Problem 1

Suppose:

\[
A = \begin{pmatrix}
-2 & 1 \\
2 & -2
\end{pmatrix}
\]  

(a) Write the characteristic polynomial for \( A \).

(b) Find the eigenvalues of \( A \).

(c) Find the eigenvectors of \( A \) corresponding to the eigenvalues. Show your work.

(d) Sketch a phase plane picture of the system \( x' = Ax \) like we did in class. Include the eigenvectors, the direction of the solutions along the lines parallel to the eigenvectors, and a few representative solutions in between.

(e) Classify the system as a nodal sink, nodal source, center, spiral sink, spiral source, line of equilibria, or saddle.

Problem 2

Suppose:

\[
A = \begin{pmatrix}
1 & 2 \\
1 & -1
\end{pmatrix}
\]  

(a) Write the characteristic polynomial for \( A \).
(b) Find the eigenvalues of $A$.

(c) Find the eigenvectors of $A$ corresponding to the eigenvalues.

(d) Sketch a phase plane picture of the system $\mathbf{x}' = A\mathbf{x}$ like we did in class. Include the eigenvectors, the direction of the solutions along the lines parallel to the eigenvectors, and a few representative solutions in between.

(e) Classify the system as a nodal sink, nodal source, center, spiral sink, spiral source, line of equilibria, or saddle.

**Problem 3**

Suppose:

$$A = \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}$$

(a) Write the characteristic polynomial for $A$.

(b) Find the eigenvalues of $A$.

(c) Find the eigenvectors of $A$ corresponding to the eigenvalues.

(d) Sketch a phase plane picture of the system $\mathbf{x}' = A\mathbf{x}$ like we did in class. Include the eigenvectors, the direction of the solutions along the lines parallel to the eigenvectors, and a few representative solutions in between.

(e) Classify the system as a nodal sink, nodal source, center, spiral sink, spiral source, line of equilibria, or saddle.

**Problem 4**

Let’s return to the two-population model from a few lectures ago. We will let $r(t)$ be the number of rabbits and $f(t)$ the number of foxes. Note that I’ve switched $c$ and $d$ from the lecture to make things cleaner.

Suppose:

$$r' = ar - bf$$
$$f' = cr + df$$

(a) Write this in matrix-vector notation: $\mathbf{x}' = A\mathbf{x}$. 

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(b) Suppose $a = 1, c = 1, d = -1$. Assume that $b$ is an unknown positive number. $b$ can be interpreted as how much the foxes suppress the rabbit population growth.

(i) For what values of $b$ will the population of foxes and rabbits oscillate?

(ii) Now suppose $b = 1/4$. Find the eigenvalues and eigenvectors. Using these, write the general solution to the problem in the form $C_1x_1(t) + C_2x_2(t)$. As $t \to \infty$, will there be more foxes or more rabbits (assuming $C_1 \neq 0, C_2 \neq 0$)?