Stochastic billiards for sampling from the boundary of a convex set

A. B. Dieker, Santosh S. Vempala

June 2013

Abstract

Stochastic billiards can be used for approximate sampling from the boundary of a bounded convex set through the Markov Chain Monte Carlo (MCMC) paradigm. This paper studies how many steps of the underlying Markov chain are required to get samples (approximately) from the uniform distribution on the boundary of the set, for sets with an upper bound on the curvature of the boundary. Our main theorem implies a polynomial-time algorithm for sampling from the boundary of such sets.

1 Introduction

High-dimensional sampling is a fundamental algorithmic task with many applications to central problems in operations research and computer science. As with optimization, sampling is algorithmically tractable for convex bodies [5, 9, 11] and their extension to logconcave densities [1, 10], using rapidly-mixing Markov chains whose state space is the body of interest.

This paper discusses the problem of sampling from the *boundary* of a convex body. There are several reasons that warrant a detailed inquiry into such sampling algorithms: (1) Sampling from the boundary of a convex set generalizes sampling from a convex set K, since samples from K can be generated by sampling from the boundary of the set $K \times [0, 1]$ in $\mathbb{R}^{n+1}$. (2) There are specific applications for sampling from the boundary of a convex set, see [14] for details and references. (3) MCMC algorithms that exploit the boundary could prove to be faster, and the underlying ideas could lead to faster algorithms for sampling from other sets. (4) Natural Markov chains in this setting can be viewed as stochastic variants of billiards, a classical topic in chaos theory. (5) New tools need to be developed, which are potentially useful in various other settings.

Shake-and-bake algorithms [2, 15] have been proposed for sampling from the boundary of a compact set through the MCMC paradigm. Underlying each of these algorithms is a Markov chain, whose state space is the boundary of a convex body and whose stationary distribution is the target distribution one seeks samples from. After running the Markov chain for a while, the distribution of the chain becomes 'close' to the target distribution and one thus obtains an approximate sample from the target distribution. Clearly, the efficiency of these (and any) MCMC algorithms critically depends on how long the Markov chain will need to be run to get close to the target distribution.

The stochastic billiard algorithm we study in this paper is a special shake-and-bake algorithm ('running shake-and-bake'), and can informally be described as follows. It traces a ball bouncing inside a set; when the ball hits the boundary of the domain, it is sent in a random direction according to a cosine distribution, depending only on the normal to the tangent at the point of contact with the boundary and *not depending* on the incoming direction. The Markov chain of hitting points on the boundary has a uniform stationary distribution. The motivation for this paper is to understand the convergence properties of stochastic billiards, i.e., for how many steps this Markov chain has to be run as a function of the body.

The main contribution of this paper is the first (to our knowledge) rapid mixing guarantee for sampling from the boundary of a convex body with bounded curvature. For such bodies, the main theorem implies a polynomial-time algorithm for sampling from a distribution arbitrarily close to uniform on the boundary. We emphasize that the guarantee is polynomial in the dimension for any body in this class. This in turn has applications, including estimating the surface area. As we note later, our mixing time bound is asymptotically the best possible.

Related work

Hit-and-run is a random walk in a convex body (not its boundary) [3, 16]. Its convergence was first analyzed by Lovász [8], who showed that it mixes rapidly from a *warm* start, i.e., a distribution close to the stationary. Later, it was shown to be rapidly mixing from any starting point for general logconcave densities [11, 12]. It is similar to stochastic billiards in that each step uses a randomly chosen line through the current point. In these and other works on MCMC sampling for convex bodies, the required number of steps for approximate convergence of the Markov chain ('mixing time') only depends on the dimension n of the body and its diameter D.

At a high level, our main proof of convergence is similar to previous work. It is based on bounding the *conductance* of the Markov chain. This is done via an isoperimetric inequality and an analysis of single steps of the chain (see, e.g., the survey [17] on geometric random walks). Beyond this high-level outline however, our analysis departs from the standard route and from that of hit-and-run. The isoperimetric inequality we need is for the boundary of a convex body, unlike most inequalities in the literature on sampling, which are for convex bodies or logconcave functions. The main challenge for proving rapid convergence comes in the analysis of single steps, which is significantly more intricate for stochastic billiards than for hit-and-run. Here we have to show two things. First that "proper" steps are substantial and not infrequent; and second that the one-step distributions from two *nearby* points have significant overlap. The latter proof has to take into account the specific geometry of stochastic billiards and the cosine law.

There is a significant body of work on convergence properties of stochastic billiards with a focus on establishing geometric ergodicity of the Markov chain, i.e., exponentially fast convergence to the stationary distribution [4, 6, 7] on the body, e.g., through the dimension n and diameter D of the body. It is this dependence that is of paramount importance from an algorithmic point of view, since this convergence rate determines if the resulting algorithm is polynomial-time or not.

Notation

We say that K is a convex body with *curvature bounded from above by* $\mathcal{C} < \infty$ if for each $x \in \partial K$, there is a ball B with radius $1/\mathcal{C}$ and center in K so that the tangent planes of K and B at x coincide and B lies in K. Throughout this paper, we study bodies with curvature bounded from above. Note that this implies that the boundary of the body is smooth.

The notation $f(n) \sim g(n)$ as $n \to \infty$ is shorthand for $\lim_{n\to\infty} f(n)/g(n) = 1$. We write Ψ for the tail of the standard Gaussian distribution, i.e., $\Psi(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$, and Ψ^{-1} for its inverse.

The *total variation distance* between measures P and Q on ∂K is defined as

$$\|P - Q\|_{\mathrm{TV}} = \sup_{A \subseteq \partial K} |P(A) - Q(A)|.$$

(Here and elsewhere, the necessary measurability assumptions are implicit.)

We write B^n and $S^{n-1} \subseteq \mathbb{R}^n$ for the unit ball and unit sphere in $\mathbb{R}^n$, respectively. For $x \in \partial K$, let S_x be the unit sphere centered at x, and n_x the inward normal at x. Write $P_x^{\cos}$ for the law with density proportional to $n_x' y$ on the halfsphere $\{y \in S_x : n_x'(y - x) \geq 0\}$, and P_x^{unif} for the law with the uniform density on this halfsphere. We refer to this law as the cosine law, since the density is proportional to $\cos(\phi_{xy})$, where where we write ϕ_{xy} for the acute angle between $x - y$ and n_x, where $x, y \in \partial K$, see Figure 1 below. We also need two-sided versions: $\tilde{P}_x^{\cos}$ is the law on S_x with density proportional to $|n_x'(y - x)|$, and $\tilde{P}_x^{\text{unif}}$ is the uniform distribution on S_x.

