Interest rates

Interest rate risk

Credit spreads and spread risk

Interest rate risk measurement
**Interest rates**

Interest rates and yield curves
Bond math: spot, forward and par yield curves

**Interest rate risk**

**Credit spreads and spread risk**

**Interest rate risk measurement**
What are interest rates?

- An **interest rate** is a rate paid for borrowing money
  - Expressed as a percent of **principal**—amount of money borrowed—per unit of time, generally per annum
  - E.g. 5 percent per annum → $5 per annum for each $100 of principal
- **Term** or **time to maturity** of the borrowing generally contractually specified
- Interest rate may be categorized by whether it varies over the term
  - **Fixed rate**: set at a particular value for all or part of the term
  - **Floating rate**: varies over the term by reference to an index
Fixed income instruments

**Loans:** bilateral contract between lender, generally a bank, and borrower
- Large loans may be **syndicated**, several lenders extend credit
- Secondary markets exist for syndicated and some other, generally large, loans, with credit extended by several lenders

**Securities** such as bonds, bills and notes
- Legal design facilitates offering them to many potential lenders at issuance and secondary market trading

**Money markets:** short-term lending, including
- **Bank deposits** and other **unsecured interbank lending**
- **Repo** and other **secured** forms (→collateral markets)
- **Short-term securities** such as Treasury bills, **commercial paper**

**Derivatives** including
- **Money market and bond futures:** exchange-traded contracts
- **Interest-rate swaps:** regular exchange of fixed for floating payments, largest OTC fixed-income market
- **Credit default swaps:** exchange of regular fixed payments for a par amount in the event of the default of an underlying security
Yield curve

- Interest rates—risk-free or for a given obligor—are not a single risk factor
- Typically vary by the term of the loan
- **Term structure of interest rates** or **yield curve**: rates as function of maturity

Typical behavior of the term structure

- Level and shape of curve based primarily on expected future short-term rates
- But also contains liquidity, credit, interest-rate and other risk premiums
- Term structure generally upward-sloping due to **risk premiums**
  - Longer-term risk-free interest rates and credit spreads generally higher than short-term
  - Particularly steep at very short end due to (→)**money premium**
- But may be downward-sloping—**yield curve inversion**—overall or in some segments, e.g.
  - Short-term rates spike, not expected to persist, e.g. emerging-market rates under foreign exchange pressure
  - High demand for safe long-term bonds, e.g. 2005 conundrum in U.S.
Equivalent ways of expressing the term structure

- Term structure can be represented in different ways, each useful in certain contexts
- **Yield curve**: yields of bonds/loans as a function of term
  - Yield-to-maturity calculations assume reinvestment of coupons at same yield
  - Ambiguity due to coupon size
- **Spot or zero-coupon curve** and **spot or zero-coupon rates**:
  - Money lent now and repaid at single specific time in the future
- **Forward curve** and **forward rates**:
  - Loans of a specified term to maturity commencing at different **settlement date** in the future
  - E.g. rates on 3-month loans settling immediately, in 1 month, in 3 months, ...
  - Can be transformed into spot curve and v.v.
Bond prices and yield to maturity

- Simple example:
  - Shortest-term issue: a 1-year bill or zero-coupon
  - Coupon bonds maturing in 2, 3 and 4 years

- Annualized interest rates, annual pay frequency and compounding

<table>
<thead>
<tr>
<th>term</th>
<th>coupon</th>
<th>price</th>
<th>yield</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 year</td>
<td>0.00</td>
<td>98.692</td>
<td>1.325</td>
</tr>
<tr>
<td>2 years</td>
<td>1.75</td>
<td>100.104</td>
<td>1.697</td>
</tr>
<tr>
<td>3 years</td>
<td>2.00</td>
<td>100.237</td>
<td>1.918</td>
</tr>
<tr>
<td>4 years</td>
<td>2.00</td>
<td>100.029</td>
<td>1.992</td>
</tr>
</tbody>
</table>
**Bond price, yield, and spot rate relationships**

- Quote data generally in the form of prices and/or yields
- Price $p_t$ of a $t$-year bond with coupon $c_t$ is related to its (generally observable) yield $y_t$ by

$$
p_t = c_t \left[ \frac{1}{1 + y_t} + \frac{1}{(1 + y_t)^2} + \cdots + \frac{1}{(1 + y_t)^t} \right] + \frac{1}{(1 + y_t)^t}
$$

