty)$, the generalized Pickands' constant can be expressed in a classical one by exploiting the self-similarity of B_H; one easily checks that $\mathcal{H}_{(\sqrt{2}/\mathcal{G})B_H} = (\sqrt{2}/\mathcal{G})^{1/H}\mathcal{H}_{B_H}$. The above formula is then found by noting that $\beta = 2H$ and

$$\frac{\mu(u)^{H+1}}{u} \sim \mathcal{G}^{1/H} \frac{u^{1/H-1}}{\mu(u)^{1-H}}.$$

□

For a standard Brownian motion ($H = 1/2$), Pickands' constant equals $\mathcal{H}_{B_{1/2}} = 1$, so that the formula reduces to

$$P\left(\sup_{t\geq 0} B_{\mu(t)} - t > u\right) \sim 2\sqrt{2\pi}\beta(2\beta-1)^{\frac{1}{2}(1/\beta-3)} \frac{u}{\sqrt{\mu(u)}} \Psi\left(\inf_{t\geq 0} \frac{u(1+t)}{\sqrt{\mu(ut)}}\right). \tag{13}$$

This probability has been extensively studied in the literature; the whole distribution of $\sup_{t\geq 0} B_{\mu(t)} - t$ is known in a number of cases. We refer to some recent contributions [7, 21, 22] for background and references.

The tail asymptotics of $\sup_{t\geq 0} B_{\mu(t)} - t$ are studied in Dębicki [10], but we believe that formula (13) does not appear elsewhere in the literature.

3.3.2 Gaussian integrated process

A Gaussian integrated process Y has the form

$$Y_t = \int_0^t Z(s)ds, \tag{14}$$

where Z is a centered stationary Gaussian process with covariance function R. We suppose that R be ultimately continuous and that $R(0) > 0$. It is easy to see that

$$\sigma^2(t) = 2\int_0^t \int_0^s R(v)dvds.$$

In the literature, μ is assumed to be of the form $\mu(t) = t/c$ for some $c > 0$, so that **M1–M4** obviously hold. For an easy comparison, we also adopt this particular choice for μ here (simple scaling arguments show that we may have assumed $c = 1$ without loss of generality). Evidently, the results of this paper allow for much more general drift functions, and the reader has no difficulties to write out the corresponding formula.

The structure of the problem ensures that **S2** and **S3** hold, and that $\sigma(t) \leq Ct^\gamma$ for some $C, \gamma > 0$ since $\sigma^2(t)/t^2 = 2\int_0^1\int_0^s R(tv)dvds$ tends to $R(0)$ as $t \downarrow 0$.

Short-range dependent case

A number of important Gaussian integrated processes have short-range dependent characteristics. Perhaps the most well-known example is an Ornstein-Uhlenbeck process, for which $R(t) = \exp(-\alpha t)$, where $\alpha > 0$ is a constant. Dębicki and Rolski [16] study the more general case where $Z = r'X$ for some k-vector r and X is the stationary solution of the stochastic differential equation

$$dX(t) = AX(t)dt + \sigma dW(t),$$

for $k \times k$ matrices A, σ (satisfying certain conditions) and a standard k-dimensional Brownian motion W. Then $R(t) = r'\Sigma e^{tA'}r$ for some covariance Σ.

By stating that a Gaussian integrated process is short-range dependent, we mean that $\mathcal{R} := \lim_{t\to\infty}\int_0^t R(s)ds$ exists as a strictly positive real number and that R is integrable, i.e., $\int_0^\infty |R(s)|ds < \infty$. We can now specialize Proposition 2 to this case.

Corollary 2 *Let Y be a Gaussian integrated process with short-range dependence. Then*

$$P\left(\sup_{t\geq 0} Y_t - ct > u\right) \sim \mathcal{H}_{\frac{c}{\sqrt{2\mathcal{R}}}Y} \sqrt{\pi} \frac{2\sqrt{\mathcal{R}}}{c^{3/2}} \sqrt{u}\Psi\left(\inf_{t\geq 0} \frac{u(1+ct)}{\sqrt{2\int_0^{ut}\int_0^s R(v)dvds}}\right). \tag{15}$$

Proof. By the existence of $\mathcal{R}$, continuity of $t \mapsto \int_0^t R(s)ds$, and bounded convergence, we have

$$\lim_{t\to\infty} \frac{\sigma^2(t/c)}{t} = \frac{2}{c}\lim_{t\to\infty}\int_0^1\int_0^{st} R(v)dvds = \frac{2\mathcal{R}}{c} < \infty,$$

so that **S1** holds with $H = 1/2$ and we are in the second case Proposition 2 with $\mathcal{G} = 2\mathcal{R}/c$. □

Notice that Corollary 2 is a modest generalization of the results of Dębicki [11]. To see this, note that (15) is asymptotically equivalent with

$$\mathcal{H}_{\frac{c}{\sqrt{2\mathcal{R}}}Y} \frac{\mathcal{R}}{c^2} \exp\left(-\frac{1}{4}\inf_{t\geq 0} \frac{u^2(1+ct)^2}{\int_0^{ut}\int_0^s R(v)dvds}\right),$$

since $t^* = H/(\beta - H) = 1$ and $\sqrt{u}\sigma(u) \sim \sqrt{2\mathcal{R}u}$. Proposition 6.1 of [11] shows that this expression is in agreement with the findings of [11].

Long-range dependent case

Consider a Gaussian integrated process as in (14), but now with a covariance function R that is regularly varying at infinity with index $2H-2$ for some $H \in (1/2, 1)$ (in addition to the regularity assumptions above). Since there is so much long term correlation that $\int_0^\infty |R(t)|dt = \infty$, the process is long-range dependent. The motivation for studying this long-range dependent case stems from the fact that it arises as a limit in heavy traffic of on-off fluid models [15].

By the direct half of Karamata's theorem (Theorem 1.5.11 of [4]), we have for $t \to \infty$,

$$\sigma^2(t) = 2\int_0^t \int_0^s R(v)dvds \sim \frac{t\int_0^t R(v)dv}{H} \sim \frac{t^2 R(t)}{H(2H-1)}.$$

Therefore, since $H > 1/2$, $\sigma^2(t)/t \to \infty$ and we are in the first case of Proposition 2.

Corollary 3 *Let Y be a Gaussian integrated process with long-range dependence. Then $P\left(\sup_{t\geq 0} Y_t - ct > u\right)$ is asymptotically equivalent to*

$$\mathcal{H}_{B_H} C_{H,1,1,H} c^{1-H} \frac{1-H}{H} [H(2H-1)]^{\frac{1}{2H}-\frac{1}{2}} \frac{u\sqrt{R(u)}}{\overleftarrow{\tau}(uR(u))} \Psi\left(\inf_{t\geq 0} \frac{u(1+ct)}{\sqrt{2\int_0^{ut}\int_0^s R(v)dvds}}\right),$$

where $\overleftarrow{\tau}$ denotes an asymptotic inverse of $t \mapsto t\sqrt{R(t)}$ (at infinity).

The case of a Gaussian integrated process with long-range dependence is also studied by Hüsler and Piterbarg [24]. The reasoning following Equation (7) of [24] shows that the formulas are the same (up to the constants; we leave it to the reader to check that these coincide).

4 A variant of Pickands' lemma

In this section, we present a generalization of a classical lemma by J. Pickands III. As we need a field version of this lemma, we let time be indexed by $\mathbb{R}^n$ for some $n \geq 1$, and we write $\mathbf{t} = (t_1, \ldots, t_n)$.

Given an even functional $\xi_\eta : \mathbb{R}^n \to \mathbb{R}$ (i.e., $\xi_\eta(\mathbf{t}) = \xi_\eta(-\mathbf{t})$ for $\mathbf{t} \in \mathbb{R}^n$), we define the centered Gaussian field η by its covariance

$$\mathbb{C}\mathrm{ov}(\eta_\mathbf{s}, \eta_\mathbf{t}) = \xi_\eta(\mathbf{s}) + \xi_\eta(\mathbf{t}) - \xi_\eta(\mathbf{s}-\mathbf{t}), \tag{16}$$

provided it is a proper covariance in the sense that the field η exists.

A central role in the lemma is played by functions g_k, ξ_η, and θ_k. These functions are in principle arbitrary, but they are assumed to satisfy certain conditions, which we now formulate. To get some feeling for these conditions, the reader may want to look in the proof of Lemma 3, for instance, to see how the functions are chosen in a particular situation.

Throughout, $\{K_u\}$ denotes a nondecreasing family of countable sets (say $K_u \subset \mathbb{Z}$), and $\{X_\mathbf{t}^{(u,k)} : \mathbf{t} \in [0,T]^n\}$, $u \in \mathbb{N}$, $k \in K_u$ denotes a collection of centered continuous separable Gaussian fields on $[0,T]^n$ for some fixed $T > 0$. We suppose that $X^{(u,k)}$ has unit variance. It is important to notice that we do not assume stationarity of the $X^{(u,k)}$.

P1 $\inf_{k\in K_u} g_k(u) \to \infty$ as $u \to \infty$,

P2 for some even functional ξ_η, $\sup_{k\in K_u} |\theta_k(u,\mathbf{s},\mathbf{t}) - 2\xi_\eta(\mathbf{s}-\mathbf{t})| \to 0$ for any $\mathbf{s}, \mathbf{t} \in [0,T]^n$,

P3 for some $\gamma_1, \ldots, \gamma_n > 0$,

$$\limsup_{u\to\infty} \sup_{k\in K_u} \sup_{\mathbf{s},\mathbf{t}\in[0,T]^n} \frac{\theta_k(u,\mathbf{s},\mathbf{t})}{\sum_{i=1}^n |s_i - t_i|^{\gamma_i}} < \infty,$$

P4 $\mathbf{t} \mapsto g_k^2(u)\mathbb{C}\mathrm{ov}\left(X_\mathbf{t}^{(u,k)}, X_\mathbf{0}^{(u,k)}\right)$ is uniformly continuous in the sense that

$$\lim_{\varepsilon\to 0} \limsup_{u\to\infty} \sup_{k\in K_u} \sup_{\substack{|\mathbf{s}-\mathbf{t}|<\varepsilon \\ \mathbf{s},\mathbf{t}\in[0,T]^n}} g_k^2(u)\mathbb{C}\mathrm{ov}\left(X_\mathbf{s}^{(u,k)} - X_\mathbf{t}^{(u,k)}, X_\mathbf{0}^{(u,k)}\right) = 0.$$

We use the following lemma in Section 6 for $n = 1$ to establish the upper bound, and in Section 7 for $n = 2$ to establish the lower bound. The main assumption of the lemma is that $\mathbb{C}\mathrm{ov}\left(X_\mathbf{s}^{(u,k)}, X_\mathbf{t}^{(u,k)}\right)$ tends uniformly to unity at rate $2\theta_k(u,\mathbf{s},\mathbf{t})/g_k^2(u)$ as $u \to \infty$.

Lemma 2 *Suppose there exist functions g_k, ξ_η, and θ_k satisfying* **P1**–**P4**. *If*

$$\lim_{u\to\infty} \sup_{k\in K_u} \sup_{\substack{\mathbf{s},\mathbf{t}\in[0,T]^n \\ \mathbf{s}\neq\mathbf{t}}} \left| g_k^2(u) \frac{\mathbb{V}\mathrm{ar}\left(X_\mathbf{s}^{(u,k)} - X_\mathbf{t}^{(u,k)}\right)}{\theta_k(u,\mathbf{s},\mathbf{t})} - 1 \right| = 0, \tag{17}$$

then for any $k \in \bigcup_u K_u$, as $u \to \infty$,

$$P\left(\sup_{\mathbf{t}\in[0,T]^n} X_\mathbf{t}^{(u,k)} > g_k(u)\right) \sim \mathcal{H}_\eta([0,T]^n)\Psi(g_k(u)), \tag{18}$$

where

$$\mathcal{H}_\eta([0,T]^n) = \mathbb{E}\exp\left(\sup_{\mathbf{t}\in[0,T]^n} \sqrt{2}\eta_\mathbf{t} - \xi_\eta(\mathbf{t})\right).$$

Moreover, we have

$$\limsup_{u\to\infty} \sup_{k\in K_u} \frac{P\left(\sup_{\mathbf{t}\in[0,T]^n} X_\mathbf{t}^{(u,k)} > g_k(u)\right)}{\Psi(g_k(u))} < \infty. \tag{19}$$

Proof. The proof is based on a standard approach in the theory of Gaussian processes; see for instance (the proof of) Lemma D.1 of Piterbarg [34].