2 Stochastic billiards and our main results

This section describes the stochastic billiards we study in this paper, and presents our main results.

Stochastic billiard on ∂K

1. Assuming the current state is $x \in \partial K$, sample $w \in S_x$ from the law $P_x^{\cos}$.
2. The next state is the unique intersection point $y \neq x$ of the line $\{x + tw : t \in \mathbb{R}\}$ with ∂K.
3. Repeat.

Note that the intersection point always exists and is unique by our assumption on the curvature. Figure 1 illustrates the dynamics. Efficiently sampling from $P_x^{\cos}$ can be done by first sampling uniformly from the $(n-1)$-dimensional unit ball centered at x in the tangent plane at x, and then projecting on S_x in the direction of n_x; see [2, 14].

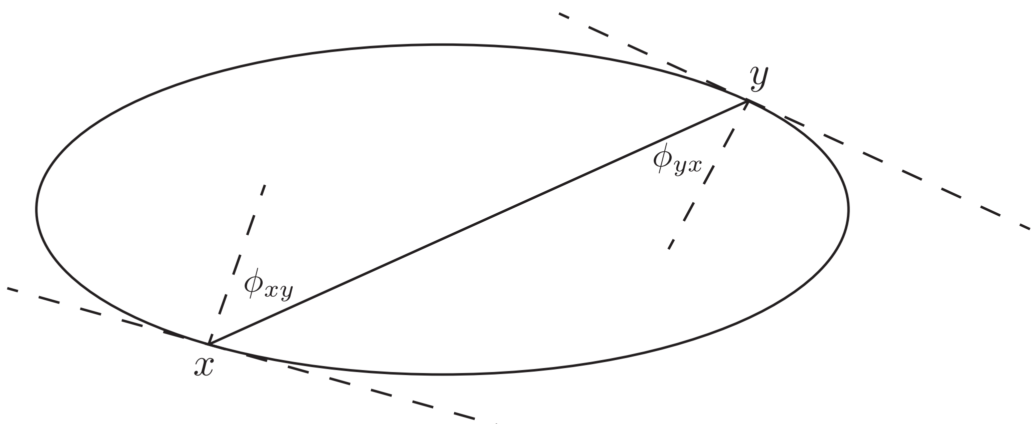

FIGURE 1: The stochastic billiard moves from x to y. The inward normals and the tangent planes at x and y are dashed.

The subsequent intersection points $\{X_k\}$ form a random sequence on ∂K, and it is immediate that this is a Markov chain. We call it the *stochastic billiard Markov chain.* The one-step distribution of the Markov chain is, for $u \in \partial K$ and $A \subseteq \partial K$, [2, 15]

$$P_u(A) = \frac{\pi^{(n-1)/2}}{|\partial K|\Gamma((n+1)/2)} \int_A \frac{\cos(\phi_{uv})\cos(\phi_{vu})}{\|u - v\|^{n-1}} dv, \tag{1}$$

where Γ is the gamma function.

It is worthwhile to understand why the cosine distribution is a natural choice for the outgoing direction in the stochastic billiard chain, since this is closely related to the fact that the uniform distribution is stationary for the chain. As can be seen in Figure 2, the more oblique the incidence of a bundle, its mass must be 'spread out' over a larger region. It is readily seen that this effect is proportional to the cosine of the incoming angle ϕ (as defined previously with respect to the normal).

As a result, the two cosines appearing in the transition density make the kernel symmetric and consequently the uniform distribution is stationary. Thus, the next lemma forms the starting point of MCMC algorithms for sampling from the uniform distribution π on ∂K, We refer to [2, 14] for proofs of this lemma.

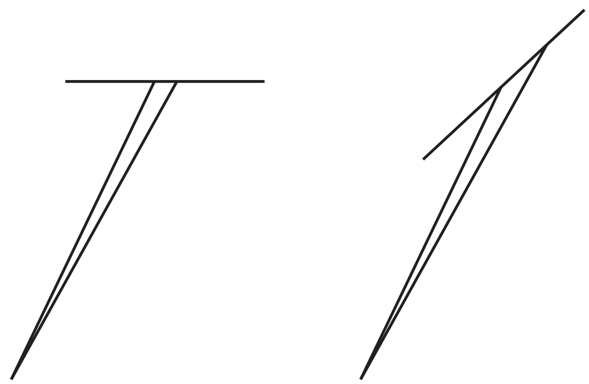

FIGURE 2: The same bundle with different incidence angles.

Lemma 2.1. *The uniform distribution π on ∂K is stationary for the stochastic billiard Markov chain. Moreover, for any initial state $x \in \partial K$, we have*

$$\lim_{k\to\infty} P(X_k \in B | X_0 = x) = \pi(B).$$

To use the stochastic billiard chain as a MCMC sampler for approximate sampling from π, the chain must be stopped after an appropriate number of steps. Our main result is that it takes order $\mathcal{C}^2 n^2 D^2$ steps to get arbitrary close to its uniform equilibrium distribution, where $\mathcal{C}$ is the upper bound on the curvature of ∂K and D is the diameter of K, i.e., the largest distance between any two points in the body. A different way of phrasing our result is to say that the mixing time of the stochastic billiard chain is order $\mathcal{C}^2 n^2 D^2$.

Theorem 2.2. *Let K be a convex body in $\mathbb{R}^n$ with diameter D. Suppose that K contains a unit ball, and that the curvature of ∂K is bounded from above by $\mathcal{C}$. Set $M = \sup_A Q_0(A)/\pi(A)$. Then there is a constant c such that, for $k \geq 0$,*

$$\|Q_k - \pi\|_{TV} \leq \sqrt{M}\left(1 - \frac{c}{\mathcal{C}^2 n^2 D^2}\right)^k.$$

We remark that an explicit expression for the constant c can be found by tracing constants in the proofs of this paper. Since it is not our objective to find the sharpest possible constant, we do not specify the constant in this statement.

We also note that the mixing time bound of $O(\mathcal{C}^2 n^2 D^2)$ is asymptotically the best possible in terms of the dimension n and the diameter D. This can be seen by considering the stochastic billiard process on the boundary of a cylinder in $\mathbb{R}^n$ consisting of the product of a unit ball in $\mathbb{R}^{n-1}$ with an interval of length D. Imagine a partition of the interval into 3 equal length subintervals, inducing three cylinders. Then it can be shown that with large probability it takes $\Omega(n^2 D^2)$ steps for the process to move from a random point in the first cylinder to the third cylinder. This is very similar to the lower bound for hit-and-run shown in [8].

