Credit risk models

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Overview of credit risk analytics

Single-obligor credit risk models
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Credit risk metrics and models
Intensity models and default time analytics

Single-obligor credit risk models
Key metrics of credit risk

**Probability of default** $\pi_t$ defined over a time horizon $t$, e.g. one year

**Exposure at default**: amount the lender can lose in default
- For a loan or bond, par value plus accrued interest
- For *OTC derivatives*, also driven by market value
  - Net present value (NPV) $\geq 0$ (→counterparty risk)
  - But exposure at default $\geq 0$

**Recovery**: creditor generally loses fraction of exposure $R < 100$ percent

**Loss given default** (LGD) equals exposure minus recovery (a fraction $1 - R$)

**Expected loss** (EL) equals default probability $\times$ LGD or fraction $\pi_t \times (1 - R)$
- Credit risk management focuses on *unexpected loss*

**Credit Value-at-Risk** related to a quantile of the credit return distribution
- Differs from market risk in *excluding* EL
- Credit VaR at confidence level of $\alpha$ defined as:

$$1 - \alpha$$-quantile of credit loss distribution $-$ EL
Estimating default probabilities

**Risk-neutral default probabilities** based on market prices, esp. credit spreads
- Data sources include credit-risky securities and CDS
- Risk-neutral default probabilities may incorporate risk premiums
- Used primarily for market-consistent pricing

**Physical default probabilities** based on fundamental analysis
- Based on historical default frequencies, scenario analysis, or credit model
- Associated with credit ratings
- Used primarily for risk measurement
Types of credit models

Differ on inputs, on what is to be derived, and on assumptions:

**Structural models** or **fundamental models** model default, derive measures of credit risk from fundamental data
- Firm’s balance sheet: volumes of assets and debt
- Standard is the **Merton default model**

**Reduced-form models** or **intensity models** take estimates of default probability or LGD as inputs
- Often used to simulate default times as one step in portfolio credit risk modeling
- Often risk-neutral
- Common example: **copula models**

**Factor models:** company, industry, economy-wide fundamentals, but highly schematized, lends itself to portfolio risk modeling.

Some models fall into several of these categories
What risks are we modeling?

**Credit risk:** models are said to operate in
- **Migration mode** taking into account credit migration as well as default, or
- **Default mode** taking into account default only

**Spread risk:** credit-risk related market risk
Rating migration rates, 1920–2016

<table>
<thead>
<tr>
<th>From/To:</th>
<th>Aaa</th>
<th>Aa</th>
<th>A</th>
<th>Baa</th>
<th>Ba</th>
<th>B</th>
<th>Caa</th>
<th>Ca–C</th>
<th>WR</th>
<th>Default</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aaa</td>
<td>86.7</td>
<td>7.8</td>
<td>0.8</td>
<td>0.2</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>4.4</td>
<td>0.0</td>
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<tr>
<td>Aa</td>
<td>1.1</td>
<td>84.2</td>
<td>7.6</td>
<td>0.7</td>
<td>0.2</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>6.1</td>
<td>0.1</td>
</tr>
<tr>
<td>A</td>
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<td>2.7</td>
<td>85.0</td>
<td>5.6</td>
<td>0.6</td>
<td>0.1</td>
<td>0.0</td>
<td>0.0</td>
<td>5.7</td>
<td>0.1</td>
</tr>
<tr>
<td>Baa</td>
<td>0.0</td>
<td>0.2</td>
<td>4.3</td>
<td>82.7</td>
<td>4.6</td>
<td>0.7</td>
<td>0.1</td>
<td>0.0</td>
<td>7.0</td>
<td>0.3</td>
</tr>
<tr>
<td>Ba</td>
<td>0.0</td>
<td>0.1</td>
<td>0.5</td>
<td>6.1</td>
<td>73.9</td>
<td>6.9</td>
<td>0.7</td>
<td>0.1</td>
<td>10.6</td>
<td>1.2</td>
</tr>
<tr>
<td>B</td>
<td>0.0</td>
<td>0.0</td>
<td>0.2</td>
<td>0.6</td>
<td>5.6</td>
<td>71.7</td>
<td>6.2</td>
<td>0.5</td>
<td>11.9</td>
<td>3.3</td>
</tr>
<tr>
<td>Caa</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.1</td>
<td>0.6</td>
<td>6.9</td>
<td>67.3</td>
<td>2.9</td>
<td>13.7</td>
<td>8.4</td>
</tr>
<tr>
<td>Ca–C</td>
<td>0.0</td>
<td>0.0</td>
<td>0.1</td>
<td>0.0</td>
<td>0.6</td>
<td>3.0</td>
<td>8.0</td>
<td>48.4</td>
<td>18.7</td>
<td>21.1</td>
</tr>
</tbody>
</table>