First note that

$$\begin{aligned} & P\left(\sup_{\mathbf{t}\in[0,T]^n} X_\mathbf{t}^{(u,k)} > g_k(u)\right) \\ = \; & \frac{1}{\sqrt{2\pi}g_k(u)} \exp\left(-\frac{1}{2}g_k^2(u)\right) \int_\mathbb{R} \exp(w)\exp\left(-\frac{1}{2}\frac{w^2}{g_k^2(u)}\right) \times \\ & P\left(\sup_{\mathbf{t}\in[0,T]^n} X_\mathbf{t}^{(u,k)} > g_k(u) \,\middle|\, X_\mathbf{0}^{(u,k)} = g_k(u) - \frac{w}{g_k(u)}\right) dw. \end{aligned} \tag{20}$$

For fixed w, we set $\chi_{u,k}(\mathbf{t}) := g_k(u)[X_\mathbf{t}^{(u,k)} - g_k(u)] + w$, so that the conditional probability that appears in the integrand equals $P(\sup_{\mathbf{t}\in[0,T]^n} \chi_{u,k}(\mathbf{t}) > w | \chi_{u,k}(\mathbf{0}) = 0)$.

We first study the field $\chi_{u,k}|\chi_{u,k}(\mathbf{0})=0$ as $u\to\infty$, starting with the finite-dimensional (cylinder) distributions. These converge uniformly in $k\in K_u$ to the corresponding distributions of $\sqrt{2}\eta-\xi_\eta$. To see this, we set $v_{u,k}(\mathbf{s},\mathbf{t}):=\mathbb{V}\mathrm{ar}(X_{\mathbf{s}}^{(u,k)}-X_{\mathbf{t}}^{(u,k)})$, so that by **P1**, **P2**, and (17), uniformly in $k\in K_u$,

$$\begin{aligned}\mathbb{E}[\chi_{u,k}(\mathbf{t})|\chi_{u,k}(\mathbf{0})=0] &= -\frac{1}{2}g_k^2(u)v_{u,k}(\mathbf{0},\mathbf{t})+\frac{1}{2}wv_{u,k}(\mathbf{0},\mathbf{t})\\ &= -\frac{1}{2}\theta_k(u,\mathbf{0},\mathbf{t})(1+o(1))+o(1)\to-\xi_\eta(\mathbf{t}),\end{aligned}$$

and similarly, also uniformly in $k\in K_u$,

$$\begin{aligned}&\mathbb{V}\mathrm{ar}(\chi_{u,k}(\mathbf{s})-\chi_{u,k}(\mathbf{t})|\chi_{u,k}(\mathbf{0})=0)\\ =\ & g_k^2(u)v_{u,k}(\mathbf{s},\mathbf{t})-\frac{1}{4}g_k^2(u)\left[v_{u,k}(\mathbf{0},\mathbf{t})-v_{u,k}(\mathbf{0},\mathbf{s})\right]^2\\ =\ & \theta_k(u,\mathbf{s},\mathbf{t})(1+o(1))+o(1)\to 2\xi_\eta(\mathbf{s}-\mathbf{t}).\end{aligned}$$

Denoting the law of a field X by $\mathcal{L}(X)$, we next show that the family $\{\mathcal{L}(\chi_{u,k}|\chi_{u,k}(\mathbf{0})=0):u\in\mathbb{N},k\in K_u\}$ is uniformly tight. Since $\mathbf{t}\mapsto\mathbb{E}(\chi_{u,k}(\mathbf{t})|\chi_{u,k}(\mathbf{0})=0)$ is uniformly continuous in the sense that **P4** holds, it suffices to show that the family of centered distributions is tight. We denote the centered $\chi_{u,k}$ by $\tilde\chi_{u,k}$, i.e., $\tilde\chi_{u,k}(\mathbf{t}):=\chi_{u,k}(\mathbf{t})-\mathbb{E}[\chi_{u,k}(\mathbf{t})|\chi_{u,k}(\mathbf{0})=0]$. It is important to notice that $\mathcal{L}(\tilde\chi_{u,k}|\tilde\chi_{u,k}(\mathbf{0})=0)$ does not depend on w.

To see that $\{\mathcal{L}(\tilde\chi_{u,k}|\tilde\chi_{u,k}(\mathbf{0})=0):u\in\mathbb{N},k\in K_u\}$ is tight, observe that for u large enough, uniformly in $\mathbf{s},\mathbf{t}\in[0,T]^n$ and $k\in K_u$, we have

$$\mathbb{V}\mathrm{ar}(\tilde\chi_{u,k}(\mathbf{s})-\tilde\chi_{u,k}(\mathbf{t})|\tilde\chi_{u,k}(\mathbf{0})=0)\le g_k^2(u)v_{u,k}(\mathbf{s},\mathbf{t})\le 2\theta_k(u,\mathbf{s},\mathbf{t}).$$

By **P3**, there exist constants $\gamma_1,\dots,\gamma_n,C'>0$ such that, uniformly for $\mathbf{s},\mathbf{t}\in[0,T]^n$ and $k\in K_u$,

$$\mathbb{V}\mathrm{ar}(\tilde\chi_{u,k}(\mathbf{s})-\tilde\chi_{u,k}(\mathbf{t})|\tilde\chi_{u,k}(\mathbf{0})=0)\le C'\sum_{i=1}^n|s_i-t_i|^{\gamma_i},$$

provided u is large enough. As a corollary of Theorem 1.4.7 in Kunita [26], we have the claimed tightness.

Since the functional $x\in C([0,T]^n)\mapsto\sup_{\mathbf{t}\in[0,T]^n}x(\mathbf{t})$ is continuous in the topology of uniform convergence, the Continuous Mapping Theorem yields for $w\in\mathbb{R}$,

$$\lim_{u\to\infty}P\left(\sup_{\mathbf{t}\in[0,T]^n}\chi_{u,k}(\mathbf{t})>w\,\middle|\,\chi_{u,k}(\mathbf{0})=0\right)=P\left(\sup_{\mathbf{t}\in[0,T]^n}[\sqrt{2}\eta_{\mathbf{t}}-\xi_\eta(\mathbf{t})]>w\right).$$

Using $\int_{\mathbb{R}}e^wP(\sup_{\mathbf{t}\in[0,T]^n}[\sqrt{2}\eta_{\mathbf{t}}-\xi_\eta(\mathbf{t})]>w)dw=\mathcal{H}_\eta([0,T]^n)$ and (7), this proves (18) once it has been shown that the integral and limit can be interchanged.

The dominated convergence theorem and Borell's inequality are used to see that this can indeed be done. For arbitrary $\delta>0$ and u large enough,

$$\begin{aligned}\sup_{k\in K_u}\sup_{\mathbf{t}\in[0,T]^n}\mathbb{E}[\chi_{u,k}(\mathbf{t})|\chi_{u,k}(\mathbf{0})=0] &\le \delta|w|,\\ \sup_{k\in K_u}\sup_{\mathbf{t}\in[0,T]^n}\mathbb{V}\mathrm{ar}[\chi_{u,k}(\mathbf{t})|\chi_{u,k}(\mathbf{0})=0] &\le 2\sup_{k\in K_u}\sup_{\mathbf{t}\in[0,T]^n}\theta_k(u,\mathbf{t},\mathbf{0}),\end{aligned}$$

and the latter quantity remains bounded as $u\to\infty$ as a consequence of **P3**; let $\overline{\xi_\eta}$ denote an upper bound. Observe that for $a\in\mathbb{R}$, again by the Continuous Mapping Theorem, we have

$$\lim_{u\to\infty}\sup_{k\in K_u}P\left(\sup_{\mathbf{t}\in[0,T]^n}\tilde\chi_{u,k}(\mathbf{t})>a\,\middle|\,\chi_{u,k}(\mathbf{0})=0\right)=P\left(\sup_{\mathbf{t}\in[0,T]^n}\sqrt{2}\eta_{\mathbf{t}}>a\right).$$

Since η is continuous (as remarked below), one can select an a independent of w,u,k such that the conditions for applying Borell's inequality (e.g., Theorem D.1 of [34]) are fulfilled. Hence, for every u,k,w,

$$P\left(\sup_{\mathbf{t}\in[0,T]^n}\chi_{u,k}(\mathbf{t})>w\,\middle|\,\chi_{u,k}(\mathbf{0})=0\right)\le 2\Psi\left(\frac{w-\delta|w|-a}{3\overline{\xi_\eta}}\right).$$

When multiplied by $\exp(w)\exp(-\frac{1}{2}w^2/g_k^2(u))$, this upper bound is integrable with respect to w for large u. This not only shows that the dominated convergence theorem can be applied, it also implies (19). Indeed, using **P1**, we have

$$\lim_{u\to\infty}\sup_{k\in K_u}\frac{e^{-\frac{1}{2}g_k^2(u)}}{g_k(u)\Psi(g_k(u))}=\sqrt{2\pi}$$

by standard bounds on Ψ. □

One observation in the proof deserves to be emphasized, namely the existence and continuity of η. If θ_k satisfies (17) and converges uniformly in k to some $2\xi_\eta$ as in **P2**, the analysis of the finite-dimensional distributions shows that there automatically exists a field η with covariance (16). Moreover, η has continuous sample paths as a consequence of **P3** and **P4** (i.e., the tightness).

A number of special cases of Lemma 2 appear elsewhere in the literature. Perhaps the best known example is the case where X is a stationary process with covariance function satisfying $r(t)=1-|t|^\alpha+o(|t|^\alpha)$ for some $\alpha\in(0,2]$ as $t\downarrow 0$, see Lemma D.1 of Piterbarg [34]. This lemma is obtained by letting K_u consist of only a single element for every u, and by setting $g(u)=u$, $X_t^{(u)}=X_{u^{-2/\alpha}t}$, $\eta=B_{\alpha/2}$ and $\xi_\eta(t)=|t|^\alpha$.

A generalization of Lemma D.1 in [34] to a stationary field $\{X(\mathbf{t}):\mathbf{t}\in\mathbb{R}^n\}$ is given in Lemma 6.1 of Piterbarg [34], and we now compare this generalization to Lemma 2. We use the notation of [34]. Lemma 2 deals with the case $A=0$ and T (in the notation of [34]) equal to $[0,T]^n$ (in our notation). Again, let K_u consist of only a single element for every u, and set $g(u)=u$, $X_{\mathbf{t}}^{(u)}=X_{g_u^{-1}\mathbf{t}}$, and $\xi_\eta(\mathbf{t})=|\mathbf{t}|_{E,\alpha}$. As the ideas of the proof are the same, Lemma 2 can readily be extended to also generalize Lemma 6.1 of [34]. However, we do not need this to derive the results of the present paper.

Theorem 2.1 of Dębicki [11] can also be considered to be a special case of Lemma 2. There, again, K_u consists of a single element, and $X_{(t_1,\dots,t_n)}^{(u)}=\frac{1}{\sqrt{n}}\sum_{i=1}^nX_i^{(u)}(t_i)$ for independent processes $X_i^{(u)}$ satisfying a condition of the type (17), but where θ does not depend on u.

Lemma 2 has some interesting consequences for the properties of Pickands' constant. For instance, Pickands' constant is readily seen to be subadditive, i.e., for $T_1,T_2>0$ and $n=1$,

$$\mathcal{H}_\eta([0,T_1+T_2])\le\mathcal{H}_\eta([0,T_1])+\mathcal{H}_\eta([0,T_2]),$$

with appropriate generalizations to the multidimensional case. This property guarantees that the limit in (6) exists.

The value of Pickands' constant is only known in two cases: $\mathcal{H}_{B_{1/2}}=1$ (Brownian motion) and $\mathcal{H}_{B_1}=1/\sqrt{\pi}$ ('degenerate' case). Further properties of Pickands' constants are explored both theoretically and numerically by Shao [38], Dębicki [8], and Dębicki *et al.* [14].

5 Four cases

We now specialize Lemma 2 according to the four types of correlation structures introduced in Section 2. Throughout this section, we suppose that **S1** and **M1** hold.

Let $T>0$ be fixed, and write $I_k^T(u)$ for the intervals $[t_u^*+kT\Delta(u)/u,t_u^*+(k+1)T\Delta(u)/u]$, where Δ is some function that depends on the correlation structure, and $\Delta(u)=o(\delta(u))$.

5.1 Case A

We say that case A applies if **A1**, **A2**, **T1**, **T2**, **N1**, and **N2** hold and Δ is given by

$$\Delta(u) := \frac{1}{\dot\nu(ut^*)}\overleftarrow{\tau}\left(\frac{\sqrt{2}\tau(\nu(u))}{\sqrt{\mathcal{D}}}\frac{\sigma(\mu(ut^*))}{u(1+t^*)}\right), \tag{21}$$

where $\overleftarrow{\tau}$ denotes an asymptotic inverse of τ at infinity (this exists when **T1** holds, see Theorem 1.5.12 of [4]). Note that the argument of $\overleftarrow{\tau}$ tends to infinity as a consequence of **A2**, and that therefore $\nu(u)\Delta(u)/u \to \infty$. It is easy to check that Δ is regularly varying with index $(H/\beta - 1)/\iota_\tau + 1 < 1$.