We prove this result by bounding the so-called conductance of the stochastic billiard chain; this is a standard tool for bounding rates of convergence to stationarity for Markov chains. The conductance is defined as

$$\phi = \inf_{A \subseteq \partial K : 0 < \pi(A) < 1/2} \frac{\int_A P_u(\partial K \backslash A) d\pi(u)}{\pi(A)}.$$

The next proposition states our main result on the conductance of the stochastic billiard chain. Theorem 2.2 immediately follows from this proposition in conjunction with Corollary 1.5 from [9]. The constant c is different from the one in Theorem 2.2.

Proposition 2.3. *Under the assumptions of Theorem 2.2, the conductance ϕ of the stochastic billiard chain $\{X_k\}$ satisfies*

$$\phi \geq \frac{c}{\mathcal{C} n D},$$

for some universal constant c.

3 Single step analysis

This section studies the one-step distribution in detail. For $x \in \partial K$, define $F(x)$ through

$$P_x(y \in \partial K : |x - y| \leq F(x)) = \frac{1}{128}.$$

We think of $F(x)$ as the 'median' step size from x. The goals of this section are two-fold: (1) to establish a pointwise lower bound on F which does not depend on x and (2) to establish that the distribution of two points on the boundary ∂K overlap considerably if the points are sufficiently close. We discuss these two parts in Sections 3.1 and 3.2, respectively.

The proofs of these two parts proceed essentially independently, but the following auxiliary lemma is used in both parts.

Lemma 3.1. *For $A \subseteq \mathbb{R}_+$, we have*

$$\begin{aligned} \lim_{n\to\infty} P_x^{\cos}(\{y \in S_x : n_x'(y-x) \in A/\sqrt{n}\}) &= \int_A x \exp(-x^2/2) dx, \\ \lim_{n\to\infty} P_x^{\mathrm{unif}}(\{y \in S_x : n_x'(y-x) \in A/\sqrt{n}\}) &= \int_A \frac{1}{\sqrt{\pi/2}} \exp(-x^2/2) dx, \end{aligned}$$

and for $A \subseteq \mathbb{R}$, we have

$$\begin{aligned} \lim_{n\to\infty} \tilde{P}_x^{\cos}(\{y \in S_x : n_x'(y-x) \in A/\sqrt{n}\}) &= \int_A \frac{x}{2} \exp(-x^2/2) dx, \\ \lim_{n\to\infty} \tilde{P}_x^{\mathrm{unif}}(\{y \in S_x : n_x'(y-x) \in A/\sqrt{n}\}) &= \int_A \frac{1}{\sqrt{2\pi}} \exp(-x^2/2) dx. \end{aligned}$$

Proof. The key ingredient in the proof is the observation that an $(n-2)$-dimensional sphere with radius r has surface measure proportional to r^{n-2}. The radius of the $(n-2)$-dimensional sphere $\{y \in S_x : n_x'(y-x) = t\}$ is $\sqrt{1-t^2}$, so for the probability under $P_x^{\cos}$ we find that

$$\lim_{n\to\infty} \frac{\int_{A/\sqrt{n}} t(\sqrt{1-t^2})^{n-2} dt}{\int_0^\infty t(\sqrt{1-t^2})^{n-2} dt} = \frac{\int_A s \exp(-s^2/2) ds}{\int_0^\infty s \exp(-s^2/2) ds} = \int_A s \exp(-s^2/2) ds.$$

Similarly, for the probability under P_x^{unif} we find that

$$\lim_{n\to\infty} \frac{\int_{A/\sqrt{n}} (\sqrt{1-t^2})^{n-2} dt}{\int_0^\infty (\sqrt{1-t^2})^{n-2} dt} = \frac{\int_A s \exp(-s^2/2) ds}{\int_0^\infty \exp(-s^2/2) ds} = \int_A \frac{1}{\sqrt{\pi/2}} \exp(-s^2/2) ds.$$

The statements for $\tilde{P}_x^{\cos}$ and $\tilde{P}_x^{\mathrm{unif}}$ are found analogously. □

3.1 Lower bound on F

The main result of this subsection is the following lemma, which guarantees that the stochastic billiard Markov chain makes 'large enough' steps with good probability.

Lemma 3.2. *If K is a convex body with curvature bounded from above by $\mathcal{C} < \infty$, then there exists a constant c such that $F(x) \geq c/(\mathcal{C}\sqrt{n})$ for $x \in \partial K$.*

We prove this lemma by comparing F with a family of functions $\{s_\gamma : 0 \leq \gamma \leq 1\}$ that is easier to bound. As with F, one similarly interprets s_γ as a step size. For $1 \geq \gamma \geq 0$ and $x \in K$, it is defined through

$$s_\gamma(x) = \sup\left\{t \geq 0 \,:\, \frac{\mathrm{vol}((x + tB^n) \cap K)}{\mathrm{vol}(tB^n)} \geq \gamma\right\}.$$

We are now ready to formulate our comparison between F and s_γ.

Lemma 3.3. *For any $u \in \partial K$ and $1/2 - 1/2048 \leq \gamma \leq 1/2$, we have for large enough n,*

$$F(u) \geq \frac{s_\gamma(u)}{2}.$$

Proof. Fix $u \in \partial K$, and write $s = s_\gamma(u)$. Let $T \subseteq S_u$ be given by

$$T = \{u + y : u + (s/2)y \in K, y \in S^{n-1}\}.$$

Writing $p = \tilde{P}_u^{\mathrm{unif}}(S_u \backslash T)$ for the fraction of $u + (s/2)S^{n-1}$ that lies outside K, we find by convexity of K that

$$\mathrm{vol}((u + sB^n) \setminus K) \geq p\mathrm{vol}(sB^n) - \mathrm{vol}\left(\frac{s}{2}B^n\right).$$

By definition of s, we also have $\mathrm{vol}((u + sB^n) \setminus K) \leq (1 - \gamma)\mathrm{vol}(sB^n)$. Since $\mathrm{vol}(\alpha B^n) = \alpha^n \mathrm{vol}(B^n)$, we deduce that for large enough n and $\gamma \geq 1023/2048$ that

$$p \leq 1 - \gamma + \frac{1}{2^n} \leq \frac{513}{1024}.$$

Note that, by definition of p,

$$P_u^{\mathrm{unif}}(S_u \backslash T) = 1 - 2(1 - p) \leq 1/512.$$

We next argue that $P_u^{\cos}(S_u \backslash T) \leq 1/128$, so that $F(u) \geq s/2$ then follows from

$$P_u\left(y \in \partial K : |u - y| < \frac{s}{2}\right) = 1 - P_u^{\cos}(T) \leq \frac{1}{128}.$$