Average one-year letter rating migration rates, 1920-2016, percent. Each row shows the probability of starting the year with the rating in row heading and ending with the rating in the column heading. “WR” denotes withdrawn rating. *Source:* Moody’s Investor Service.
Modeling default time

- Occurrence of default event for single company over discrete time horizon $t$ can be modeled as Bernoulli distribution
  - Default occurrence the random variable
- Alternatively: default intensity models, model the time—specific instant $\tau$—at which default occurs
  - Default time the random variable
  - Look out over horizon from now (time 0) to time $t$
  - Default probability over $[0, t)$: $P[0 \leq \theta < t]$, firm defaults by time $t$
- Example of jump or Poisson process with exactly one jump possible
- Default probability increases as horizon grows longer
- Every firm defaults eventually: $\lim_{t \to \infty} P[0 \leq \tau < t] = 1$
- Survival probability is $1 - P[0 \leq \tau < t]$: Obligor remains solvent until at least time $t$
Default time distributions

- Default probabilities can be expressed through **cumulative default time distribution function**

- Simple form: \( P[0 \leq \tau < t] = 1 - e^{-\lambda t} \)
  - Survival probability is \( e^{-\lambda t} \)
  - If \( t \) expressed in years, 1-year default probability is \( 1 - e^{-\lambda} \)

- Corresponding p.d.f. is derivative w.r.t. \( t \): \( \lambda e^{-\lambda t} \)
  - For tiny time interval \( dt \)

\[
P[t \leq \tau < t + dt] \approx \lambda e^{-\lambda t}
\]
Hazard rates

- Default can only occur once \( \Rightarrow \)

\[
P[t \leq \tau < t + dt] = P[t \leq \tau < t + dt \cup t < \tau]
\]

- Conditional probability of default, given it has not occurred before \( t \):

\[
P[t \leq \tau < t + dt \mid t < \tau] = \frac{P[t \leq \tau < t + dt \cup t < \tau]}{P[t < \tau]}
\]

\[
= \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}}
\]

\[
= \lambda
\]

- \( \lambda \) called **hazard rate** or **default intensity**
  - Viewed from time 0, probability of default over \( dt \) is \( \lambda dt \)
  - \( \lambda \) can be modeled as a constant or as changing over time
  - In insurance, **force of mortality**, probability of death of a population member over next short time interval
Conditional default probability

- **Conditional default probability**: probability of default over a future time horizon, *given* no default before then
- With constant hazard rate:
  - Unconditional one-year default probability lower for more remote years
  - But time to default *memoryless*: *if* no default occurs next year, probability of default over subsequent year is same as next year
- \( \lambda \): *instantaneous* conditional default probability
  - Probability of default over next instant, given no prior default
### Default probability analytics: example

<table>
<thead>
<tr>
<th>Hazard rate</th>
<th>$\lambda$</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-yr. default probability</td>
<td>$1 - e^{-\lambda}$</td>
<td>0.1393</td>
</tr>
<tr>
<td>2-yr. default probability</td>
<td>$1 - e^{-2\lambda}$</td>
<td>0.2592</td>
</tr>
<tr>
<td>1-yr. survival probability</td>
<td>$e^{-\lambda}$</td>
<td>0.8607</td>
</tr>
<tr>
<td>1-yr. conditional default probability</td>
<td>$1 - e^{-\lambda}$</td>
<td>0.1393</td>
</tr>
</tbody>
</table>