The next lemma shows that this particular choice of Δ 'balances' the correlation structure on the intervals $I_k^T(u)$ (note that the interval $I_k^T(u)$ depends on Δ).

Lemma 3 *Let* **S1** *and* **M1** *hold and suppose that case A applies. Let δ be such that $\delta(u) = o(u)$ and $\Delta(u) = o(\delta(u))$. For any u and $-\frac{\delta(u)}{T\Delta(u)} \le k \le \frac{\delta(u)}{T\Delta(u)}$, pick some $t_k^\circ(u) \in I_k^T(u)$. Then we have for $u \to \infty$,*

$$P\left(\sup_{t\in I_k^T(u)} \frac{Y_{\mu(ut)}}{\sigma(\mu(ut))} > \frac{u(1+t_k^\circ(u))}{\sigma(\mu(ut_k^\circ(u)))}\right) \sim \mathcal{H}_{B_{\iota_\tau}}(T)\Psi\left(\frac{u(1+t_k^\circ(u))}{\sigma(\mu(ut_k^\circ(u)))}\right),$$

where $\mathcal{H}_{B_{\iota_\tau}}(T)$ is defined as in (6). Moreover,

$$\limsup_{u\to\infty} \sup_{-\frac{\delta(u)}{T\Delta(u)}\le k\le\frac{\delta(u)}{T\Delta(u)}} \frac{P\left(\sup_{t\in I_k^T(u)} \frac{Y_{\mu(ut)}}{\sigma(\mu(ut))} > \frac{u(1+t_k^\circ(u))}{\sigma(\mu(ut_k^\circ(u)))}\right)}{\Psi\left(\frac{u(1+t_k^\circ(u))}{\sigma(\mu(ut_k^\circ(u)))}\right)} < \infty. \tag{22}$$

Proof. The main argument in the proof is, of course, Lemma 2. Set

$$\kappa_k(u) := \sqrt{\frac{\mathcal{D}}{2}}\frac{u(1+t_k^\circ(u))}{\sigma(\mu(ut_k^\circ(u)))}\frac{\tau(\dot\nu(ut^*)\Delta(u))}{\tau(\nu(u))} \tag{23}$$

and note that by the UCT and (21),

$$\sup_{-\frac{\delta(u)}{T\Delta(u)}\le k\le\frac{\delta(u)}{T\Delta(u)}} \sup_{s,t\in I_k^T(u)} \left|\kappa_k^2(u) - 1\right| \to 0.$$

Equation (3) implies that {**A1**}

$$\sup_{\substack{s,t\in[t_u^*\pm\delta(u)/u]\\ |s-t|\le T\Delta(u)/u}} \left|\frac{2\kappa_k^2(u)\tau^2(\nu(u))}{\mathcal{D}\tau^2(\dot\nu(ut^*)\Delta(u))}\frac{\mathbb{V}\mathrm{ar}\left(\frac{Y_{\mu(us)}}{\sigma(\mu(us))} - \frac{Y_{\mu(ut)}}{\sigma(\mu(ut))}\right)}{2\tau^2(|\nu(us)-\nu(ut)|)/\tau^2(\dot\nu(ut^*)\Delta(u))} - 1\right| \to 0. \tag{24}$$

The preceding display suggests certain choices for the functions g_k and θ_k of Lemma 2, cf. (17); we now show that **P1**–**P4** are indeed satisfied.

As for **P1**, one readily checks that

$$g_k(u) := \sqrt{\frac{2}{\mathcal{D}}}\frac{\kappa_k(u)\tau(\nu(u))}{\tau(\dot\nu(ut^*)\Delta(u))} = \frac{u(1+t_k^\circ(u))}{\sigma(\mu(ut_k^\circ(u)))}$$

tends to infinity uniformly in k. We set for $s,t\in[0,T]$ and $-\frac{\delta(u)}{T\Delta(u)} \le k \le \frac{\delta(u)}{T\Delta(u)}$

$$\theta_k(u,s,t) := 2\frac{\tau^2(|\nu(ut_u^* + (kT+s)\Delta(u)) - \nu(ut_u^* + (kT+t)\Delta(u))|)}{\tau^2(\dot\nu(ut^*)\Delta(u))}. \tag{25}$$

To check that $\theta_k(u,s,t)$ converges uniformly in k as $u\to\infty$, we note that by the Mean Value Theorem {**N2**} there exists some $t_k^\wedge(u,s,t)\in[0,T]$ such that

$$\nu(ut_u^* + (kT+s)\Delta(u)) - \nu(ut_u^* + (kT+t)\Delta(u)) = \Delta(u)\dot\nu(ut_u^* + [kT + t_k^\wedge(u,s,t)])(s-t).$$

Now note that we have for $s,t\in[0,T]$,

$$\begin{aligned}
&\sup_k \left|\theta_k(u,s,t) - 2|s-t|^{2\iota_\tau}\right| \\
\le\ & \sup_k \left|\theta_k(u,s,t) - 2\left(\frac{\dot\nu(ut_u^* + [kT + t_k^\wedge(u,s,t)])}{\dot\nu(ut^*)}\right)^{2\iota_\tau}|s-t|^{2\iota_\tau}\right| \\
+\ & 2\sup_k \left|\left(\frac{\dot\nu(ut_u^* + [kT + t_k^\wedge(u,s,t)])}{\dot\nu(ut^*)}\right)^{2\iota_\tau} - 1\right| |s-t|^{2\iota_\tau} \\
=:\ & I(u) + II(u).
\end{aligned}$$

As a consequence of the UCT {**N1**, **N2**}, we have

$$\lim_{u\to\infty} \sup_{s,t\in[0,T]} \sup_{-\frac{\delta(u)}{T\Delta(u)}\le k\le\frac{\delta(u)}{T\Delta(u)}} \frac{\dot\nu(ut_u^* + [kT + t_k^\wedge(u,s,t)])}{\dot\nu(ut^*)}(s-t) = T. \tag{26}$$

Since $\dot\nu(u)\Delta(u)$ tends to infinity {**A2**}, this shows that $I(u)$ is majorized by

$$\sup_{t\in[0,2T]} \left|\frac{\tau^2(\dot\nu(ut^*)\Delta(u)t)}{\tau^2(\dot\nu(ut^*)\Delta(u))} - t^{2\iota_\tau}\right| \to 0.$$

$II(u)$ also tends to zero by the UCT. Hence, **P2** holds with $\xi_\eta(t) = |t|^{2\iota_\tau}$, so that η is a fractional Brownian motion with Hurst parameter ι_τ.

A similar reasoning is used to check **P3**. Notice that $\tau^2(t)t^{-2\gamma'}$ is bounded on intervals of the form $(0,\cdot]$ {**T2**}, and that we may suppose that $\gamma' < \iota_\tau$ without loss of generality. Again using (26) and the UCT, we observe that for large u,

$$\begin{aligned}
&\sup_k \sup_{\substack{s,t\in(0,T]\\ s>t}} \theta_k(u,s,t)(s-t)^{-2\gamma'} \\
=\ & \sup_k \sup_{\substack{s,t\in(0,T]\\ s>t}} 2\frac{\tau^2\left(\Delta(u)\dot\nu(ut_u^* + [kT + t_k^\wedge(u,s,t)])(s-t)\right)}{\tau^2(\dot\nu(ut^*)\Delta(u))}(s-t)^{-2\gamma'} \\
\le\ & 2 \sup_{t\in\left[0,\left(\frac{3}{2}\right)^{1/(2\iota_\tau-2\gamma')}T\right]} \frac{\tau^2(\dot\nu(ut^*)\Delta(u)t)}{\tau^2(\dot\nu(ut^*)\Delta(u))}t^{-2\gamma'} \\
\le\ & 4T^{2\iota_\tau-2\gamma'},
\end{aligned}$$

which is clearly finite (the factor 4 turns up again in the proof of Lemma 9 below).

It remains to check **P4**. For this, observe that it suffices to show that

$$\lim_{\varepsilon\to0}\limsup_{u\to\infty} \sup_{s,s',t\in[t_u^*\pm\delta(u)/u]} \sup_{\substack{|s-t|<\varepsilon T\Delta(u)/u\\ |s'-t|<\varepsilon T\Delta(u)/u}} g_k^2(u)\mathbb{C}\mathrm{ov}\left(\frac{Y_{\mu(us)}}{\sigma(\mu(us))} - \frac{Y_{\mu(us')}}{\sigma(\mu(us'))}, \frac{Y_{\mu(ut)}}{\sigma(\mu(ut))}\right) = 0,$$

and hence that

$$\lim_{\varepsilon\to0}\limsup_{u\to\infty} \sup_{s,t\in[t_u^*\pm\delta(u)/u]} \sup_{|s-t|<\varepsilon\Delta(u)/u} g_k^2(u)\mathbb{V}\mathrm{ar}\left(\frac{Y_{\mu(us)}}{\sigma(\mu(us))} - \frac{Y_{\mu(ut)}}{\sigma(\mu(ut))}\right) = 0. \tag{27}$$

Observe that for large u, by (24) and the Mean Value Theorem, uniformly for $s,t \in [t_u^* \pm \delta(u)/u]$,

$$\begin{aligned}\sup_{|s-t|<\varepsilon\Delta(u)/u} g_k^2(u)\mathbb{V}\text{ar}\left(\frac{Y_{\mu(us)}}{\sigma(\mu(us))} - \frac{Y_{\mu(ut)}}{\sigma(\mu(ut))}\right) &\leq 4 \sup_{|s-t|<\varepsilon\Delta(u)/u} \frac{\tau^2(|\nu(us)-\nu(ut)|)}{\tau^2(\dot\nu(ut^*)\Delta(u))} \\ &\leq 4 \sup_{t<2\varepsilon\Delta(u)/u} \frac{\tau^2(u\dot\nu(ut^*)t)}{\tau^2(\dot\nu(ut^*)\Delta(u))} \\ &\leq 8(2\varepsilon)^{2\iota_\tau} \to 0,\end{aligned}$$

as $\varepsilon \to 0$.

Having checked that Lemma 2 can be applied, we use the definition of $\Delta(u)$ to see that

$$\begin{aligned}P\left(\sup_{t\in I_k^T(u)} \frac{Y_{\mu(ut)}}{\sigma(\mu(ut))} > \frac{u(1+t_k^\circ(u))}{\sigma(\mu(ut_k^\circ(u)))}\right) &= P\left(\sup_{t\in I_k^T(u)} \frac{Y_{\mu(ut)}}{\sigma(\mu(ut))} > \sqrt{\frac{2}{\mathcal{D}}}\frac{\kappa_k(u)\tau(\nu(u))}{\tau(\dot\nu(ut^*)\Delta(u))}\right) \\ &\sim \mathcal{H}_{B_{\iota_\tau}}(T)\Psi\left(\sqrt{\frac{2}{\mathcal{D}}}\frac{\kappa_k(u)\tau(\nu(u))}{\tau(\dot\nu(ut^*)\Delta(u))}\right) \\ &= \mathcal{H}_{B_{\iota_\tau}}(T)\Psi\left(\frac{u(1+t_k^\circ(u))}{\sigma(\mu(ut_k^\circ(u)))}\right),\end{aligned}$$

as claimed. □

5.2 Case B

Case B is different from the other cases in the sense that no (asymptotic) inverse is involved in the definition of Δ. As a consequence, a non-classical Pickands' constant appears in the asymptotics.

We say that case B applies when **B1**, **B2**, **T1**, **T2**, **N1**, and **N2** hold and Δ is given by

$$\Delta(u) := \frac{1}{\dot\nu(ut^*)}. \tag{28}$$

Moreover, we set

$$\mathcal{F} := \frac{\mathcal{D}(1+t^*)^2}{2\mathcal{G}^2(t^*)^{2H/\beta}}.$$

Under these assumptions, $\lim_{u\to\infty} \nu(u)\Delta(u)/u$ exists in $(0,\infty)$.