To show that $P_u^{\cos}(S_u \backslash T) \leq 1/128$, we use a change-of-measure argument. Let $f^{\cos}$ and f^{unif} be the densities on S_u of $P^{\cos}$ and P^{unif}, respectively, and write E_u^{unif} for the expectation operator of P^{unif}. We let X denote the random variable given by $X(\omega) = \omega$ for $\omega \in S_u$. After noting that

$$\frac{f^{\cos}(y)}{f^{\mathrm{unif}}(y)} = \frac{\cos(\phi_{uy})}{E_u^{\mathrm{unif}}[n_u' X]},$$

we write

$$
\begin{aligned}
P_u^{\cos}(S_u\backslash T) &= E_u^{\text{unif}}\left[\frac{f^{\cos}(X)}{f^{\text{unif}}(X)}; X \in S_u\backslash T\right] \\
&\leq \sup_{A:P_u^{\text{unif}}(A)\leq 1/512} E_u^{\text{unif}}\left[\frac{f^{\cos}(X)}{f^{\text{unif}}(X)}; X \in A\right] \\
&= \sup_{A:P_u^{\text{unif}}(A)\leq 1/512} \frac{E_u^{\text{unif}}[\cos(\phi_{uX}); X \in A]}{E_u^{\text{unif}}[n_u' X]} \\
&= \frac{E_u^{\text{unif}}[\cos(\phi_{uX}); X \in C^{1/512}]}{E_u^{\text{unif}}[n_u' X]},
\end{aligned}
$$

where $C^{1/512} = \{x \in S_u : n_u'(x-u) \geq c_n'/\sqrt{n}\}$ is the cap of S_u in 'direction' n_u, where c_n' is chosen so that $P_u^{\text{unif}}(C^{1/512}) = 1/512$. Note that $c_n' \to \Psi^{-1}(1/1024)$ as $n \to \infty$ by Lemma 3.1. The asymptotic behavior as $n \to \infty$ of the denominator is readily found:

$$
E_u^{\text{unif}}[n_u' X] = \int_0^1 \sqrt{1-r^2}\,(n-1)r^{n-2}dr = \frac{\sqrt{\pi}(n-1)\Gamma((n-1)/2)}{4\Gamma(n/2+1)} \sim \sqrt{\frac{\pi}{2n}}.
$$

As for the numerator, we find by again applying Lemma 3.1 that

$$
E_u^{\text{unif}}[\cos(\phi_{uX}); X \in C^{1/512}] \sim \frac{1}{\sqrt{n}} \int_{\Psi^{-1}(1/1024)}^{\infty} s\exp(-s^2/2)ds = \frac{\exp(-\Psi^{-1}(1/1024)^2/2)}{\sqrt{n}}.
$$

Since $\sqrt{2/\pi}\exp(-\Psi^{-1}(1/1024)^2/2) < 1/128$, we find that for large n,

$$
P_u^{\cos}(S_u\backslash T) \leq 1/128,
$$

which proves the claim. □

Lemma 3.2 follows by combining the preceding lemma with the following result, which establishes a lower bound on s_γ.

Lemma 3.4. *Let $\gamma \in (0, 1/2)$. If K is a convex body with curvature bounded from above by $\mathcal{C} < \infty$, then*

$$
s_\gamma(x) \geq \frac{c_\gamma}{\mathcal{C}\sqrt{n}},
$$

where c_γ is a constant only depending on γ.

Proof. Fix $x \in \partial K$. In view of the definition of s_γ, it suffices to show that

$$
\text{vol}((x+tB)\cap K) \geq \gamma \text{vol}(tB),
$$

for $t = c_\gamma/(\mathcal{C}\sqrt{n})$.

Let $B^{\mathcal{C}}$ be a ball with radius $1/\mathcal{C}$ and center in K so that the tangent planes of K and $B^{\mathcal{C}}$ at x coincide and $B^{\mathcal{C}}$ lies in K. Let B_t be a ball with radius $t < 1/\mathcal{C}$ centered at x. Let x be the origin of a new coordinate system, in which the center of the larger ball is $(1/\mathcal{C}, 0, \ldots, 0)$.

We first characterize the points (in the new coordinate system) where the boundaries of the two balls intersect. All of these points have the same first coordinate, namely equal to y satisfying

$$
t^2 - y^2 = \frac{1}{\mathcal{C}^2} - \left(\frac{1}{\mathcal{C}} - y\right)^2,
$$

and the solution is $y = \mathcal{C}t^2/2$, and we write $H_{\mathcal{C}t^2/2}$ for the halfspace of all points with first coordinate exceeding y.

We now use the property that, for a unit ball B, the volume of all points with first coordinate exceeding $1/\sqrt{n}$ takes up a constant fraction of its volume. Consequently, if $t = c_\gamma/(\mathcal{C}\sqrt{n})$ for an appropriate constant c_γ, then $\mathcal{C}t^2/2 = c_\gamma t/(2\sqrt{n})$ and therefore $\mathrm{vol}(B_t \cap H_{\mathcal{C}t^2/2}) = \gamma \mathrm{vol}(B_t)$. Upon noting that, for this choice of the radius t,

$$\mathrm{vol}((x + tB) \cap K) \geq \mathrm{vol}(B_t \cap H_{\mathcal{C}t^2/2}) = \gamma \mathrm{vol}(B_t) = \gamma \mathrm{vol}(tB),$$

we obtain the claim. □

3.2 Overlap for points that are close

It is the aim of this subsection to prove the following lemma, which states that the transition probabilities for points that are sufficiently close must be similar. This is formalized as having (total variation) 'overlap' at least $1/\kappa > 0$, for some κ.

Lemma 3.5. *Let $u, v \in \partial K$, and let n be large. If*

$$|u - v| < \frac{1}{100\sqrt{n}} \max(F(u), F(v)),$$

then we have

$$\|P_u - P_v\|_{TV} \leq 1 - \frac{1}{\kappa},$$

where κ is an absolute constant determined in the proof.

Proof. Let $u, v \in \partial K$ be as in the hypothesis of the lemma with $F(u) \geq F(v)$. Our main idea is to compare the transition densities of from u and v on a set of full measure. We first introduce four subsets $A_1, \ldots, A_4$ of ∂K which we exclude from this comparison.

The set A_1. Let A_1 be the subset of ∂K that is close to u, i.e.,

$$A_1 = \{x \in \partial K \,:\, |u - x| \leq F(u)\}.$$

Note that $P_u(A_1) = 1/128$ by definition of $F(u)$.