- Hazard rate $\lambda = 0.15$
- 1-yr. default probability $1 - e^{-0.15} = 0.1393$
- 2-yr. default probability $1 - e^{-2 	imes 0.15} = 0.2592$
- 1-yr. survival probability $e^{-0.15} = 0.8607$
- 1-yr. conditional default probability $1 - e^{-0.15} = 0.1393$
Default time distribution

Cumulative default time distribution function $\pi_t$, constant hazard rate $\lambda = 0.15$, $t$ measured in years, $\pi_t$ and $\lambda$ at an annual rate.
Overview of credit risk analytics

**Single-obligor credit risk models**
- Merton default model
- Single-factor model
- Conditional independence in the single-factor model
Merton model: overview

- Widely-used structural model based on fluctuations in debt-issuing firm’s asset value
- Default occurs when asset value falls below default threshold, at which
  - Equity value extremely low or zero
  - Asset value close to par value of debt (plus accrued interest)
- Simplest version:
  - Default occurs when equity value hits zero
  - Default threshold equals par value of debt (plus accrued interest)
Equity and debt as options

- Assets assumed to display return volatility, so can apply option-pricing theory
- Equity can be viewed as a long call on the firm’s assets, with a strike price equal to the par value of the debt
- Debt can be viewed as a portfolio:
  - A riskless bond with the same par value as the debt
  - Plus an implicit short put on the firm’s assets, with a strike price equal to the par value of the debt
- If the lender bought back the short put, it would immunize itself against credit risk
  - \( \Rightarrow \) The value of the implicit short a measure of credit risk
Credit risk models

Merton default model

Left: 15 daily-frequency sample paths of the geometric Brownian motion process of the firm’s assets with a drift of 15 percent and an annual volatility of 25 percent, starting from a current value of 145. Right: probability density of the firm’s asset value on the maturity date, one year hence, of the debt. The grid line represents the debt’s par value (100) plus accrued interest at 8 percent.
Applying the Merton default model

- Immediate consequence: higher volatility (risk) benefits equity at expense of debtholders
- Model can be used to compute credit spread, expected recovery rate
- Two ways to frame model, depending on how mean of underlying return process interpreted
  - **Risk-neutral default probability**: expected value equal to firm’s dividend rate
  - **Physical default probability**: expected value equal to asset rate of return
- Model timing of default, compute default probability
- KMV Moody’s (and other practitioner applications):
  - Equity vol plus leverage → asset vol
  - Plus book value of liabilities → default threshold
  - Historical data + secret sauce to map into default frequency
Structure of single-factor model

- Basic similarity to Merton model
  - Default occurs when asset value falls below default threshold
- Asset returns depend on two random variables:
  Market risk factor $m$ affects all firms, but not in equal measure
  - Expresses influence of general business conditions, state of economy on default risk
  - Latent factor: not directly observed, but influences results indirectly via model parameters
  Idiosyncratic risk factor $\epsilon$ affects just one firm
  - Expresses influence of individual firm’s situation on default risk
- Fixed time horizon, e.g. one year
- Returns and shocks are measured as deviations from expectations or from a “neutral” state
- Most often used to model portfolio credit risk rather than single obligor
Parameters of single-factor model

**Default probability** \( \pi \) or, equivalently, default threshold \( k \)
- Combination of adverse market and idiosyncratic shocks sufficient to push borrower into default

**Correlation** \( \beta \) of asset return to market risk factor \( m \)
- High correlation implies strong influence of general business conditions on firm’s default risk
- Correlations of individual firms’ asset returns key driver of extent to which defaults of different firms coincide
- → Portfolio credit models and **default correlation**
Single-factor model: asset return behavior

