Lecture notes on risk management, public policy, and the financial system

Incorporating extreme events into risk measurement

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Incorporating extreme events into risk measurement

Outline

Stress testing and scenario analysis

Expected shortfall

Extreme value theory
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Stress testing and scenario analysis

Expected shortfall

Extreme value theory
What are stress tests?

- Stress tests analyze performance under extreme loss scenarios
- Heuristic portfolio analysis
- Steps in carrying out a stress test
  1. Determine appropriate scenarios
  2. Calculate shocks to risk factors in each scenario
  3. Value the portfolio in each scenario
- Objectives of stress testing
  - Address tail risk
  - Reduce model risk by reducing reliance on models
  - “Know the book”: stress tests can reveal vulnerabilities in specific positions or groups of positions
- Criteria for appropriate stress scenarios
  - Should be tailored to firm’s specific key vulnerabilities
  - And avoid assumptions that favor the firm, e.g. competitive advantages in a crisis
  - Should be extreme but not implausible
Approaches to formulating stress scenarios

**Historical scenarios** based on actual past events
- Issues: time frame of individual returns, treatment of correlation
- Omitting key risk factor, such as option implied volatility for option portfolio

**“What if” or hypothetical scenarios** based on assessment of potential large market and credit events
- Must be calibrated so as to achieve appropriate severity
- May be based on models and/or macroeconomic scenario, must then be translated into asset returns

**Factor-push approach** in place of or (better) in addition to judgement
- Compute impact of many combinations of risk factor returns
- Stress losses define as largest portfolio losses
Strengths and weaknesses of stress tests

- **Pros**
  - Avoid reliance on models, model risk
  - Easy to communicate

- **Cons**
  - Scenarios are not directly associated with probabilities
  - Arbitrariness in scenario design
  - Difficulty including, configuring large number of risk factors
Portfolio sensitivity analysis

- Often categorized as a form of stress testing
- But focused on small changes
- Help know the book, identify concentrations, understanding drivers of P&L and risk
- Can capture nonlinearity by displaying convexities, but may miss nonlinearities that only appear if large market moves realized
Stress testing and scenario analysis

**Expected shortfall**
Estimating expected shortfall

**Extreme value theory**
Definition of expected shortfall

- Expected shortfall (or conditional Value-at-Risk or tail Value-at-Risk or expected tail loss) is defined as expected value of losses, given that VaR loss is exceeded.

- Expected value of realizations of the random variable $X$, representing portfolio losses expressed as a positive number, in the tail of the distribution, left of the VaR scenario:

$$E[X | X < \text{VaR}(t, \alpha, \tau)]$$

Expected shortfall at a 95 percent confidence level is area under density to left of VaR, divided by probability (0.05) that VaR is breached.
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Expected shortfall

Estimating expected shortfall

Estimation of expected shortfall

- VaR a *quantile* of loss distribution $\leftrightarrow$ expected shortfall a *moment* of *truncated* loss distribution
- Expected shortfall for a single position can be computed using the same basic approaches used to compute VaR
  - Parametric and Monte Carlo simulation approaches can employ same specific distributional hypothesis as VaR
  - Monte Carlo and historical simulation approaches can employ same set of simulated values as VaR
- Parametric expected shortfall computed analytically
- Monte Carlo and historical simulation estimates of expected shortfall: mean of simulated losses $> \text{VaR}(t, \alpha, \tau)$
- **Example**: 1-day expected shortfall of long position in S&P 500 index with initial value $1,000,000$ as of close on 28Aug2013
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**Expected shortfall**

**Estimating expected shortfall**

**Parametric estimates of expected shortfall**

- Assume returns lognormally distributed, use EWMA estimate of volatility 0.00691049 as of 28Aug2013
- Moments of truncated normal distribution can be computed analytically
- In lognormal model, ratio of expected shortfall to the VaR is

\[
\frac{\phi(z_{1-\alpha})}{(1-\alpha)z_{1-\alpha}}
\]

- \(\phi(\cdot)\) represents standard normal density function, e.g.
  \(\phi(z_{0.05}) = \phi(-1.645) = 0.103136\)

<table>
<thead>
<tr>
<th>conf. level</th>
<th>VaR</th>
<th>exp. shortfall</th>
<th>ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.900</td>
<td>8 817.04</td>
<td>12 074.24</td>
<td>1.3694</td>
</tr>
<tr>
<td>0.950</td>
<td>11 302.38</td>
<td>14 173.64</td>
<td>1.2540</td>
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<tr>
<td>0.975</td>
<td>13 452.99</td>
<td>16 046.44</td>
<td>1.1928</td>
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<tr>
<td>0.990</td>
<td>15 947.66</td>
<td>18 270.67</td>
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Parametric estimates of 1-day VaR and expected shortfall of $1\,000\,000 long position in S&P 500 on 28Aug2013.
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Expected shortfall