- Converts quote in price terms to yield terms and v.v.
- Bond price also related to its (generally unobservable) spot rates $r_1, r_2, \ldots, r_t$:

$$
p_t = c_t \left[ \frac{1}{1 + r_1} + \frac{1}{(1 + r_2)^2} + \cdots + \frac{1}{(1 + r_t)^t} \right] + \frac{1}{(1 + r_t)^t}
$$

- Derives unobserved spot rates from observed quote data
Bootstrapping the spot curve from bond prices

- Derived yield curves used in pricing, valuation, risk analysis
  - Market quotes for different securities on same curve may employ different pay frequency, day count and other conventions
  - Derived curves—spot curve or other forms—impose consistency
- **Bootstrapping** a technique for obtaining spot curve, starting
  - Other techniques include spline interpolation, least-squares fitting
- Bootstrapping starts with shortest-maturity security, uses each successively longer security to capture one longer-term spot rate
- For each bond maturity, assumes we have either a price or yield (or both—and coupon rate)
Bootstrapping spot rates from prices: example

- 1-year bond has no coupon (i.e. $c_0 = 0$), so $p_1$ and $r_1$ satisfy

$$p_1 = 0.986923 = \frac{1}{1 + r_1} = \frac{1}{1.01325}$$

- Next, solve for $r_2 = 0.0170$ from:

$$p_2 = 1.00104 = c_2 \left[ \frac{1}{1 + r_1} + \frac{1}{(1 + r_2)^2} \right] + \frac{1}{(1 + r_2)^2}$$

$$= \frac{0.0175}{1.01325} + \frac{0.0175 + 1}{(1 + r_2)^2}$$

- Solve the same type of equation to get $r_3 = 0.01925$:

$$p_3 = 1.00237 = c_2 \left[ \frac{1}{1 + r_1} + \frac{1}{(1 + r_2)^2} + \frac{1}{(1 + r_3)^3} \right] + \frac{1}{(1 + r_3)^3}$$

$$= 0.02 \times 1.95377 + \frac{0.02 + 1}{(1 + r_3)^3}$$

- And, finally, $r_4 = 0.02$
Equivalent ways of expressing the spot curve

**Discount curve** and **discount factors**: present values corresponding to spot rates

- Discount factor corresponding to $r_1$ is $(1 + r_1)^{-1}$, discount factor corresponding to $r_2$ is $(1 + r_2)^{-2}$, etc.

**Par yield curve**: hypothetical bonds with coupons equal to yields

- Computed from spot curve by setting price to par and solving for coupon rate

<table>
<thead>
<tr>
<th>term</th>
<th>discount factor</th>
<th>par yield</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 year</td>
<td>0.9869</td>
<td>1.325</td>
</tr>
<tr>
<td>2 years</td>
<td>0.9668</td>
<td>1.697</td>
</tr>
<tr>
<td>3 years</td>
<td>0.9444</td>
<td>1.918</td>
</tr>
<tr>
<td>4 years</td>
<td>0.9238</td>
<td>1.993</td>
</tr>
</tbody>
</table>
Forward and spot rates: example

- Forward rate $f_{t,t+1}$ from $t$ to $t+1$ related to $r_t$ $r_{t+1}$ by

$$f_{t,t+1} = \frac{(1 + r_{t+1})^{t+1}}{(1 + r_t)^t} - 1$$

- Forward rate $f_{0,1}$ settling immediately is identical to shortest-term spot rate.

- Forward higher than spot rates if yield curve positively sloped.

- Forward curve falling due to **convexity** of spot curve.
  - Slope of spot curve positive but declining.

<table>
<thead>
<tr>
<th>term</th>
<th>spot</th>
<th>forward</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 year</td>
<td>1.325</td>
<td></td>
</tr>
<tr>
<td>2 years</td>
<td>1.700</td>
<td>2.0764</td>
</tr>
<tr>
<td>3 years</td>
<td>1.925</td>
<td>2.3765</td>
</tr>
<tr>
<td>4 years</td>
<td>2.000</td>
<td>2.2253</td>
</tr>
</tbody>
</table>
Interest rates

**Interest rate risk**
- Sources of interest rate risk
  - Inflation risk
  - Annuities and market risk

Credit spreads and spread risk

Interest rate risk measurement
Sources of fixed income return

**Deterministic sources:** returns with an unchanged curve

- **Yield** of the security
- **Rolldown** or **theta:** term to maturity shortens over time as maturity date nears
  - Generally positive, since yield curves generally upward-sloping
  - **Example:** a 3-year bond held for 1 year becomes a 2-year bond, typically with a lower yield and higher market value