- Merton model framework: default threshold is hit when firm’s asset return $r$ large and negative
- Asset return a function of market and idiosyncratic risk factors $m$, $\epsilon$:
  \[ r = \beta m + \sqrt{1 - \beta^2} \epsilon \]
- $\beta$: correlation between firm’s asset return and market factor $m$
- $m$ and $\epsilon$ uncorrelated standard normal variates:
  \[ m \sim \mathcal{N}(0, 1) \]
  \[ \epsilon \sim \mathcal{N}(0, 1) \]
  \[ \text{Cov}[m, \epsilon] = 0 \]
- Therefore $r$ is a standard normal variate, expressed in “volatility units”: $r \sim \mathcal{N}(0, 1)$, with
  - $r$ and $m$ expressed as deviations from neutral state of business cycle
  \[ \mathbb{E}[r] = 0 \]
  \[ \text{Var}[r] = \beta^2 + 1 - \beta^2 = 1 \]
Each panel shows a sequence of 100 simulations from the single-factor model. **Cyan** plot: returns on the market index $m$. **Purple** plot: associated returns $r = \beta m + \sqrt{1 - \beta^2} \epsilon$ on firm’s assets with the specified $\beta$ to the market. Plots are generated by simulating $m$ and $\epsilon$ as a pair of uncorrelated $\mathcal{N}(0, 1)$ variates, using the same random seed for both panels.
Single-factor model: default probability

- Default probability an assigned parameter
  - Rather than an output, as in the Merton model, the default probability is an input in the single-factor model
- Expressed via default threshold $k$ or distance-to-default
  - Default threshold a negative number, distance-to-default initially equal to $-k$
  - Default if $r$ negative and large enough to wipe out equity:
    $$\beta m + \sqrt{1 - \beta^2} \epsilon \leq k$$
  - Or, equivalently, distance-to-default $-k = |k|$
- Finding the initial default threshold: set $k$ to match stipulated default probability $\pi$ via
  $$\pi = P [r \leq k] \iff k = \Phi^{-1}(\pi),$$
  where $\Phi(\cdot)$ is the standard normal CDF
- Example:

<table>
<thead>
<tr>
<th>$\pi$</th>
<th>Distance-to-default ($-k$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>2.33</td>
</tr>
<tr>
<td>0.10</td>
<td>1.28</td>
</tr>
</tbody>
</table>
Single-factor model: default probability

Vertical grid lines mark the default threshold corresponding to default probabilities of 0.01 and 0.10.
Single-factor model and CAPM

- Single-factor model vs. CAPM beta
  - Since \( \text{Var}[r] = 1 \), \( \beta \) analogous to the correlation of market and firm, rather than CAPM beta
  - Relationship of asset rather than equity values to market factor
- Systematic and idiosyncratic risk: fraction of asset return variance explained by variances of
  - Market risk factor: \( \beta^2 \)
  - Idiosyncratic risk factors: \( 1 - \beta^2 \)
- Example:

<table>
<thead>
<tr>
<th></th>
<th>( \beta = 0.40 )</th>
<th>( \beta = 0.90 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Market factor ( \beta^2 )</td>
<td>0.16</td>
<td>0.81</td>
</tr>
<tr>
<td>Idiosyncratic factor ( 1 - \beta^2 )</td>
<td>0.84</td>
<td>0.19</td>
</tr>
</tbody>
</table>
Market factor and conditional independence

- Suppose we know the “state of the economy,” i.e. the particular realization $\bar{m}$ of $m$
- Obligor $i$ asset return $r_i$ now has only one random driver: idiosyncratic factor $\epsilon_i$

$$r_i = \beta_i \bar{m} + \sqrt{1 - \beta_i^2} \epsilon_i, \quad i = 1, 2, \ldots$$