Estimating expected shortfall

Estimating expected shortfall via historical simulation

- Use 2 years \((T = 503)\) of return observations from 28Aug2011 to 28Aug2013
- Estimated as mean of observed losses in excess of VaR scenario
- As with VaR, expected shortfall estimates may vary widely with historical look-back period

<table>
<thead>
<tr>
<th>conf. level</th>
<th>rank</th>
<th>VaR</th>
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<th>ratio</th>
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<td>50</td>
<td>11348.34</td>
<td>18439.68</td>
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<td>25</td>
<td>16147.23</td>
<td>23280.36</td>
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<td>22966.30</td>
<td>27450.94</td>
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<tr>
<td>0.990</td>
<td>5</td>
<td>26705.46</td>
<td>30872.39</td>
<td>1.1560</td>
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</table>

Historical simulation estimates of 1-day VaR and expected shortfall of $1,000,000 long position in S&P 500 on 28Aug2013. The rank stated in the table is that of the smallest loss included in expected shortfall and is one less than that of the VaR scenario.
### Computation of expected shortfall by historical simulation

<table>
<thead>
<tr>
<th>$i$</th>
<th>$t$</th>
<th>$S_t$</th>
<th>date</th>
<th>$S_{t-1}$</th>
<th>$\tilde{r}(i)$</th>
<th>$\tilde{r}_{\text{arith.}}(i)$</th>
<th>P&amp;L</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>52</td>
<td>1229.10</td>
<td>09Nov2011</td>
<td>1275.92</td>
<td>-0.03739</td>
<td>-0.03670</td>
<td>-36695.09</td>
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<tr>
<td>2</td>
<td>18</td>
<td>1129.56</td>
<td>22Sep2011</td>
<td>1166.76</td>
<td>-0.03240</td>
<td>-0.03188</td>
<td>-31883.16</td>
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<tr>
<td>3</td>
<td>17</td>
<td>1166.76</td>
<td>21Sep2011</td>
<td>1202.09</td>
<td>-0.02983</td>
<td>-0.02939</td>
<td>-29390.48</td>
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<tr>
<td>4</td>
<td>25</td>
<td>1099.23</td>
<td>03Oct2011</td>
<td>1131.42</td>
<td>-0.02886</td>
<td>-0.02845</td>
<td>-28450.97</td>
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<tr>
<td>5</td>
<td>46</td>
<td>1218.28</td>
<td>01Nov2011</td>
<td>1253.30</td>
<td>-0.02834</td>
<td>-0.02794</td>
<td>-27942.23</td>
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<tr>
<td>6</td>
<td>9</td>
<td>1154.23</td>
<td>09Sep2011</td>
<td>1185.90</td>
<td>-0.02707</td>
<td>-0.02671</td>
<td>-26705.46</td>
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</tr>
</tbody>
</table>

The entries in the penultimate column are the ordered arithmetic historical returns $\tilde{r}(i)$, $i = 1, \ldots, T$ and $T = 503$. The P&L realizations are $x \left( \tilde{S}(i) - S_t \right)$. 
Relationship of expected shortfall to VaR

- Expected shortfall is always at least as great as the VaR: average of loss levels greater than the VaR
  - Can be much larger than VaR if return distribution is heavy-tailed and skewed
- Ratio of expected shortfall to VaR is higher at lower confidence levels and falls toward 1 for very high confidence levels
  - Many large observations beyond VaR at lower confidence level
- The normal distribution is thin-tailed →
  - Parametric estimates of ratio of expected shortfall to VaR relatively low at lower confidence level
  - Highlights disadvantage of standard model: implies risk of very large losses relatively low
- Empirical distributions heavy-tailed →
  - Historical simulation expected shortfall generally quite high relative to VaR
  - Disparity shrinks as confidence level rises
Estimation of expected shortfall by historical simulation

Ratio of expected shortfall to VaR as a function of confidence level.