Lemma 4 *Let* **S1** *and* **M1** *hold and suppose that case B applies. Let δ be such that $\delta(u) = o(u)$ and $\Delta(u) = o(\delta(u))$. For any u and $-\frac{\delta(u)}{T\Delta(u)} \leq k \leq \frac{\delta(u)}{T\Delta(u)}$, pick some $t_k^\circ(u) \in I_k^T(u)$. For T large enough, we have for $u \to \infty$,*

$$P\left(\sup_{t\in I_k^T(u)} \frac{Y_{\mu(ut)}}{\sigma(\mu(ut))} > \frac{u(1+t_k^\circ(u))}{\sigma(\mu(ut_k^\circ(u)))}\right) \sim \mathcal{H}_{\mathcal{F}\tau^2}(T)\Psi\left(\frac{u(1+t_k^\circ(u))}{\sigma(\mu(ut_k^\circ(u)))}\right),$$

where $\mathcal{H}_{\mathcal{F}\tau^2}(T)$ is defined as in (6). Moreover, (22) holds.

Proof. Define

$$\kappa_k(u) := \sqrt{\frac{\mathcal{D}}{2\mathcal{F}}}\frac{u(1+t_k^\circ(u))}{\tau(\nu(u))\sigma(\mu(ut_k^\circ(u)))}$$

which converges uniformly in k to unity as a consequence of the fact that by **B2**,

$$\frac{2\mathcal{F}\tau^2(\nu(u))}{\mathcal{D}} = \frac{(1+t^*)^2}{\mathcal{G}^2(t^*)^{2H/\beta}}\tau^2(\nu(u)) \sim \frac{u^2(1+t^*)^2}{\sigma^2(\mu(ut^*))}.$$

Therefore, as in Lemma 3, we have by (3),

$$\sup_{\substack{-\frac{\delta(u)}{T\Delta(u)}\leq k\leq\frac{\delta(u)}{T\Delta(u)} \\ k\in\mathbb{Z}}} \sup_{s,t\in I_k^T(u)} \left|\frac{2\mathcal{F}\kappa_k^2(u)\tau^2(\nu(u))}{\mathcal{D}} \frac{\mathbb{V}\text{ar}\left(\frac{Y_{\mu(us)}}{\sigma(\mu(us))} - \frac{Y_{\mu(ut)}}{\sigma(\mu(ut))}\right)}{2\mathcal{F}\tau^2(|\nu(us)-\nu(ut)|)} - 1\right| \to 0.$$

Again, this should be compared to (17). Set $g_k(u) := \sqrt{2\mathcal{F}/\mathcal{D}}\kappa_k(u)\tau(\nu(u))$, and

$$\theta_k(u,s,t) := 2\mathcal{F}\tau^2(|\nu(ut_u^* + (kT+s)\Delta(u)) - \nu(ut_u^* + (kT+t)\Delta(u))|). \tag{29}$$

Obviously, we have **P1**. We now check that **P2** holds with $\xi_\eta(t) = \mathcal{F}\tau^2(|t|)$. Let $s,t \in [0,T]$, and observe that by the Mean Value Theorem there exist $t_k^\wedge(u,s,t) \in [0,T]$ such that for every $\epsilon > 0$,

$$\begin{aligned}&\sup_k |\theta_k(u,s,t) - 2\mathcal{F}\tau^2(s-t)| \\ &= 2\sup_k |\mathcal{F}\tau^2(\Delta(u)\dot\nu(ut_u^* + [kT + t_k^\wedge(u,s,t)])|s-t|) - \mathcal{F}\tau^2(|s-t|)| \\ &\leq 2\mathcal{F} \sup_{s\in[1-\epsilon,1+\epsilon]} \sup_{t\in[0,T]} |\tau^2(st) - \tau^2(t)| \\ &\leq 2\mathcal{F} \sup_{\substack{s,t\in[0,2T] \\ |s-t|\leq\epsilon T}} |\tau^2(s) - \tau^2(t)|,\end{aligned}$$

where we used the definition of Δ and the UCT. By continuity of τ {**T1**}, this upper bound (which is a modulus of continuity) tends to zero as $\epsilon \to 0$. As for **P3**, the same arguments show that for large T (by the UCT) {**T1**, **T2**}

$$\sup_k \sup_{s,t\in[0,T]} \frac{\theta_k(u,s,t)}{|s-t|^{2\gamma'}} \leq 2\mathcal{F} \sup_{t\in\left[0,\left(\frac{3}{2}\right)^{1/(2\iota_\tau - 2\gamma')}T\right]} \frac{\tau^2(t)}{t^{2\gamma'}} \leq 4\mathcal{F}T^{2(\iota_\tau-\gamma')}.$$

It remains to verify **P4**. As in the proof of Lemma 3, it suffices to show that (27) holds. By again applying the UCT, one can check that for $s,t \in [t_u^* \pm \delta(u)/u]$,

$$\sup_k \sup_{|s-t|<\varepsilon\Delta(u)/u} g_k^2(u)\mathbb{V}\text{ar}\left(\frac{Y_{\mu(us)}}{\sigma(\mu(us))} - \frac{Y_{\mu(ut)}}{\sigma(\mu(ut))}\right) \leq 2\mathcal{F}\sup_{t\in[0,2\varepsilon]}\tau^2(t),$$

showing **P4** since τ^2 is continuous at zero.

In conclusion, Lemma 2 can be applied and thus

$$\begin{aligned}P\left(\sup_{t\in I_k^T(u)} \frac{Y_{\mu(ut)}}{\sigma(\mu(ut))} > \frac{u(1+t_k^\circ(u))}{\sigma(\mu(ut_k^\circ(u)))}\right) &= P\left(\sup_{t\in I_k^T(u)} \frac{Y_{\mu(ut)}}{\sigma(\mu(ut))} > \sqrt{\frac{2\mathcal{F}}{\mathcal{D}}}\kappa_k(u)\tau(\nu(u))\right) \\ &\sim \mathcal{H}_{\mathcal{F}\tau^2}(T)\Psi\left(\sqrt{\frac{2\mathcal{F}}{\mathcal{D}}}\kappa_k(u)\tau(\nu(u))\right) \\ &= \mathcal{H}_{\mathcal{F}\tau^2}(T)\Psi\left(\frac{u(1+t_k^\circ(u))}{\sigma(\mu(ut_k^\circ(u)))}\right),\end{aligned}$$

as claimed. □

5.3 Case C

We say that case C applies when **C1**–**C3**, **T1**, **N1**, and **N2** hold and Δ is given by

$$\Delta(u) := \frac{1}{\dot\nu(ut^*)} \overleftarrow{\tau}\left(\frac{\sqrt{2}\tau(\nu(u))}{\sqrt{\mathcal{D}}}\frac{\sigma(\mu(ut^*))}{u(1+t^*)}\right), \tag{30}$$

where $\overleftarrow{\tau}$ denotes an asymptotic inverse of τ at zero (which exists due to **T1**, see Theorem 1.5.12 of [4]). Here, the argument of $\overleftarrow{\tau}$ tends to zero as a consequence of **C2**, and therefore $\nu(u)\Delta(u)/u \to 0$. Note that we do not impose **T2**, since i is automatically satisfied once **C3** holds.

The following lemma is the analog of Lemma 3 and Lemma 4 for case C.

Lemma 5 *Let* **S1** *and* **M1** *hold and suppose that case C applies. Let δ be such that $\delta(u) = o(u)$ and $\Delta(u) = o(\delta(u))$. For any u and $-\frac{\delta(u)}{T\Delta(u)} \le k \le \frac{\delta(u)}{T\Delta(u)}$, pick some $t_k^\circ(u) \in I_k^T(u)$. Then we have for $u \to \infty$,*

$$P\left(\sup_{t\in I_k^T(u)} \frac{Y_{\mu(ut)}}{\sigma(\mu(ut))} > \frac{u(1+t_k^\circ(u))}{\sigma(\mu(ut_k^\circ(u)))}\right) \sim \mathcal{H}_{B_{\tilde\iota_\tau}}(T)\Psi\left(\frac{u(1+t_k^\circ(u))}{\sigma(\mu(ut_k^\circ(u)))}\right),$$

where $\mathcal{H}_{B_{\tilde\iota_\tau}}(T)$ is defined as in (6). Moreover, (22) holds.

Proof. The proof is exactly the same as the proof of Lemma 3, except that now ι_τ is replaced by $\tilde\iota_\tau$. □

5.4 Case D

We say that case D applies when **D1**, **D2**, **N1**, **N2** hold and Δ is given by

$$\Delta(u) := \frac{u}{\iota_\nu (t^*)^{\iota_\nu - 1}} \overleftarrow{\tau}\left(\frac{\sqrt{2}\sigma(\mu(ut^*))}{u(1+t^*)}\right). \tag{31}$$

The local behavior is described by the following lemma.

Lemma 6 *Let* **S1** *and* **M1** *hold and suppose that case D applies. Let δ be such that $\delta(u) = o(u)$ and $\Delta(u) = o(\delta(u))$. For any u and $-\frac{\delta(u)}{T\Delta(u)} \le k \le \frac{\delta(u)}{T\Delta(u)}$, pick some $t_k^\circ(u) \in I_k^T(u)$. Then we have for $u \to \infty$,*

$$P\left(\sup_{t\in I_k^T(u)} \frac{Y_{\mu(ut)}}{\sigma(\mu(ut))} > \frac{u(1+t_k^\circ(u))}{\sigma(\mu(ut_k^\circ(u)))}\right) \sim \mathcal{H}_{B_{\tilde\iota_\tau}}(T)\Psi\left(\frac{u(1+t_k^\circ(u))}{\sigma(\mu(ut_k^\circ(u)))}\right),$$

where $\mathcal{H}_{B_{\tilde\iota_\tau}}(T)$ is defined as in (6). Moreover, (22) holds.

Proof. The arguments are similar to those in the proof of Lemma 3. Therefore, we only show how the functions in Lemma 2 should be chosen in order to match (4) with (17).

Define for $-\frac{\delta(u)}{T\Delta(u)} \le k \le \frac{\delta(u)}{T\Delta(u)}$

$$\kappa_k(u) := \frac{u\tau(\iota_\nu(t^*)^{\iota_\nu-1}\Delta(u)/u)(1+t_k^\circ(u))}{\sqrt{2}\sigma(\mu(ut_k^\circ))}, \quad g_k(u) := \frac{\sqrt{2}\kappa_k(u)}{\tau(\iota_\nu(t^*)^{\iota_\nu-1}\Delta(u)/u)},$$

and

$$\theta_k(u,s,t) := 2\frac{\tau^2(\nu(|ut_u^* + (kT+s)\Delta(u)) - \nu(ut_u^* + (kT+t)\Delta(u))|/\nu(u))}{\tau^2(\iota_\nu(t^*)^{\iota_\nu-1}\Delta(u)/u)}.$$

It follows from Lemma 2 with $\eta = B_{\tilde\iota_\tau}$ that

$$\begin{aligned} P\left(\sup_{t\in I_k^T(u)} \frac{Y_{\mu(ut)}}{\sigma(\mu(ut))} > \frac{u(1+t_k^\circ(u))}{\sigma(\mu(ut_k^\circ(u)))}\right) &= P\left(\sup_{t\in I_k^T(u)} \frac{Y_{\mu(ut)}}{\sigma(\mu(ut))} > \frac{\sqrt{2}\kappa_k(u)}{\tau(\iota_\nu(t^*)^{\iota_\nu-1}\Delta(u)/u)}\right) \\ &\sim \mathcal{H}_{B_{\tilde\iota_\tau}}(T)\Psi\left(\frac{\sqrt{2}\kappa_k(u)}{\tau(\iota_\nu(t^*)^{\iota_\nu-1}\Delta(u)/u)}\right) \\ &= \mathcal{H}_{B_{\tilde\iota_\tau}}(T)\Psi\left(\frac{u(1+t_k^\circ(u))}{\sigma(\mu(ut_k^\circ(u)))}\right), \end{aligned}$$

as claimed. □

6 Upper bounds

In this section, we prove the upper bound part of Theorem 1 in each of the four cases. Since the proof is almost exactly the same for each of the regimes, we only give it once by using the following notation in both the present and the next section.

We denote the Pickands' constants $\mathcal{H}_{B_{\iota_\tau}}(T)$, $\mathcal{H}_{\mathcal{DM}\tau^2}(T)$, and $\mathcal{H}_{B_{\tilde\iota_\tau}}(T)$ by $\mathcal{H}(T)$. The abbreviation $\mathcal{H} := \lim_{T\to\infty}\mathcal{H}(T)/T$ is used for the corresponding limits. The definition of Δ also depends on the regime; it is defined in (21), (28), (30), and (31) for the cases A, B, C, and D, respectively. Notice that the dependence on Δ is suppressed in the notation $I_k^T(u) = [t_u^* + kT\Delta(u)/u, t_u^* + (k+1)T\Delta(u)/u]$. It is convenient to define $\underline{t}_k^T(u)$ and $\overline{t}_k^T(u)$ as the left and right end of $I_k^T(u)$ respectively. In the proofs of the upper and lower bounds, we write

$$\mathcal{C} := \frac{1}{2}\frac{d^2}{dt^2}\frac{(1+t)^2}{t^{2H/\beta}}\bigg|_{t=t^*} = (t^*)^{-2H/\beta-1}. \tag{32}$$

We start with an auxiliary lemma, which shows that it suffices to focus on *local* behavior near t_u^*. This observation is important since the lemmas of the previous section only yield local uniformity (note that $I_k^T(u) \subset [t_u^* \pm \delta(u)/u]$ and $\delta(u) = o(u)$).