The set A_2. Next we define A_2 as the subset of points that are far from being orthogonal to $[u, v]$, the line through u and v (interpreting u as the origin):

$$A_2 = \left\{ x \in \partial K \,:\, |(x - u)^T (v - u)| \geq \frac{3}{\sqrt{n}} |x - u||v - u| \right\}.$$

Claim 3.6. *We have $P_u(A_2) \leq 1/64$.*

To prove this claim, consider the two-dimensional plane consisting of u, v, and n_u. Recall that S_u stands for the unit sphere centered at u. Define $C_y = \{x \in S_u : |y'(x - u)|/|y| \geq 3/\sqrt{n}\}$ as the union of two caps centered around the line determined by y. Note that $x \in C_v$ if and only if either $x \in A_2$ or $-x \in A_2$, and that the latter are mutually exclusive statements. Thus we have $P_u(A_2) = \tilde{P}_u^{\cos}(C_v) \leq \sup_{y \in S_u} \tilde{P}_u^{\cos}(C_y)$, and the supremum is attained for $y = n_u$ due to the form of the density of $\tilde{P}_u^{\cos}$. Therefore, as $n \to \infty$, we have by Lemma 3.1 that

$$P_u(A_2) = \tilde{P}_u^{\cos}(C_{n_u}) \to \int_3^\infty s \exp(-s^2/2) ds = \exp(-3^2/2).$$

The right-hand side is less than 1/64.

The set A_3**.** Let A_3 be given by

$$A_3 = \left\{x \in \partial K\ : \sqrt{n}\cos(\phi_{ux}) \notin (c_1, c_2)\right\} \cup \left\{x \in \partial K\ : \sqrt{n}\cos(\phi_{vx}) \notin (c_1, c_2)\right\},$$

where c_1 and c_2 satisfy

$$\begin{aligned} P_u(x \in \partial K\ : \sqrt{n}\cos(\phi_{ux}) < c_1) &= 1/64 \\ P_u(x \in \partial K\ : \sqrt{n}\cos(\phi_{ux}) > c_2) &\leq 1/64. \end{aligned}$$

Note that the probabilities on the left-hand side are equal to $1 - \exp(-c_1^2/2)$ and $\exp(-c_2^2/2)$, respectively, by Lemma 3.1. We can thus set $c_1 = \sqrt{-2\log(63/64)} \approx 0.18$ and $c_2 = \sqrt{-2\log(1/64)} \approx 2.88$.

Claim 3.7. *We have* $P_u(A_1^c \cap A_3) \leq 30/64$ *and therefore* $P_u(A_1 \cup A_3) \leq 61/128$.

By definition of c_1 and c_2, it suffices to show that $P_u(x \in A_1^c : \sqrt{n}\cos(\phi_{vx}) \notin (c_1, c_2)) \leq 28/64$. We first show that $P_u(x \in A_1^c : \sqrt{n}\cos(\phi_{vx}) > c_2) \leq \exp(-(c_2 - 1/100)^2/2)$. Since $\cos(\phi_{vx}) = n_v'(x-v)/|x-v|$, we have to bound $P_u(x \in A_1^c : n_v'(x-v) > c_2|x-v|/\sqrt{n})$. The key ingredient is the following observation: if $n_v'(x-v) \geq c_2|x-v|/\sqrt{n}$ and $|x-u| > F(u)$, then

$$n_v'(x-u) \geq \frac{c_2}{\sqrt{n}}|x-v| + n_v'(v-u) \geq \frac{c_2}{\sqrt{n}}|x-u| - \left(1 + \frac{c_2}{\sqrt{n}}\right)|v-u| > \left(\frac{c_2 - 1/100}{\sqrt{n}} - \frac{c_2}{100n}\right)|x-u|,$$

where the last inequality uses $|v-u| < F(u)/(100\sqrt{n}) \leq |x-u|/(100\sqrt{n})$, which holds since $x \in A_1^c$. We thus deduce that, for large enough n,

$$P_u(x \in A_1^c : n_v'(x-v) > c_2|x-v|/\sqrt{n}) \leq P_u\left(x \in \partial K : n_v'(x-u) > \left(\frac{c_2 - 1/50}{\sqrt{n}}\right)|x-u|\right). \quad (2)$$

We now bound this probability. For a unit vector y, write $C_y' = \{x \in S_u : y'(x-u) \geq (c_2 - 1/50)/\sqrt{n}\}$, which is a cap of S_u with 'center' y. The right-hand side of (2) equals $P_u^{\cos}(C_{n_v}')$. Due to the form of the density of $P_u^{\cos}$, we have $P_u^{\cos}(C_{n_v}') \leq \sup_{y \in S_u} P_u^{\cos}(C_y') = P_u^{\cos}(C_{n_u}') \to \exp(-(c_2 - 1/50)^2/2)$.

We next bound $P_u(x \in A_1^c : \sqrt{n}\cos(\phi_{vx}) < c_1)$, which is equal to

$$P_u(x \in A_1^c : 0 \leq n_v'(x-v) \leq c_1|x-v|/\sqrt{n}).$$

We use a similar argument as before. If $x \in A_1^c$ and $0 \leq n_v'(x-v) \leq c_1|x-v|/\sqrt{n}$, we have

$$n_v'(x-u) \geq n_v'(v-u) \geq -|v-u| > -|x-u|/(100\sqrt{n})$$

and

$$n_v'(x-u) \leq \frac{c_1}{\sqrt{n}}|x-v| + n_v'(v-u) \leq \frac{c_1}{\sqrt{n}}|x-u| + \left(1 + \frac{c_1}{\sqrt{n}}\right)|v-u| < \left(\frac{c_1 + 1/50}{\sqrt{n}}\right)|x-u|.$$

For a unit vector y, write $C_y'' = \{x \in S_u : -1/(100\sqrt{n}) \leq y'(x-u) \leq (c_1 + 1/50)/\sqrt{n}\}$. We have now shown that

$$P_u(x \in A_1^c : \sqrt{n}\cos(\phi_{vx}) < c_1) \leq P_u^{\cos}(C_{n_v}'').$$

Note that by Lemma 3.1, we have

$$\tilde{P}_u^{\text{unif}}(C_{n_v}'') \to \frac{1}{\sqrt{2\pi}}\int_{-1/100}^{c_1 + 1/50} \exp(-y^2/2)dy.$$

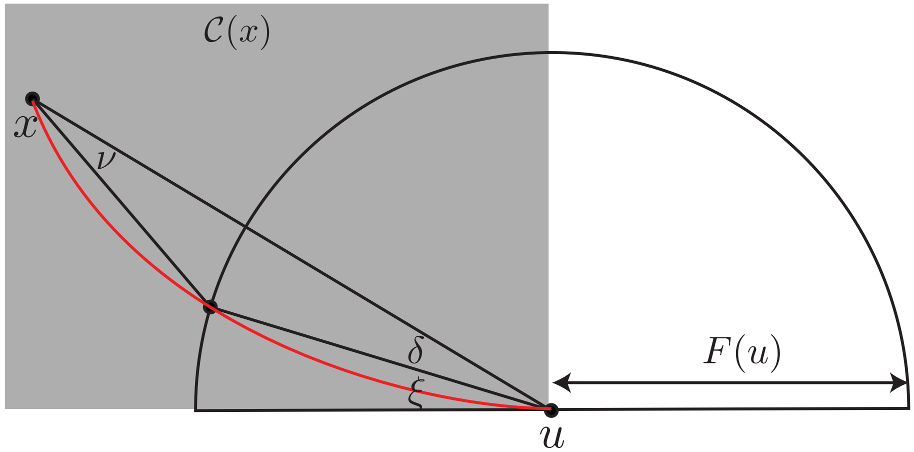

FIGURE 3: Incidence angles and the cone $\mathcal{C}(x)$. Part of the boundary ∂K is depicted in red.