Advantages and disadvantages of expected shortfall

- Expected shortfall in principle oriented toward tail-risk measurement
  - More appropriate than VaR for use in setting \((\rightarrow)\) economic capital
  - States average extreme event if extreme event should occur
- But unlikely \textit{per se} to provide significant improvement in tail risk measurement
  - Not really an \textit{alternative} to VaR
  - Alternative statistic \textit{within} VaR framework
  - If data and model not providing good tail risk estimate, expected shortfall won’t help much
Backtesting expected shortfall is difficult

- Difficult to backtest, since extremes by definition more infrequent than observations in the body of the distribution
- VaR backtesting involves counting episodes in which a quantile is exceeded over some past period
- VaR ES involves counting episodes in which a conditional mean size of exceedance on each date is exceeded over some past period
  - At any useful confidence level, relatively few exceedances of VaR
  - Each one provides one observation on exceedance size
  - But you need many to have an estimate of the conditional mean
- In addition to limitations of VaR backtesting
- Revised Basel standards backtest VaR at 97.5- and 99-percent confidence levels as internal model check
  - No requirement to test ES itself
The elicitability problem

- **Elicitability** is a desirable property of a statistic of a random variable such as P&L.
- Forecasts of elicitable statistics can be tracked day by day to see how close they are to their realizations (*scoring*).
- VaR is elicitable using this *scoring function*:
  - Each day, measure the absolute value of the difference between VaR and P&L.
  - Weight is $\alpha$ if there is an exception, otherwise $1 - \alpha$.
  - A low score indicates few exceptions, thus validates VaR.
  - But note that a zero score would indicate P&L has no variability.
- ES is not elicitable, i.e. no such scoring function can be formulated.
- But ES can nonetheless be evaluated through backtesting.
Stress testing and scenario analysis

Expected shortfall

**Extreme value theory**

Overview of extreme value theory
Estimation
Modeling extremes

- **Extreme value theory**: (EVT) set of models focused on extreme events rather than entire return distribution
- Losses a random variable $\tilde{x}$ with observed realizations
  - Treat losses as positive numbers for convenience
  - E.g. multiply returns by -1 for long position
- Goals of EVT
  - Extract information about extreme losses that go beyond what has been observed empirically
  - Determine probability distributions of extremes for application in risk analysis
- What is an extreme? Two standard definitions
  - **Block maximum**: maximum value in a set of successive observations over a given time horizon: $\max(X_1, X_2, \ldots X_T)$
  - **Peaks over threshold**: realizations exceeding a given high threshold: $\{X_t | X_t > u\}$, with $u$ a “large number”
Extreme value distributions

- Probability distributions of block maxima converge to **generalized extreme value (GEV) distribution** as \( T \to \infty \)
  - Few assumptions regarding distribution, just i.i.d. from some *distribution*
  - Extreme value must be normalized in some way

- Analogous to **central limit theorem (CLT)**:
  - Sum of independent random variates with finite mean and variance converges \( \to \) normal distribution
  - CLT applies to normalized random variates

- GEV distribution comes in 3 variants
  - Thin, e.g. mortality: no possibility of exceeding some finite limit
  - “Normal”
  - Fat-tailed, e.g. most asset returns: low but material probability of very large (loss) realization
Power laws and the tail index

- If a GEV-distributed random variable falls into fat-tailed category, then large losses follow a power law:

\[ P \left[ X_t \geq x \right] = L(X_t)X_t^{-\iota}, \quad x, \iota > 0, \]

- \( L(X) \) is a normalizing function that varies little with \( t \)
- For example, a constant or the logarithmic function, which rises very gradually for large values of the argument
- \( \iota \) called the tail index
  - Fat-tailed asset returns/losses have tail index in excess of 2
Estimating the tail index

- Simple approach: **Hill’s estimator**
- To apply Hill’s estimator to loss on long position in single asset
  - Multiply set of returns by $-1$
  - Set (somewhat arbitrarily) threshold $u$ for extreme loss
    \[
    \{X_t\}_{t=1,...,T} = \{-r_t | r_t < -u\}
    \]
  - Or, equivalently, include $k$ largest losses in the data set: largest $k$ order statistics, denoted $\{X_{(1)}, \ldots, X_{(k)}\}$, with $X_{(2)} \geq \ldots \geq X_{(k)}$
- Estimator is the reciprocal of the mean log excess of losses over the threshold $u$ or $X_{(k)}$:
  \[
  \hat{\gamma} = \left[ \frac{1}{k} \sum_{j=1}^{k} \ln(X_{(j)}) - \ln(X_{(k)}) \right]^{-1}
  \]
- **Example:** S&P 500 daily returns since 1928
  - Bad news: varies considerably with “user input,” the threshold
  - Good news: converges to about 3.0
Estimating the tail index

Estimates of the tail index $\alpha$ using Hill’s estimator and $k$ largest-magnitude negative returns, with $k = 5, \ldots, 300$. Purple plot: $\hat{\alpha}$; gray plot: $k$. Data source: Bloomberg Financial L.P.