Lemma 7 *Suppose that* **S1**–**S4**, *and* **M1**–**M4** *hold for some $\beta > H$. Let δ be such that $\delta(u) = o(u)$ and $\sigma(\mu(u)) = o(\delta(u))$. Then we have*

$$P\left(\sup_{t\notin[t_u^*\pm\delta(u)/u]} \frac{Y_{\mu(ut)}}{1+t} > u\right) = o\left(\frac{\sigma(\mu(u))}{\Delta(u)}\Psi\left(\inf_{t\ge 0}\frac{u(1+t)}{\sigma(\mu(ut))}\right)\right). \tag{33}$$

Proof. The proof consists of three parts: we show that the intervals $[0,\omega]$, $[\omega,T]\backslash[t_u^*\pm\delta(u)/u]$ and $[T,\infty]$ play no role in the asymptotics, where $\omega, T > 0$ are chosen appropriately.

We start with the interval $[T,\infty)$. If T is chosen as in **S4**, this interval is asymptotically negligible by assumption.

As for the remaining intervals, by **S4** we can find some $\epsilon, C \in (0,\infty)$, $\gamma \in (0,2]$ such that for each $s,t \in [0,(1+\epsilon)T^{1/\beta}]$

$$\mathbb{V}\mathrm{ar}\,(Y_{us} - Y_{ut}) \le C\sigma^2(u)|s-t|^\gamma, \tag{34}$$

where u is large. Starting with $[0,\omega]$, we select ω so that for large u,

$$\sup_{t\in[0,\omega]} \frac{\sigma(\mu(ut))}{1+t} \le \frac{1}{2}\frac{\sigma(\mu(ut_u^*))}{1+t_u^*}. \tag{35}$$

The main argument is Borell's inequality, but we first have to make sure that it can be applied. For $a > 0$, there exists constants c_γ, C independent of u and a such that for large u, $\{\mathbf{M2}\}$

$$\begin{aligned} P\left(\sup_{t\in[0,\omega]} \frac{Y_{\mu(ut)}}{\sigma(\mu(u))(1+t)} > a\right) &\leq P\left(\sup_{t\in\left[0,\left(\frac{\mu(u\omega)}{\mu(u)}\right)^\beta\right]} \frac{Y_{\mu(u)t^{1/\beta}}}{\sigma(\mu(u))} > a\right) \\ &\leq P\left(\sup_{t\in[0,2\omega]} \frac{Y_{\mu(u)t^{1/\beta}}}{\sigma(\mu(u))} > a\right) \\ &\leq 4\exp\left(-\frac{c_\gamma a^2}{C}\right), \end{aligned}$$

where the last inequality follows from (34) and Fernique's lemma [28, p. 219] as $\gamma \in (0, 2]$. By choosing a sufficiently large, we have by Borell's inequality (e.g., Theorem D.1 of [34])

$$P\left(\sup_{t\in[0,\omega]} \frac{Y_{\mu(ut)}}{1+t} > u\right) \leq 2\Psi\left(\frac{1 - a\sigma(\mu(u))/u}{\sup_{t\in[0,\omega]} \frac{\sigma(\mu(ut))}{u(1+t)}}\right).$$

Since (35) holds, there exist constants $\mathcal{K}_1, \mathcal{K}_2 < \infty$ such that

$$P\left(\sup_{t\in[0,\omega]} \frac{Y_{\mu(ut)}}{1+t} > u\right) \leq \mathcal{K}_1 \exp\left(-2\frac{u^2(1+t_u^*)^2}{\sigma^2(\mu(ut_u^*))} + \mathcal{K}_2 \frac{u(1+t_u^*)}{\sigma(\mu(ut_u^*))}\right).$$

This shows that the interval $[0, \omega]$ is asymptotically negligible in the sense of (33).

We next consider the contribution of the set $[\omega, T]\backslash[t_u^* \pm \delta(u)/u]$ to the asymptotics. Define

$$\overline{\sigma}(u) = \sup_{t\in[\omega,T]\backslash[t_u^*\pm\delta(u)/u]} \frac{\sigma(\mu(ut))}{1+t} = \max\left(\frac{\sigma(\mu(ut_u^* - \delta(u)))}{1 + t_u^* - \delta(u)/u}, \frac{\sigma(\mu(ut_u^* + \delta(u)))}{1 + t_u^* + \delta(u)/u}\right),$$

where the last equality holds for large u. Now observe that by the UCT $\{\mathbf{M1}\}$, for large u,

$$\begin{aligned} P\left(\sup_{t\in[\omega,T]\backslash[t_u^*\pm\delta(u)/u]} \frac{Y_{\mu(ut)}}{1+t} > u\right) &\leq P\left(\sup_{t\in[\omega,T]\backslash[t_u^*\pm\delta(u)/u]} \frac{Y_{\mu(ut)}}{\sigma(\mu(ut))} > \frac{u}{\overline{\sigma}(u)}\right) \\ &\leq P\left(\sup_{t\in\left[\omega^{1/\beta}/2, 2T^{1/\beta}\right]} \frac{Y_{\mu(u)t}}{\sigma(\mu(u)t)} > \frac{u}{\overline{\sigma}(u)}\right). \end{aligned}$$

In order to bound this quantity further, we use (34) and the inequality $2ab \leq a^2 + b^2$: for $s, t \in \left[\omega^{1/\beta}/2, 2T^{1/\beta}\right]$, $\{\mathbf{M2}\}$

$$\begin{aligned} \mathbb{V}\mathrm{ar}\left(\frac{Y_{\mu(u)s}}{\sigma(\mu(u)s)} - \frac{Y_{\mu(u)t}}{\sigma(\mu(u)t)}\right) &\leq \frac{\mathbb{V}\mathrm{ar}\left(Y_{\mu(u)s} - Y_{\mu(u)t}\right)}{\sigma(\mu(u)s)\sigma(\mu(u)t)} \\ &\leq \sup_{v\in[\omega^{1/\beta}/2, 2T^{1/\beta}]} \frac{\mathbb{V}\mathrm{ar}\left(Y_{\mu(u)s} - Y_{\mu(u)t}\right)}{\sigma^2(\mu(u)v)} \\ &\leq \frac{2^{1+2H}\omega^{-2H/\beta}}{\sigma^2(\mu(u))} \mathbb{V}\mathrm{ar}\left(Y_{\mu(u)s} - Y_{\mu(u)t}\right) \\ &\leq \mathcal{K}'|s-t|^\gamma, \end{aligned}$$

where $\mathcal{K}' < \infty$ is some constant (depending on ω and T). Hence, by Theorem D.4 of Piterbarg [34] there exists a constant $\mathcal{K}''$ depending only on $\mathcal{K}'$ and γ such that

$$P\left(\sup_{t\in[\omega,T]\backslash[t_u^*\pm\delta(u)/u]} \frac{Y_{\mu(ut)}}{1+t} > u\right) \leq T\mathcal{K}'' \left(\frac{u}{\overline{\sigma}(u)}\right)^{2/\gamma} \Psi\left(\frac{u}{\overline{\sigma}(u)}\right).$$

Consider the expression

$$u^2\left(\frac{(1 + t_u^* + \delta(u)/u)^2}{\sigma^2(\mu(ut_u^* + \delta(u)))} - \frac{(1+t_u^*)^2}{\sigma^2(\mu(ut_u^*))}\right) \Big/ \mathcal{C}\left[\frac{\delta(u)}{\sigma(\mu(u))}\right]^2, \tag{36}$$

where $\mathcal{C}$ is given by (32). By Taylor's Mean Value Theorem $\{\mathbf{S3}, \mathbf{M4}\}$, there exists some $t_\# = t_\#(u) \in [t_u^*, t_u^* + \delta(u)/u]$ such that this expression equals

$$\frac{1}{2}\delta^2(u) \frac{d^2}{dt^2} \frac{(1+t)^2}{\sigma^2(\mu(ut))}\Big|_{t=t_\#} \Big/ \mathcal{C}\left[\frac{\delta(u)}{\sigma(\mu(u))}\right]^2.$$

Recall that $\sigma^2(\mu(\cdot))$ is regularly varying with index $2H/\beta > 0$, and that (under the present conditions) both its first and second derivative are regularly varying with respective indices $2H/\beta - 1$ and $2H/\beta - 2$. The UCT now yields

$$\lim_{u\to\infty} \frac{\sigma^2(\mu(u))}{2} \frac{d^2}{dt^2} \frac{(1+t)^2}{\sigma^2(\mu(ut))}\Big|_{t=t_\#} = \mathcal{C}.$$

Since $\sigma(\mu(u)) = o(\delta(u))$, the expression in (36) converges to one as $u \to \infty$. Hence, we have

$$\frac{\Psi\left(\frac{u}{\overline{\sigma}(u)}\right)}{\Psi\left(\frac{u(1+t_u^*)}{\sigma(\mu(ut_u^*))}\right)} = \exp\left(-\frac{1}{2}\mathcal{C}\frac{\delta^2(u)}{\sigma^2(\mu(u))}(1 + o(1))\right)(1 + o(1)),$$

showing that the interval $[\omega, T]\backslash[t_u^* \pm \delta(u)/u]$ plays no role in the asymptotics. □

We can now prove the upper bounds. In the proof, it is essential that $\sigma(\mu(u))/\Delta(u) \to \infty$ in all four cases. To see that this holds, note that this function is regularly varying with index $(1 - H/\beta)(1/\iota_\tau - 1) > 0$ in case A and B (use $\iota_\nu = (1 - H/\beta)/\iota_\tau$ in the latter case). In case C, the index of variation is

$$\frac{H}{\beta} + \iota_\nu - 1 + \frac{1 - \iota_\tau\iota_\nu - H/\beta}{\tilde{\iota}_\tau} > \left(1 - \iota_\tau\iota_\nu - \frac{H}{\beta}\right)\left(\frac{1}{\tilde{\iota}_\tau} - 1\right) > 0.$$

Finally, it is regularly varying with index $(1 - H/\beta)(1/\tilde{\iota}_\tau - 1) > 0$ in case D.

The upper bounds are formulated in the following proposition.

Proposition 4 *Let μ and σ satisfy assumptions* **M1**–**M4** *and* **S1**–**S4** *for some $\beta > H$. Moreover, let case A, B, C, or D apply. We then have*

$$\limsup_{u\to\infty} \frac{P\left(\sup_{t\geq 0} Y_{\mu(t)} - t > u\right)}{\frac{\sigma(\mu(u))}{\Delta(u)}\Psi\left(\inf_{t\geq 0} \frac{u(1+t)}{\sigma(\mu(ut))}\right)} \leq \mathcal{H}\sqrt{\frac{2\pi}{\mathcal{C}}}.$$

Proof. Select some δ such that $\delta(u) = o(u)$, $\sigma(\mu(u)) = o(\delta(u))$, $\Delta(u) = o(\delta(u))$, and $u = o(\delta(u)\nu(u))$. While the specific choice is irrelevant, it is left to the reader that such δ exists in each of the four cases. In view of Lemma 7, we need to show that

$$\limsup_{u\to\infty} \frac{P\left(\sup_{t\in[t_u^*\pm\delta(u)/u]} \frac{Y_{\mu(ut)}}{1+t} > u\right)}{\frac{\sigma(\mu(u))}{\Delta(u)}\Psi\left(\frac{u(1+t_u^*)}{\sigma(\mu(ut_u^*))}\right)} \leq \mathcal{H}\sqrt{\frac{2\pi}{\mathcal{C}}}. \tag{37}$$