Call the ratio on the right-hand side ρ, and we find that $\rho \approx 0.082$. Application of Lemma 3.1 (twice) yields

$$P_u^{\cos}(C''_{n_v}) \leq \sup_{D:\tilde{P}_u^{\text{unif}}(D)=\rho} P_u^{\cos}(D) = P_u^{\cos}(\{x \in S_u : n_u'(x-u) \geq \Psi^{-1}(\rho)/\sqrt{n}\}) = \exp(-\Psi^{-1}(\rho)^2/2),$$

where $\Psi(x) = \int_x^\infty \exp(-y^2/2)dy/\sqrt{2\pi}$. We conclude that

$$\limsup_{n\to\infty} P_u(A_1^c \cap A_3) \leq \frac{2}{64} + \exp(-(c_2 - 1/50)^2/2) + \exp(-\Psi^{-1}(\rho)^2/2).$$

It is readily verified that the right hand side approximately equals 0.40, and the first part of Claim 3.7 follows. For the second part, note that $P_u(A_1 \cup A_3) \leq P(A_1) + P(A_1^c \cap A_3)$.

The set A_4**.** For the following argument, we interpret u as the origin of our coordinate system, so that (for instance) cones are defined with respect to u. For $x \in \partial K$, let $\mathcal{C}(x)$ be the cone generated by the orthogonal projection of x on the hyperplane $\{z : n_u'(z-u) = 0\}$ and the normal $u + n_u$ at u. Write $\xi(x)$ for the angle between the point in $\mathcal{C}(x) \cap (u + F(u)B^n) \cap \partial K$ and the aforementioned hyperplane, see Figure 3. Write

$$A_4 = \{x \in \partial K : \xi(x) \geq c_1/\sqrt{n}\}$$

and $B = \{x \in \partial K : \sqrt{n}\cos(\phi_{ux}) < c_1\}$. Since $P_u(B) = 1/64$ and $P_u(A_1) = 1/128$, we find that the conditional probability $P_u(A_1^c|B)$ is at least $1 - P_u(A_1)/P_u(B) = 1/2$.

The angle $\xi(x)$ is only a function of x through $\mathcal{C}(x)$, i.e., $\xi(x)$ is determined once $\mathcal{C}(x)$ is given. Interpreting $\mathcal{C}$ and ξ as random variables on the sample space ∂K, the distribution of $\mathcal{C}$ under P_u is the uniform distribution over all such cones. Since the distribution of $\mathcal{C}$ under $P_u(\cdot|B)$ is also uniform, the distribution of ξ is the same under P_u and under $P_u(\cdot|B)$. We conclude that

$$P_u(A_4^c) = P_u(\xi < c_1/\sqrt{n}) = P_u(\xi < c_1/\sqrt{n}|B) \geq P_u(A_1^c|B) \geq 1/2,$$

so that $P_u(A_4) \leq 1/2$.

A set on which P_v **majorizes** P_v**.** Let $A = \partial K \setminus A_1 \setminus A_2 \setminus A_3 \setminus A_4$. Then we have

$$P_u(A) \geq 1 - \frac{1}{128} - \frac{1}{64} - \frac{30}{64} - \frac{1}{2} = \frac{1}{128}.$$

We will show that for any subset $S \subseteq A$, we have

$$P_v(S) \geq \frac{128 P_u(S)}{\kappa}, \tag{3}$$

where κ is some positive constant. This implies that for any subset S of ∂K, we have

$$\begin{aligned}
P_u(S) - P_v(S) &\leq P_u(S) - P_v(S \setminus A_1 \setminus A_2 \setminus A_3 \setminus A_4) \\
&\leq P_u(S) - \frac{128}{\kappa} P_u(S \setminus A_1 \setminus A_2 \setminus A_3 \setminus A_4) \\
&\leq P_u(S) - \frac{128}{\kappa}[P_u(S) - P_u(A_1 \cup A_2 \cup A_3 \cup A_4)] \\
&\leq P_u(S) - \frac{128}{\kappa}[P_u(S) - 127/128] \\
&\leq 1 - \frac{1}{\kappa},
\end{aligned}$$

and therefore we obtain the conclusion of the lemma from (3).

We prove (3) using the formula for the one-step distribution from v as given in (1):

$$P_v(S) = \frac{\pi^{(n-1)/2}}{|\partial K| \Gamma((n+1)/2)} \int_S \frac{\cos(\phi_{vx}) \cos(\phi_{xv})}{|v-x|^{n-1}} dx,$$

and we compare the three terms in the integrand with the corresponding quantities for v replaced with u. This rests on the following three claims for $x \in A$, which show that we can take $\kappa > 0$ to satisfy

$$\frac{128}{\kappa} = e^{-7/2} \frac{c_1}{c_2} \left(1 - \frac{1}{100(c_2 - c_1)}\right).$$

Claim 3.8. *For $x \in A$, we have*

$$|v - x| \leq \left(1 + \frac{7}{2n}\right) |u - x|.$$

To prove Claim 3.8, we note that for $x \in A$,

$$|x - u| \geq F(u) \geq \sqrt{n} |u - v|$$

and

$$|(x-u)^T (v-u)| \leq \frac{3}{\sqrt{n}} |x-u||v-u|.$$

Using these, we deduce that

$$\begin{aligned}
|x - v|^2 &= |x-u|^2 + |u-v|^2 + 2(x-u)^T(u-v) \\
&\leq |x-u|^2 + |u-v|^2 + \frac{6}{\sqrt{n}} |x-u||u-v| \\
&\leq |x-u|^2 + \frac{1}{n}|x-u|^2 + \frac{6}{n}|x-u|^2 \\
&\leq \left(1 + \frac{7}{n}\right) |x-u|^2,
\end{aligned}$$

which completes the proof of Claim 3.8.