- Distance-to-default—the default-triggering return—becomes $-k_i + \beta_i \bar{m}$
- $\epsilon_i$ independent $\Rightarrow$ conditional returns of two different obligors

$$\sqrt{1 - \beta_i^2} \epsilon_i, \sqrt{1 - \beta_j^2} \epsilon_j, i \neq j$$

are independent

- $\Rightarrow$ **Conditional independence**: defaults of two firms independent
  - Conditioning is on realization of market risk factor
  - ↔ discussion of hazard rates, in which “conditional” default probability refers to non-default in prior period
Conditional default distribution of a single obligor

- Conditional on \( m = \bar{m} \):
  - **Mean** of the return distribution **changes**: \( 0 \rightarrow \beta_i \bar{m} \)
  - **Variance** of the return distribution **reduced**: \( 1 \rightarrow 1 - \beta_i^2 \)
    - Because we have eliminated market factor as source of variation
    - And **standard deviation** from \( 1 \rightarrow \sqrt{1 - \beta_i^2} \)
  - **Distance-to-default changes**: \( -k_i \rightarrow -(k_i - \beta_i \bar{m}) \)
    - In standard units: \( -k_i \rightarrow -\frac{k_i - \beta_i \bar{m}}{\sqrt{1 - \beta_i^2}} \)
  - **Default probability changes**: \( \pi_i = \Phi(k_i) \rightarrow \Phi \left( \frac{k_i - \beta_i \bar{m}}{\sqrt{1 - \beta_i^2}} \right) \)
    - With \( \Phi(x) \) the CDF of a standard normal variate \( x \)

- \( \Rightarrow \) **Conditional default probability distribution function**:

\[
p_i(m) = P [r_i \leq k_i | m] = \Phi \left( \frac{k_i - \beta_i m}{\sqrt{1 - \beta_i^2}} \right), \quad i = 1, 2, \ldots
\]
Conditional default probability: given market shock

Density and cumulative probability as a function of idiosyncratic shock. Graph assumes $\beta_i = 0.4, k_i = -2.33 \ (\iff \pi_i = 0.01)$, and $\bar{m} = -2.33$. The unconditional default distribution is a standard normal, while the conditional default distribution is $\mathcal{N}(\beta_i \bar{m}, \sqrt{1 - \beta_i^2}) = \mathcal{N}(-0.9305, 0.9165)$. The orange area in the density plot and horizontal grid line in the cumulative distribution plot identify $p(\bar{m})$, as in the example.
Conditional default distributions: example

- **Firm:** $\beta_i = 0.4$, $k_i = -2.33$ (so $\pi_i = 0.01$)
- **Market shock:** $\bar{m} = -2.33$ (sharp downturn)

<table>
<thead>
<tr>
<th></th>
<th>Unconditional</th>
<th>Conditional</th>
<th>Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean return</td>
<td>0</td>
<td>-0.9305</td>
<td>-0.9305</td>
</tr>
<tr>
<td>Return variance</td>
<td>1</td>
<td>0.8400</td>
<td>-0.1600</td>
</tr>
<tr>
<td>Return std. deviation</td>
<td>1</td>
<td>0.9165</td>
<td>-0.0835</td>
</tr>
<tr>
<td>Distance-to-default</td>
<td>2.33</td>
<td>1.3958</td>
<td>-0.9305</td>
</tr>
<tr>
<td>(standardized)</td>
<td>2.33</td>
<td>1.5230</td>
<td>-0.8034</td>
</tr>
<tr>
<td>Default probability</td>
<td>0.01</td>
<td>0.0639</td>
<td>0.0539</td>
</tr>
</tbody>
</table>
Properties of the conditional distribution

- Once the market factor is realized, the default distributions of individual loans/obligors are independent.
- But the market factor continues to be a random variable—together with idiosyncratic risk—driving default.
- Both parameters $\beta_i$ and $k_i$ continue to influence the shape of the distribution function.
Conditional default distributions

Probability of default of a single obligor, conditional on the realization of $m$ (x axis). Default probability 1 percent ($k = -2.33$). Conditional cumulative distribution function of default $p(m)$. Values of the distribution function run from 1 to 0 because it is plotted against $m$ rather than $\frac{k - \beta m}{\sqrt{1 - \beta^2}}$. 