For this, notice that by definition of t_u^* and continuity of σ and μ, for large u,

$$
\begin{aligned}
& P\left(\sup_{t\in[t_u^*\pm\delta(u)/u]}\frac{Y_{\mu(ut)}}{1+t}>u\right)\\
\le & \sum_{-\frac{\delta(u)}{T\Delta(u)}\le k\le\frac{\delta(u)}{T\Delta(u)}} P\left(\sup_{t\in I_k^T(u)}\frac{Y_{\mu(ut)}}{1+t}>u\right)\\
\le & \sum_{0\le k\le\frac{\delta(u)}{T\Delta(u)}} P\left(\sup_{t\in I_k^T(u)}\frac{Y_{\mu(ut)}}{\sigma(\mu(ut))}>\frac{u(1+\underline{t}_k^T(u))}{\sigma(\mu(u\underline{t}_k^T(u)))}\right)\\
& + \sum_{-\frac{\delta(u)}{T\Delta(u)}\le k<0} P\left(\sup_{t\in I_k^T(u)}\frac{Y_{\mu(ut)}}{\sigma(\mu(ut))}>\frac{u(1+\overline{t}_k^T(u))}{\sigma(\mu(u\overline{t}_k^T(u)))}\right).
\end{aligned}
\tag{38}
$$

By Lemmas 3–6, the UCT, and (7), as $u\to\infty$,

$$
\begin{aligned}
& \frac{\Delta(u)}{\sigma(\mu(u))}\sum_{0\le k\le\frac{\delta(u)}{T\Delta(u)}}\frac{P\left(\sup_{t\in I_k^T(u)}\frac{Y_{\mu(ut)}}{\sigma(\mu(ut))}>\frac{u(1+\overline{t}_k^T(u))}{\sigma(\mu(u\overline{t}_k^T(u)))}\right)}{\Psi\left(\frac{u(1+t_u^*)}{\sigma(\mu(ut_u^*))}\right)}\\
= & \mathcal{H}(T)\frac{\Delta(u)}{\sigma(\mu(u))}\sum_{0\le k\le\frac{\delta(u)}{T\Delta(u)}}\left[\frac{\Psi\left(\frac{u(1+\overline{t}_k^T(u))}{\sigma(\mu(u\overline{t}_k^T(u)))}\right)}{\Psi\left(\frac{u(1+t_u^*)}{\sigma(\mu(ut_u^*))}\right)}(1+o(1))\right]\\
= & \mathcal{H}(T)\frac{\Delta(u)}{\sigma(\mu(u))}\sum_{0\le k\le\frac{\delta(u)}{T\Delta(u)}}\left[\frac{\exp\left(-\frac{1}{2}\frac{u^2(1+\overline{t}_k^T(u))^2}{\sigma^2(\mu(u\overline{t}_k^T(u)))}\right)}{\exp\left(-\frac{1}{2}\frac{u^2(1+t_u^*)^2}{\sigma^2(\mu(ut_u^*))}\right)}(1+o(1))\right].
\end{aligned}
\tag{39}
$$

As in the proof of Lemma 7, one can show that, uniformly in k by the UCT,

$$
u^2\left(\frac{(1+\overline{t}_k^T(u))^2}{\sigma^2(\mu(u\overline{t}_k^T(u)))}-\frac{(1+t_u^*)^2}{\sigma^2(\mu(ut_u^*))}\right)\Big/\,\mathcal{C}\left[\frac{(k+1)T\Delta(u)}{\sigma(\mu(u))}\right]^2\to 0,
$$

where $\mathcal{C}$ is given in (32). Hence, (39) can be written as

$$
\frac{\mathcal{H}(T)}{T}\frac{T\Delta(u)}{\sigma(\mu(u))}\sum_{0\le k\le\frac{\delta(u)}{T\Delta(u)}}\left[\exp\left(-\frac{1}{2}\mathcal{C}\frac{[(k+1)T\Delta(u)]^2}{\sigma^2(\mu(u))}(1+o(1))\right)(1+o(1))\right].
$$

By Lemmas 3–6, the fact that $\sigma(\mu(u))=o(u)$, and the dominated convergence theorem, this tends to

$$
\frac{\mathcal{H}(T)}{T}\int_0^\infty\exp\left(-\frac{1}{2}\mathcal{C}x^2\right)dx=\frac{\mathcal{H}(T)}{T}\sqrt{\frac{\pi/2}{\mathcal{C}}}.
$$

The second term in (38) is bounded from above similarly. Hence, we have shown that for any $T>0$,

$$
\limsup_{u\to\infty}\frac{\Delta(u)}{\sigma(\mu(u))}\frac{P\left(\sup_{t\in[t_u^*+\delta(u)/u]}\frac{Y_{\mu(ut)}}{1+t}>u\right)}{\Psi\left(\frac{u(1+t_u^*)}{\sigma(\mu(ut_u^*))}\right)}\le\frac{\mathcal{H}(T)}{T}\sqrt{\frac{2\pi}{\mathcal{C}}}.
$$

The claim is obtained by letting $T\to\infty$. □

7 Lower bounds

In this section, we prove the lower bound part of Theorem 1 using an appropriate modification of the corresponding argument in the double sum method. For notational conventions, see Section 6.

Proposition 5 *Let μ and σ satisfy assumptions* **M1**–**M4** *and* **S1**–**S4** *for some $\beta>H$. Moreover, let case A, B, C, or D apply. We then have*

$$
\liminf_{u\to\infty}\frac{P\left(\sup_{t\ge 0}Y_{\mu(t)}-t>u\right)}{\frac{\sigma(\mu(u))}{\Delta(u)}\Psi\left(\inf_{t\ge 0}\frac{u(1+t)}{\sigma(\mu(ut))}\right)}\ge\mathcal{H}\sqrt{\frac{2\pi}{\mathcal{C}}}.
$$

The proof of this proposition requires some auxiliary observations, resulting in a bound on probabilities involving the supremum on a two-dimensional field. The first step in establishing those bounds is to study the variances; it is therefore convenient to introduce the notation

$$
\underline{\sigma}_{k,\ell}^2(u):=\inf_{(s,t)\in I_k^T(u)\times I_\ell^T(u)}\mathbb{V}\mathrm{ar}\left(\frac{Y_{\mu(us)}}{\sigma(\mu(us))}-\frac{Y_{\mu(ut)}}{\sigma(\mu(ut))}\right)
$$

and

$$
\overline{\sigma}_{k,\ell}^2(u):=\sup_{(s,t)\in I_k^T(u)\times I_\ell^T(u)}\mathbb{V}\mathrm{ar}\left(\frac{Y_{\mu(us)}}{\sigma(\mu(us))}-\frac{Y_{\mu(ut)}}{\sigma(\mu(ut))}\right).
$$

Lemma 8 *Suppose that one of the cases A, B, C, or D applies, and that both $\delta(u)=o(u)$ and $\Delta(u)=o(\delta(u))$. Then there exist constants $\zeta\in(0,2)$ and $\mathcal{K}\in(0,\infty)$, independent of T, such that for large T the following holds. Given $\epsilon>0$, there exists some u_0 such that for all $u\ge u_0$ and all $-\frac{\delta(u)}{T\Delta(u)}\le k,\ell\le\frac{\delta(u)}{T\Delta(u)}$ with $|\ell-k|>1$,*

$$
\underline{\sigma}_{k,\ell}^2(u)\ge(1-\epsilon)^3\mathcal{K}\left[\left(\frac{T(|k-\ell|-1)}{2}\right)^\zeta-\epsilon\right]\frac{\sigma^2(\mu(u))}{u^2}.
$$

Moreover,

$$
\sup_{\substack{-\frac{\delta(u)}{T\Delta(u)}\le k,\ell\le\frac{\delta(u)}{T\Delta(u)}\\ |k-\ell|>1}}\overline{\sigma}_{k,\ell}^2(u)\to 0.
$$

Proof. Let $\epsilon>0$ be given. By (3), the first claim is proven for case A, B, and C once it has been shown that for large u, uniformly in $\alpha\in\left[1,\frac{\delta(u)}{T\Delta(u)}\right]$,

$$
\inf_{\substack{s,t\in[t_u^*\pm\delta(u)/u]\\ |s-t|\ge\alpha T\Delta(u)/u}}\frac{\tau^2(|\nu(us)-\nu(ut)|)}{\tau^2(\nu(u))}\ge(1-\epsilon)^2\frac{\mathcal{K}}{\mathcal{D}}\left[\left(\frac{\alpha T}{2}\right)^\zeta-\epsilon\right]\frac{\sigma^2(\mu(u))}{u^2},
$$

since one can then set $\alpha=|k-\ell|-1$. By the Mean Value Theorem {**N2**} we have, for certain $t^\wedge(u,s,t)\in[t_u^*\pm\delta(u)/u]$,

$$
\begin{aligned}
\inf_{\substack{s,t\in[t_u^*\pm\delta(u)/u]\\ |s-t|\ge\alpha T\Delta(u)/u}}\frac{\tau^2(|\nu(us)-\nu(ut)|)}{\tau^2(\nu(u))} &= \inf_{\substack{s,t\in[t_u^*\pm\delta(u)/u]\\ |s-t|\ge\alpha T\Delta(u)/u}}\frac{\tau^2(u\dot\nu(ut^\wedge(u,s,t))|s-t|)}{\tau^2(\nu(u))}\\
&\ge \inf_{\substack{s,t\in[t_u^*\pm\delta(u)/u]\\ |s-t|\ge\frac{1}{2}\alpha T\Delta(u)/u}}\frac{\tau^2(u\dot\nu(ut^*)|s-t|)}{\tau^2(\nu(u))}\\
&\ge \inf_{t\ge\alpha T/2}\frac{\tau^2(\dot\nu(ut^*)\Delta(u)t)}{\tau^2(\nu(u))}
\end{aligned}
$$

where the first inequality follows from the UCT $\{\mathbf{N1}\}$; the details are left to the reader.

We investigate the lower bound in each of the three cases. In case A, $\dot\nu(ut^*)\Delta(u)$ tends to infinity. By the UCT and the definition of Δ, we have for any $\alpha \geq 1$

$$\begin{aligned}\inf_{t\geq \alpha T/2} \frac{\tau^2(\dot\nu(ut^*)\Delta(u)t)}{\tau^2(\nu(u))} &\geq (1-\epsilon)\frac{\tau^2(\dot\nu(ut^*)\Delta(u))}{\tau^2(\nu(u))}\left[\left(\frac{\alpha T}{2}\right)^{2\iota_\tau} - \epsilon\right] \\ &\geq (1-\epsilon)^2 \frac{2}{\mathcal{D}(1+t^*)^2}\frac{\sigma^2(\mu(ut^*))}{u^2}\left[\left(\frac{\alpha T}{2}\right)^{2\iota_\tau} - \epsilon\right].\end{aligned}$$

Case C is similar, except that now $\dot\nu(ut^*)\Delta(u)$ tends to zero (so that one can apply the UCT as τ is continuous and regularly varying at zero):

$$\inf_{t\geq \alpha T/2} \frac{\tau^2(\dot\nu(ut^*)\Delta(u)t)}{\tau^2(\nu(u))} \geq (1-\epsilon)^2 \frac{2}{\mathcal{D}(1+t^*)^2}\frac{\sigma^2(\mu(ut^*))}{u^2}\left[\left(\frac{\alpha T}{2}\right)^{2\tilde\iota_\tau} - \epsilon\right].$$

In case B, we note that $\sigma(\mu(u))\tau(\nu(u)) \sim \mathcal{G}u$ implies that for small $\zeta > 0$, there exists some t_0 such that for $t \geq t_0$, $\tau^2(t) \geq t^\zeta$. Therefore, for T large enough, since $\dot\nu(ut^*)\Delta(u) = 1$, uniformly in $\alpha \geq 1$,

$$\inf_{t\geq \alpha T/2} \frac{\tau^2(\dot\nu(ut^*)\Delta(u)t)}{\tau^2(\nu(u))} \geq \inf_{t\geq \alpha T/2}\frac{t^\zeta}{\tau^2(\nu(u))} = \frac{(\alpha T/2)^\zeta}{\tau^2(\nu(u))} \geq (1-\epsilon)^2\left(\frac{\alpha T}{2}\right)^\zeta \frac{1}{\mathcal{G}^2}\frac{\sigma^2(\mu(u))}{u^2},$$

implying the stated.

We leave the proof of the assertion for case D to the reader; one then exploits the regular variation of τ at zero and uses the definition of Δ.