Claim 3.9. *For $x \in A$, we have*

$$\frac{\cos(\phi_{vx})}{\cos(\phi_{ux})} \geq \frac{c_1}{c_2}.$$

Claim 3.9 immediately follows upon noting that for $x \in A$, since $x \notin A_3$, $\sqrt{n}\cos(\phi_{vx}) \geq c_1$ and $\sqrt{n}\cos(\phi_{ux}) \leq c_2$; therefore $\cos(\phi_{ux})$ and $\cos(\phi_{vx})$ are within a factor of c_2/c_1.

Claim 3.10. *For $x \in A$, we have*

$$\frac{\cos(\phi_{xv})}{\cos(\phi_{xu})} \geq 1 - \frac{1}{100(c_2 - c_1)}.$$

To prove Claim 3.10, we need to derive a lower bound on $\cos(\phi_{xv})/\cos(\phi_{xu})$. Fixing u, $\cos(\phi_{xv})/\cos(\phi_{xu})$ achieves its lowest possible value when v lies in $\mathcal{C}(x)$ with the highest possible angle with the inward normal at x. Henceforth we consider this case. Write $\alpha = \phi_{xv} - \phi_{xu}$, and note that $\alpha \leq 1/(100C\sqrt{n})$ since $|u - v| \leq F(u)/(100\sqrt{n})$.

Referring to Figure 3, we next argue that $\nu \geq (c_2 - c_1)/(C\sqrt{n})$. From the sine rule we get $\sin(\delta + \nu) = C\sin(\nu)$, so that $\cot(\nu) = (C - \cos(\delta))/\sin(\delta) \leq C/\sin(\delta)$. Since $x \in A$, we have $\delta \geq (c_2 - c_1)/\sqrt{n}$ and therefore $\tan(\nu) \geq (c_2 - c_1)/(C\sqrt{n})$ and thus $\nu \geq (c_2 - c_1)/(C\sqrt{n})$.

We conclude that

$$\frac{\cos(\phi_{xv})}{\cos(\phi_{xu})} \geq \frac{\sin(\nu - \alpha)}{\sin(\nu)} \geq \frac{\sin((c_2 - c_1 - 1/100)/(C\sqrt{n}))}{\sin((c_2 - c_1)/(C\sqrt{n}))} \geq 1 - \frac{1}{100(c_2 - c_1)} > 0,$$

where we use that $\sin(\nu - \alpha)/\sin(\nu)$ is increasing in ν and decreasing in α. Claim 3.10 follows.

This concludes the proof of Lemma 3.5. □

4 Conductance

It is the aim of this section to prove our conductance bound in Proposition 2.3. Apart from the single-step analysis of the previous section, a key ingredient is a certain isoperimetric inequality for the boundary of a convex body. Such inequalities have been studied for several decades, see for instance [18]. We need an 'integrated' form of this inequality, and we include a proof showing how this lemma follows from a classical isoperimetric inequality. For the state-of-the-art in this area, we refer to the recent work of E. Milman [13].

Lemma 4.1. *Let K be a convex body in $\mathbb{R}^n$. Suppose ∂K is partitioned into measurable sets S_1, S_2, S_3. We then have, for some constant $c > 0$,*

$$\mathrm{vol}(S_3) \geq \frac{c}{D}\, d(S_1, S_2) \min(\mathrm{vol}(S_1), \mathrm{vol}(S_2)),$$

where d denotes the geodesic distance on ∂K.

Proof. Recall the definition of the ϵ-extension A^ϵ of a set A with respect to the geodesic metric. Abusing notation, we write A^ϵ for $A^\epsilon \cap S$. μ^+ denotes Minkowski's exterior boundary measure, defined through $\mu^+(A) = \liminf_{\epsilon \downarrow 0}(|A^\epsilon| - |A|)/\epsilon$. The isoperimetric constant for manifolds with nonnegative Ricci curvature can be bounded by c/D for some constant c (e.g., [13]). This yields that, for any $A \subseteq S$,

$$\mu^+(A) \geq \frac{c}{D} \min(\mathrm{vol}(A), \mathrm{vol}(S) - \mathrm{vol}(A)).$$

For $\epsilon < d(S_1, S_2)$, the inequalities $\mathrm{vol}(S_1^\epsilon) \geq |S_1|$ and $\mathrm{vol}(S) - \mathrm{vol}(S_1^\epsilon) \geq \mathrm{vol}(S_2)$ imply that

$$\min(\mathrm{vol}(S_1^\epsilon), \mathrm{vol}(S) - \mathrm{vol}(S_1^\epsilon)) \geq \min(\mathrm{vol}(S_1), \mathrm{vol}(S_2)). \tag{4}$$

The function $x \mapsto |S_1^x|$ is nondecreasing and continuous on $(0, \infty)$. To see why it is continuous, let $x > 0$ and suppose without loss of generality that S_1^x contains an r-neighborhood B of the origin for some $r > 0$. Then $S_1^{x+\epsilon} = S_1^x + (\epsilon/r)B \subseteq (1 + \epsilon/r)S_1^x$, so that $|S_1^x| \leq |S_1^{x+\epsilon}| \leq (1 + \epsilon/r)^n |S_1^x|$. Consequently, we have for $x > 0$,

$$\begin{aligned}\mathrm{vol}(S_1^x) - \mathrm{vol}(S_1) &\geq \liminf_{\epsilon \downarrow 0} \left[\frac{1}{\epsilon}\int_x^{x+\epsilon} \mathrm{vol}(S_1^\eta) d\eta - \frac{1}{\epsilon}\int_0^\epsilon \mathrm{vol}(S_1^\eta) d\eta\right] \\ &= \liminf_{\epsilon \downarrow 0} \int_0^x \frac{1}{\epsilon}[\mathrm{vol}(S_1)^{\eta+\epsilon} - \mathrm{vol}(S_1)^\eta] d\eta \geq \int_0^x \mu^+(S_1^\eta) d\eta,\end{aligned}$$

where the last inequality follows from Fatou's lemma. Combining the above, we deduce from (4) that

$$\begin{aligned}\mathrm{vol}(S_3) &\geq \mathrm{vol}(S_1^{d(S_1,S_2)}) - \mathrm{vol}(S_1) \geq \int_0^{d(S_1,S_2)} \mu^+(S_1^\epsilon) d\epsilon \\ &\geq \frac{c}{D}\int_0^{d(S_1,S_2)} \min(\mathrm{vol}(S_1^\epsilon), \mathrm{vol}(S) - \mathrm{vol}(S_1^\epsilon)) d\epsilon \geq \frac{c}{D} d(S_1, S_2) \min(\mathrm{vol}(S_1), \mathrm{vol}(S_2)),\end{aligned}$$

as required. □

We are now ready to prove our conductance bound of Proposition 2.3, which concludes the proof of our main result.