To prove the second claim of the lemma in case A, B, and C, we use the Mean Value Theorem and the UCT: $\{\mathbf{N1}, \mathbf{N2}\}$

$$\begin{aligned}\sup_{s,t\in[t_u^*\pm\delta(u)/u]} \mathbb{V}\mathrm{ar}\left(\frac{Y_{\mu(us)}}{\sigma(\mu(us))} - \frac{Y_{\mu(ut)}}{\sigma(\mu(ut))}\right) &\sim \sup_{s,t\in[t_u^*\pm\delta(u)/u]}\frac{\mathcal{D}\tau^2(|\nu(us)-\nu(ut)|)}{\tau^2(\nu(u))} \\ &\leq \sup_{s,t\in[t_u^*\pm 2\delta(u)/u]}\frac{\mathcal{D}\tau^2(u\dot\nu(ut^*)|s-t|)}{\tau^2(\nu(u))} \\ &= \sup_{t\in[0,2]}\frac{\mathcal{D}\tau^2(\delta(u)\dot\nu(ut^*)t)}{\tau^2(\nu(u))}.\end{aligned}$$

Since $\delta(u)\dot\nu(ut^*)$ tends to infinity by assumption, **T1** implies that the latter expression is of order $\tau^2(\delta(u)\dot\nu(u))/\tau^2(\nu(u))$. In particular, it tends to zero as $u \to \infty$.

We do not prove the claim for case D, since the same arguments apply. □

The two statements of Lemma 8 on the correlation structure are exploited in the next lemma. Let $\kappa_{k,\ell}$ be arbitrary functions of u which converge uniformly for $-\frac{\delta(u)}{T\Delta(u)} \leq k, \ell \leq \frac{\delta(u)}{T\Delta(u)}$ to $2(1+t^*)$.

Lemma 9 *Suppose that one of the cases A, B, C, or D applies, and that $\delta(u) = o(u)$. There exist constants $\alpha, \mathcal{K}' < \infty$, independent of k, ℓ, such that for large u, uniformly for k, ℓ with $|k-\ell| > 1$,*

$$P\left(\sup_{(s,t)\in I_k^T(u)\times I_\ell^T(u)} \frac{Y_{\mu(us)}}{\sigma(\mu(us))} + \frac{Y_{\mu(ut)}}{\sigma(\mu(ut))} > \frac{u\kappa_{k,\ell}(u)}{\sigma(\mu(ut^*))}\right) \leq \mathcal{K}'T^\alpha \Psi\left(\frac{\frac{u\kappa_{k,\ell}(u)}{\sigma(\mu(ut^*))}}{\sqrt{4-\underline{\sigma}^2_{k,\ell}(u)}}\right). \quad (40)$$

Proof. Define

$$Y^*_{(s,t)}(u) := \frac{\frac{Y_{\mu(us)}}{\sigma(\mu(us))} + \frac{Y_{\mu(ut)}}{\sigma(\mu(ut))}}{\sqrt{\mathbb{V}\mathrm{ar}\left(\frac{Y_{\mu(us)}}{\sigma(\mu(us))} + \frac{Y_{\mu(ut)}}{\sigma(\mu(ut))}\right)}}, \quad u^*_{k,\ell} = \frac{\frac{u\kappa_{k,\ell}(u)}{\sigma(\mu(ut^*))}}{\sqrt{4-\underline{\sigma}^2_{k,\ell}(u)}}$$

so that the left hand side of (40) is majorized by

$$P\left(\sup_{(s,t)\in I_k^T(u)\times I_\ell^T(u)} Y^*_{(s,t)}(u) > u^*_{k,\ell}\right). \quad (41)$$

As a consequence of (the second claim in) Lemma 8, we have for large u

$$\inf_{k,\ell}\inf_{(s,t)\in I_k^T(u)\times I_\ell^T(u)} \mathbb{V}\mathrm{ar}\left(\frac{Y_{\mu(us)}}{\sigma(\mu(us))} + \frac{Y_{\mu(ut)}}{\sigma(\mu(ut))}\right) \geq 2.$$

The proof closely follows the reasoning on page 102 of Piterbarg [34]. In particular, for $(s,t), (s',t') \in I_k^T(u) \times I_\ell^T(u)$, we have

$$\begin{aligned}&\mathbb{V}\mathrm{ar}\left(Y^*_{(s,t)}(u) - Y^*_{(s',t')}(u)\right) \\ &\leq 4\mathbb{V}\mathrm{ar}\left(\frac{Y_{\mu(us)}}{\sigma(\mu(us))} - \frac{Y_{\mu(us')}}{\sigma(\mu(us'))}\right) + 4\mathbb{V}\mathrm{ar}\left(\frac{Y_{\mu(ut)}}{\sigma(\mu(ut))} - \frac{Y_{\mu(ut')}}{\sigma(\mu(ut'))}\right).\end{aligned} \quad (42)$$

Define

$$\upsilon(u) := \begin{cases} \sqrt{2}\tau(\nu(u))\frac{\sigma(\mu(ut^*))}{u(1+t^*)} & \text{in case A, C and D;} \\ \frac{1}{2}\sqrt{2\mathcal{D}} & \text{in case B.}\end{cases}$$

Now we have to distinguish between case D and the other cases. First we focus on the cases A, B, and C; then one can use (3) to see that (42) is asymptotically at most

$$4\frac{\mathcal{D}\tau^2(|\nu(us)-\nu(us')|)}{\tau^2(\nu(u))} + 4\frac{\mathcal{D}\tau^2(|\nu(ut)-\nu(ut')|)}{\tau^2(\nu(u))}. \quad (43)$$

As shown in the proof Lemmas 3–5,

$$\limsup_{u\to\infty}\sup_{-\frac{\delta(u)}{T\Delta(u)}\leq k\leq\frac{\delta(u)}{T\Delta(u)}}\sup_{(s,t)\in I_k^T(u)}\frac{\mathcal{D}\tau^2(|\nu(us)-\nu(ut)|)}{\upsilon^2(u)}\left(\frac{u}{\Delta(u)}(s-t)\right)^{-2\gamma'} \leq 2T^{\alpha'},$$

where $\alpha' = 2(\iota_\tau - \gamma')$ in case A and B, and $\alpha' = 2(\tilde\iota_\tau - \gamma')$ in case C. Therefore, we find the following asymptotic upper bound for (43) and hence for (42):

$$8T^{\alpha'}\frac{\upsilon^2(u)}{\tau^2(\nu(u))}\left[\left(\frac{u}{\Delta(u)}(s-s')\right)^{2\gamma'} + \left(\frac{u}{\Delta(u)}(t-t')\right)^{2\gamma'}\right]. \quad (44)$$

We now show that (44) is also an asymptotic upper bound in case D. For this, we note that in this case (42) is asymptotically at most

$$4\tau^2\left(\frac{|\nu(us)-\nu(us')|}{\nu(u)}\right) + 4\tau^2\left(\frac{|\nu(ut)-\nu(ut')|}{\nu(u)}\right),$$

and the reader can check with the Mean Value Theorem and the UCT that (44) holds for $\gamma' = \tilde\iota_\tau/2$ and $\alpha' = \tilde\iota_\tau$ (say).

For any u, we now introduce two independent centered Gaussian stationary processes $\vartheta_1^{(u)}$ and $\vartheta_2^{(u)}$. These processes have unit variance and covariance function equal to

$$r_\vartheta^{(u)}(t) := \mathbb{C}\mathrm{ov}\left(\vartheta_i^{(u)}(t), \vartheta_i^{(u)}(0)\right) = \exp\left(-32\frac{\upsilon^2(u)}{\tau^2(\nu(u))}t^{2\gamma'}\right).$$

Observe that $\upsilon^2(u)/\tau^2(\nu(u)) \to 0$ in each of the four cases, so that for $s,t,s',t' \in [0,T]$ and u large enough,

$$\begin{aligned}
&\mathbb{V}\mathrm{ar}\left(\frac{1}{\sqrt{2}}\left[\vartheta_1^{(u)}(s) + \vartheta_2^{(u)}(t) - \vartheta_1^{(u)}(s') - \vartheta_2^{(u)}(t')\right]\right) \\
&= 2 - \exp\left(-32\frac{\upsilon^2(u)}{\tau^2(\nu(u))}|s-s'|^{2\gamma'}\right) - \exp\left(-32\frac{\upsilon^2(u)}{\tau^2(\nu(u))}|t-t'|^{2\gamma'}\right) \\
&\geq 16\frac{\upsilon^2(u)}{\tau^2(\nu(u))}|s-s'|^{2\gamma'} + 16\frac{\upsilon^2(u)}{\tau^2(\nu(u))}|t-t'|^{2\gamma'}.
\end{aligned}$$

We now apply Slepian's inequality (e.g., Theorem C.1 of [34]) to compare the suprema of the two fields Y^* and $2^{-1/2}\left[\vartheta_1^{(u)} + \vartheta_2^{(u)}\right]$: (41) is majorized for $-\frac{\delta(u)}{T\Delta(u)} \leq k,\ell \leq \frac{\delta(u)}{T\Delta(u)}$ by

$$\begin{aligned}
&P\left(\sup_{(s,t)\in[0,T]^2} \frac{1}{\sqrt{2}}\left[\vartheta_1^{(u)}(T^{\alpha'/(2\gamma')}s) + \vartheta_2^{(u)}(T^{\alpha'/(2\gamma')}t)\right] > u^*_{k,\ell}\right) \\
&= P\left(\sup_{(s,t)\in[0,T^{\alpha'/(2\gamma')+1}]^2} \frac{1}{\sqrt{2}}\left[\vartheta_1^{(u)}(s) + \vartheta_2^{(u)}(t)\right] > u^*_{k,\ell}\right). \qquad (45)
\end{aligned}$$

Lemma 2 is used to investigate the asymptotics of this probability, yielding the desired bound. For notational convenience, we set $T' = T^{\alpha'/(2\gamma')+1}$. Observe that the map

$$(\alpha_1,\alpha_2) \mapsto [2 - \exp(-\alpha_1) - \exp(-\alpha_2)]/[\alpha_1+\alpha_2] - 1$$

is nonpositive and that the minimum over the set $[0,\theta]^2$ is achieved at $(\alpha_1,\alpha_2) = (\theta,\theta)$. Therefore,

$$\sup_{(s,t),(s',t')\in[0,T']^2}\left|\frac{2 - r_\vartheta^{(u)}(|s-s'|) - r_\vartheta^{(u)}(|t-t'|)}{32\frac{\upsilon^2(u)}{\tau^2(\nu(u))}\left[|s-s'|^{2\gamma'} + |t-t'|^{2\gamma'}\right]} - 1\right| = 1 - \frac{2 - r_\vartheta^{(u)}(T') - r_\vartheta^{(u)}(T')}{64\frac{\upsilon^2(u)}{\tau^2(\nu(u))}(T')^{2\gamma'}},$$

which tends to zero if $u\to\infty$. Moreover, we have

$$\sup_{-\frac{\delta(u)}{T\Delta(u)}\leq k,\ell\leq\frac{\delta(u)}{T\Delta(u)}}\left|\frac{\sigma^2(\mu(ut^*))(u^*_{k,\ell})^2}{u^2(1+t^*)^2} - 1\right| \to 0.$$

To see that Lemma 2 can be applied, set $g_{k,\ell}(u) = u^*_{k,\ell}$, and

$$\theta_{k,\ell}(u,s,s',t,t') := 32(1+t^*)^2\frac{u^2\upsilon^2(u)}{\sigma^2(\mu(ut^*))\tau^2(\nu(u))}\left[|s-s'|^{2\gamma'} + |t-t'|^{2\gamma'}\right].$$

P1 obviously holds, and $\theta_{k,\ell}(u,s,s',t,t')$ tends to

$$2\xi_\eta(s,s',t,t') := \begin{cases} 64\left[|s-s'|^{2\gamma'} + |t-t'|^{2\gamma'}\right] & \text{in case A, C, and D;} \\ \frac{16(1+t^*)^2}{(t^*)^{2H/\beta}\mathcal{G}^2}\left[|s-s'|^{2\gamma'} + |t-t'|^{2\gamma'}\right] & \text{in case B,}\end{cases}$$

showing that **P2** holds. As **P3** is immediate, it remains to investigate whether **P4** holds. The reasoning in the proof of Lemma 3 shows that it suffices to show that

$$\lim_{\varepsilon\to0}\limsup_{u\to\infty}\sup_{k,\ell}\sup_{|s-s'|^{2\gamma'}+|t-t'|^{2\gamma'}<\varepsilon}\theta_{k,\ell}(u,s,s',t,t') < \infty,$$

which is trivial. Define for $s,t\in[0,T']$,

$$\eta(s,t) := B^1_{\gamma'}(s) + B^2_{\gamma'}(t),$$

where $B^1_{\gamma'}$ and $B^2_{\gamma'}$ are independent fractional Brownian motions with Hurst parameter γ'. Then, the probability in (45) is asymptotically equivalent to

$$\begin{cases} \mathbb{E}\exp\left(\sup_{(s,t)\in[0,T']^2} 8\eta(s,t) - 32s^{2\gamma'} - 32t^{2\gamma'}\right)\Psi(u^*_{k,\ell}) & \text{in case A, C, D;} \\ \mathbb{E}\exp\left(\sup_{(s,t)\in[0,T']^2}\frac{4(1+t^*)}{(t^*)^{H/\beta}\mathcal{G}}\eta(s,t) - \frac{8(1+t^*)^2}{(t^*)^{2H/\beta}\mathcal{G}^2}\left[s^{2\gamma'}+t^{2\gamma'}\right]\right)\Psi(u^*_{k,\ell}) & \text{in case B.}\end{cases}$$

By exploiting the self-similarity of fractional Brownian motion one can see that the expectation equals $(T')^2\mathcal{K}'$ for some constant $\mathcal{K}' < \infty$. □

Proof of Proposition 5. Note that

$$\begin{aligned}
&P\left(\sup_{t\in[t_u^*\pm\delta(u)/u]}\frac{Y_{\mu(ut)}}{1+t} > u\right) \\
&\geq \sum_{-\frac{\delta(u)}{T\Delta(u)}\leq k\leq\frac{\delta(u)}{T\Delta(u)}} P\left(\sup_{t\in I_k^T(u)}\frac{Y_{\mu(ut)}}{1+t} > u;\ \sup_{t\in[t_u^*\pm\delta(u)/u]\setminus I_k^T(u)}\frac{Y_{\mu(ut)}}{1+t} \leq u\right) \\
&= \sum_{-\frac{\delta(u)}{T\Delta(u)}\leq k\leq\frac{\delta(u)}{T\Delta(u)}} P\left(\sup_{t\in I_k^T(u)}\frac{Y_{\mu(ut)}}{1+t} > u\right) \\
&\quad - \sum_{-\frac{\delta(u)}{T\Delta(u)}\leq k\leq\frac{\delta(u)}{T\Delta(u)}} P\left(\sup_{t\in I_k^T(u)}\frac{Y_{\mu(ut)}}{1+t} > u;\ \sup_{t\in[t_u^*\pm\delta(u)/u]\setminus I_k^T(u)}\frac{Y_{\mu(ut)}}{1+t} > u\right). \qquad (46)
\end{aligned}$$

A similar reasoning as in the proof of Proposition 4 can be used to see that

$$\lim_{T\to\infty}\liminf_{u\to\infty}\frac{\sum_{-\frac{\delta(u)}{T\Delta(u)}\leq k\leq\frac{\delta(u)}{T\Delta(u)}} P\left(\sup_{t\in I_k^T(u)}\frac{Y_{\mu(ut)}}{1+t} > u\right)}{\frac{\sigma(\mu(u))}{\Delta(u)}\Psi\left(\frac{u(1+t_u^*)}{\sigma(\mu(ut_u^*))}\right)} \geq \mathcal{H}\sqrt{\frac{2\pi}{\mathcal{C}}}.$$

It remains to find an appropriate upper bound for the second term in (46). For this, observe that

$$\begin{aligned}
&P\left(\sup_{t\in I_k^T(u)}\frac{Y_{\mu(ut)}}{1+t} > u;\ \sup_{t\in[t_u^*\pm\delta(u)/u]\setminus I_k^T(u)}\frac{Y_{\mu(ut)}}{1+t} > u\right) \\
&\leq P\left(\sup_{t\in I_k^T(u)}\frac{Y_{\mu(ut)}}{1+t} > u;\ \sup_{t\in\left[t_u^*-\frac{\delta(u)}{u},\underline{t}_k^T(u)-\sqrt{T}\frac{\Delta(u)}{u}\right)\cup\left(\bar{t}_k^T(u)+\sqrt{T}\frac{\Delta(u)}{u},t_u^*+\frac{\delta(u)}{u}\right]}\frac{Y_{\mu(ut)}}{1+t} > u\right) \\
&\quad + P\left(\sup_{t\in\left[\underline{t}_k^T(u)-\sqrt{T}\frac{\Delta(u)}{u},\underline{t}_k^T(u)\right)}\frac{Y_{\mu(ut)}}{1+t} > u\right) + P\left(\sup_{t\in\left(\bar{t}_k^T(u),\bar{t}_k^T(u)+\sqrt{T}\frac{\Delta(u)}{u}\right]}\frac{Y_{\mu(ut)}}{1+t} > u\right) \\
&=: p_1(u,k,T) + p_2(u,k,T) + p_3(u,k,T).
\end{aligned}$$

One can apply the arguments that are detailed in the proof of Proposition 4 to infer that

$$\limsup_{u\to\infty} \frac{\sum_{-\frac{\delta(u)}{T\Delta(u)}\le k\le \frac{\delta(u)}{T\Delta(u)}} p_2(u,k,T)}{\frac{\sigma(\mu(u))}{\Delta(u)}\Psi\left(\frac{u(1+t_u^*)}{\sigma(\mu(ut_u^*))}\right)} \le \frac{\mathcal{H}\left(\sqrt{T}\right)}{T}\sqrt{\frac{2\pi}{\mathcal{C}}},$$

which converges to zero as $T\to\infty$. The term $p_3(u,k,T)$ is bounded from above similarly.

We now study $\sum_k p_1(u,k,T)$ in more detail; for this we need the technical lemmas that were established earlier. Observe that it is majorized by

$$\begin{aligned} & \sum_{-\frac{\delta(u)}{T\Delta(u)}\le k\le \frac{\delta(u)}{T\Delta(u)}} \sum_{\substack{-\frac{\delta(u)}{T\Delta(u)}\le \ell\le \frac{\delta(u)}{T\Delta(u)} \\ |k-\ell|>1}} P\left(\sup_{t\in I_k^T(u)} \frac{Y_{\mu(ut)}}{1+t}>u;\ \sup_{t\in I_\ell^T(u)}\frac{Y_{\mu(ut)}}{1+t}>u\right) \\ + & \sum_{-\frac{\delta(u)}{T\Delta(u)}\le k\le \frac{\delta(u)}{T\Delta(u)}} P\left(\sup_{t\in I_k^T(u)} \frac{Y_{\mu(ut)}}{1+t}>u;\ \sup_{t\in[\overline{t}_u^k+\sqrt{T}\frac{\Delta(u)}{u},\overline{t}_u^k+(T+\sqrt{T})\frac{\Delta(u)}{u}]}\frac{Y_{\mu(ut)}}{1+t}>u\right) \\ + & \sum_{-\frac{\delta(u)}{T\Delta(u)}\le k\le \frac{\delta(u)}{T\Delta(u)}} P\left(\sup_{t\in I_k^T(u)} \frac{Y_{\mu(ut)}}{1+t}>u;\ \sup_{t\in[\underline{t}_u^k-(T+\sqrt{T})\frac{\Delta(u)}{u},\underline{t}_u^k-\sqrt{T}\frac{\Delta(u)}{u}]}\frac{Y_{\mu(ut)}}{1+t}>u\right) \\ =: & \ I(u,T)+II(u,T)+III(u,T). \end{aligned}$$

By symmetry, $I(u,T)$ is bounded from above by

$$2\sum_{\substack{-\frac{\delta(u)}{T\Delta(u)}\le k\le \frac{\delta(u)}{T\Delta(u)} \\ |k-\ell|>1,\sup_{t\in I_k^T(u)}\frac{\sigma(\mu(ut))}{1+t}\le \sup_{t\in I_\ell^T(u)}\frac{\sigma(\mu(ut))}{1+t}}} \sum_{-\frac{\delta(u)}{T\Delta(u)}\le \ell\le \frac{\delta(u)}{T\Delta(u)}} P\left(\sup_{t\in I_k^T(u)} \frac{Y_{\mu(ut)}}{1+t}>u;\ \sup_{t\in I_\ell^T(u)}\frac{Y_{\mu(ut)}}{1+t}>u\right).$$

Each of the summands cannot exceed

$$P\left(\sup_{(s,t)\in I_k^T(u)\times I_\ell^T(u)} \frac{Y_{\mu(us)}}{\sigma(\mu(us))}+\frac{Y_{\mu(ut)}}{\sigma(\mu(ut))} > \inf_{t\in I_k^T(u)}\frac{2u(1+t)}{\sigma(\mu(ut))}\right),$$

and we are in the setting of Lemma 9. Hence, there exist constants $\mathcal{K}',\alpha$ such that $I(u,T)$ is majorized by

$$2\mathcal{K}'T^\alpha \sum_{-\frac{\delta(u)}{T\Delta(u)}\le k\le \frac{\delta(u)}{T\Delta(u)}} \sum_{\substack{-\frac{\delta(u)}{T\Delta(u)}\le \ell\le \frac{\delta(u)}{T\Delta(u)} \\ |k-\ell|>1}} \Psi\left(\frac{\inf_{t\in I_k^T(u)}\frac{2u(1+t)}{\sigma(\mu(ut))}}{\sqrt{4-\underline{\sigma}_{k,\ell}^2(u)}}\right), \tag{47}$$

which is the 'double sum' in the double sum method. Since

$$-\frac{\inf_{t\in I_k^T(u)}\frac{u^2(1+t)^2}{\sigma^2(\mu(ut))}}{1-\frac{1}{4}\underline{\sigma}_{k,\ell}^2(u)} \le -\frac{1}{4}\inf_{t\in I_k^T(u)}\frac{u^2(1+t)^2}{\sigma^2(\mu(ut))}\underline{\sigma}_{k,\ell}^2(u) - \inf_{t\in I_k^T(u)}\frac{u^2(1+t)^2}{\sigma^2(\mu(ut))},$$

the summand in (47) is bounded from above by

$$\exp\left(-\frac{1}{8}\inf_{t\in I_k^T(u)}\frac{u^2(1+t)^2}{\sigma^2(\mu(ut))}\underline{\sigma}_{k,\ell}^2(u)\right)\Psi\left(\inf_{t\in I_k^T(u)}\frac{u(1+t)}{\sigma(\mu(ut))}\right)(1+o(1)),$$

where the $o(1)$ term is uniformly in k,ℓ as a consequence of the second claim of Lemma 8, cf. Equation (7). By the first claim of Lemma 8 for $\epsilon=1/2$, say, and the UCT, there exist constants $\mathcal{K}'',\zeta$ such that

$$\begin{aligned} & \sum_{\substack{-\frac{\delta(u)}{T\Delta(u)}\le \ell\le \frac{\delta(u)}{T\Delta(u)} \\ |k-\ell|>1}} \exp\left(-\frac{1}{8}\inf_{t\in I_k^T(u)}\frac{u^2(1+t)^2}{\sigma^2(\mu(ut))}\underline{\sigma}_{k,\ell}^2(u)\right) \\ \le & \sum_{\substack{-\frac{\delta(u)}{T\Delta(u)}\le \ell\le \frac{\delta(u)}{T\Delta(u)} \\ |k-\ell|>1}} \exp\left(-\mathcal{K}''\left[T^\zeta(|k-\ell|-1)^\zeta-2^{\zeta-1}\right]\right) \\ \le & \ 2e^{\mathcal{K}''2^{\zeta-1}}\sum_{j=1}^\infty \exp\left(-\mathcal{K}''T^\zeta j^\zeta\right) \\ \le & \ \mathcal{K}'''\exp\left(-T^\zeta\right), \end{aligned}$$

where $\mathcal{K}'''<\infty$ is some constant.

Therefore, (47) cannot be not larger than

$$\begin{aligned} & 2\mathcal{K}'\mathcal{K}'''T^\alpha\exp\left(-T^\zeta\right)\sum_{-\frac{\delta(u)}{T\Delta(u)}\le k\le \frac{\delta(u)}{T\Delta(u)}} \Psi\left(\inf_{t\in I_k^T(u)}\frac{u(1+t)}{\sigma(\mu(ut))}\right)(1+o(1)) \\ = & \ 2\sqrt{\frac{2\pi}{\mathcal{C}}}\mathcal{H}(T)\mathcal{K}'\mathcal{K}'''T^\alpha\exp\left(-T^\zeta\right)\frac{\sigma(\mu(u))}{\Delta(u)}\Psi\left(\frac{u(1+t_u^*)}{\sigma(\mu(ut_u^*))}\right)(1+o(1)), \end{aligned}$$

where the last equality was shown in the proof of Proposition 4. Now first send $u\to\infty$, and then $T\to\infty$ to see that $I(u,T)$ plays no role in the asymptotics. One can also see that $II(u,T)$ and $III(u,T)$ can be neglected, but one needs suitable analogs of Lemma 8 and Lemma 9 to see this. Except that there is no summation over ℓ, the arguments are exactly the same as for $I(u,T)$. Since it is notationally more involved, we leave this to the reader. □

Acknowledgments

The author is grateful to Jürg Hüsler for valuable discussions, and to Michel Mandjes for carefully reading the manuscript. He wishes to acknowledge Marc Yor for providing references related to (13).

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