Proof of Proposition 2.3. This part of the proof of Theorem 2.2 is quite standard, but we include details here for completeness.

Let $K = S_1 \cup S_2$ be a partition into measurable sets. We will prove that

$$\int_{S_1} P_x(S_2)\, dx \quad \geq \quad \frac{c}{\mathcal{C}nD} \min\{\mathrm{vol}(S_1), \mathrm{vol}(S_2)\}. \tag{5}$$

In this proof, the constant c can vary from line to line. The constant κ stands for the constant from Lemma 3.5. Consider the points that are deep inside these sets, i.e., unlikely to jump out of the set:

$$S_1' = \left\{x \in S_1 : P_x(S_2) < \frac{1}{2\kappa}\right\}, \qquad S_2' = \left\{x \in S_2 : P_x(S_1) < \frac{1}{2\kappa}\right\}.$$

Set $S_3' = K \setminus S_1' \setminus S_2'$.

Suppose $\mathrm{vol}(S_1') < \mathrm{vol}(S_1)/2$. Then

$$\int_{S_1} P_x(S_2)\, dx \geq \frac{1}{2\kappa}\mathrm{vol}(S_1 \setminus S_1') \geq \frac{1}{4\kappa}\mathrm{vol}(S_1)$$

which proves (5).

So we can assume that $\mathrm{vol}(S_1') \geq \mathrm{vol}(S_1)/2$ and similarly $\mathrm{vol}(S_2') \geq \mathrm{vol}(S_2)/2$. For any $u \in S_1'$ and $v \in S_2'$,

$$\|P_u - P_v\|_{\mathrm{TV}} \geq 1 - P_u(S_2) - P_v(S_1) > 1 - \frac{1}{\kappa}.$$

Thus, by Lemma 3.5, we must then have

$$d(u, v) \geq \frac{1}{100\sqrt{n}} \max\{F(u), F(v)\}.$$

In particular, we have $d(S_1', S_2') \geq \inf_{x \in \partial K} F(x)/(100\sqrt{n})$.

We next apply Lemma 4.1 to obtain

$$\begin{aligned} \frac{\mathrm{vol}(S_3')}{\min\{\mathrm{vol}(S_1'), \mathrm{vol}(S_2')\}} &\geq \frac{c}{D\sqrt{n}} \inf_{x \in \partial K} F(x) \\ &\geq \frac{c}{2D\sqrt{n}} \inf_{x \in \partial K} s_\gamma(x), \end{aligned}$$

where the last inequality follows from Lemma 3.3. By Lemma 3.4, this is bounded from below by $c/(\mathcal{C}nD)$. Therefore,

$$\begin{aligned} \int_{S_1} P_x(S_2)\, dx &\geq \frac{1}{2} \cdot \frac{1}{2\kappa} \mathrm{vol}(S_3') \\ &\geq \frac{c}{\mathcal{C}nD} \min\{\mathrm{vol}(S_1'), \mathrm{vol}(S_2')\} \\ &\geq \frac{c}{2\mathcal{C}nD} \min\{\mathrm{vol}(S_1), \mathrm{vol}(S_2)\} \end{aligned}$$

which again proves (5). □

References

[1] D. Applegate and R. Kannan, *Sampling and integration of near log-concave functions*, in STOC '91: Proceedings of the twenty-third annual ACM symposium on Theory of computing, New York, NY, USA, 1991, ACM, pp. 156–163.

[2] C. G. E. Boender, R. J. Caron, A. H. G. Rinnooy Kan, and et al., *Shake-and-bake algorithms for generating uniform points on the boundary of bounded polyhedra*, Oper. Res., 39 (1991), pp. 945–954.

[3] A. Boneh and A. Golan, *Constraints' redundancy and feasible region boundedness by random feasible point generator (RFPG)*, Third European Congress on Operations Research (EURO III), (1979).

[4] F. Comets, S. Popov, G. M. Schütz, and M. Vachkovskaia, *Billiards in a general domain with random reflections*, Arch. Ration. Mech. Anal., 191 (2009), pp. 497–537.

[5] M. E. Dyer, A. M. Frieze, and R. Kannan, *A random polynomial time algorithm for approximating the volume of convex bodies*, in STOC, 1989, pp. 375–381.

[6] S. N. Evans, *Stochastic billiards on general tables*, Ann. Appl. Probab., 11 (2001), pp. 419–437.

[7] S. Lalley and H. Robbins, *Stochastic search in a convex region*, Probab. Theory Related Fields, 77 (1988), pp. 99–116.

[8] L. Lovász, *Hit-and-run mixes fast*, Math. Prog, 86 (1998), pp. 443–461.

[9] L. Lovász and M. Simonovits, *Random walks in a convex body and an improved volume algorithm*, Random Structures Algorithms, 4 (1993), pp. 359–412.

[10] L. Lovász and S. Vempala, *Fast algorithms for logconcave functions: Sampling, rounding, integration and optimization*, in FOCS '06: Proceedings of the 47th Annual IEEE Symposium on Foundations of Computer Science, Washington, DC, USA, 2006, IEEE Computer Society, pp. 57–68.

[11] ———, *Hit-and-run from a corner*, SIAM J. Computing, 35 (2006), pp. 985–1005.

[12] ———, *Simulated annealing in convex bodies and an $O^*(n^4)$ volume algorithm*, J. Comput. Syst. Sci., 72 (2006), pp. 392–417.

[13] E. Milman, *Isoperimetric bounds on convex manifolds*, in Concentration, functional inequalities and isoperimetry, vol. 545 of Contemp. Math., Amer. Math. Soc., Providence, RI, 2011, pp. 195–208.

[14] H. E. Romeijn, *Shake-and-bake algorithms for the identification of nonredundant linear inequalities*, Statist. Neerlandica, 45 (1991), pp. 31–50.

[15] ———, *A general framework for approximate sampling with an application to generating points on the boundary of bounded convex regions*, Statist. Neerlandica, 52 (1998), pp. 42–59.

[16] R. L. Smith, *Efficient Monte Carlo procedures for generating points uniformly distributed over bounded regions*, Oper. Res., 32 (1984), pp. 1296–1308.

[17] S. Vempala, *Geometric random walks: A survey*, MSRI Combinatorial and Computational Geometry, 52 (2005), pp. 573–612.

[18] S. T. Yau, *Isoperimetric constants and the first eigenvalue of a compact Riemannian manifold*, Ann. Sci. École Norm. Sup. (4), 8 (1975), pp. 487–507.