**Interest rate risk:** yield curve may change over time, affecting security values

- Changes in interest rates random and arise for many reasons
### Decomposing interest rates

**Risk-free** or **pure rate of interest** compensates for passing of time

**Expected future interest rates:** longer-term yield is based on short-term rates expected to prevail up to maturity

**Term premium:** additional yield compensating for bearing interest-rate risk of holding longer-term securities

- Aversion to term risk may change without change in expected future interest rates

**Spreads** over the risk-free rate compensate primarily for

**Credit risk:** compensation for risk of default and credit migration losses

**Liquidity risk:** compensation for cost and risk of exiting, adjusting or maintaining position

**Nominal rates** express principal in units of money

**Real rate** expressed in units of goods, e.g. gold or determined with reference to a price index

**Inflation compensation** or **breakeven inflation** rate: difference between nominal and real rates
**Spreads**

- **Risk-free curve:** a base/benchmark relative to which risky yield curves measured
- **Risky curve:** generally expressed in terms of spreads relative to risk-free
- **Credit spread:** difference between interest rate on a risky and congruent credit risk-free security

**Risk-free:** U.S. on-the-run Treasury curve; **risky:** Republic of Turkey U.S. dollar-denominated yield curve, 16Jan2018. **Source:** Bloomberg Financial LP.
Attributing spreads to risks

- Each source of risk has an expected value and risk impact
- Wider spreads make bonds cheaper, and thus increase prospective future returns
  - Higher future return provides risk premium compensating for expected and unexpected losses
- Models needed to decompose spreads into various sources
- Spreads can also be measured across currencies
Credit risk premium

- Credit risk generally the main driver of spreads and changes in spreads over risk-free yields
- Credit spread compensates for expected value of default—default probability times loss given default
- But also compensates for “putting up with” default risk
- Includes the market risk of changes in credit-related portion of spread
Liquidity risk premium

- Liquidity risk can have countervailing impact on spreads
- **Liquidity premium**: aversion to risk of holding longer-term securities may change
  - More likely to affect corporate and sovereign issues outside advanced market economies
  - Sovereign issues of advanced market economies may have negative liquidity risk premium
- Financial stress likely to impair liquidity conditions, induce flight to quality, lowering yields on most-liquid sovereign bonds
- **Money premium** due to provision of money services by safe very short-term debt
  - Very short end of Treasury yield curve steeper than otherwise
Inflation risk

- **Inflation rate risk** is the risk of loss from a rise in the general price level
  - Directly affects securities with payoffs defined in nominal terms
  - Indirectly affects real assets by affecting macroeconomic conditions
- Inflation difficult to hedge
  - U.S. (since 1997), U.K., other countries issue **inflation-protected** or **inflation-linked bonds** that increase principal based on price index
  - **Inflation swaps** and other derivatives
  - **Gold clause** specifying gold value of principal and interest of certain U.S. Treasury bonds invalidated 1933 by Joint Resolution of Congress
Insurance company exposure to inflation

- Insurers may benefit from inflation
- Long-term liabilities generally defined in nominal terms
  - Generally not fully hedged against changes in interest rates
  - And substantial allocation to real assets: real estate, equities
- Permanent risk in inflation rate reduces real value of liabilities
Types of annuities

- **Annuities** are contracts for exchange of a specified sequence of payments between an **annuitant** and intermediary, generally an insurance company.

- Very wide variety of types

- Payments by annuitant may be a lump sum or periodic over a future time interval
  - Annuities with periodic future payments may **lapse** or include **early surrender penalties**

- Payments by insurance company may be fixed or vary:
  - **Fixed annuity**: payments or interest rate fixed over time
  - **Variable annuity**: payments vary with return on a specified portfolio, generally equity-focused

- Annuities may include guarantees by insurance company, such as **guaranteed minimum benefits**
Risks of annuity issuance

- Market risks interact with risks arising from guarantees and policyholder behavior
- Variable annuities generally provide guaranteed minimum return
  - Economically equivalent to sale of put option on equity market by insurer to policyholder
  - Annuity is underpriced if value of put not fully incorporated
- Large losses to U.S. insurers in 2008
  - Hartford Life became a Troubled Asset Relief Program (TARP) recipient
- Fixed annuity issuance exposed to convexity risk
  - Assets generally duration-matched to liabilities
  - But liabilities exhibit greater convexity due to guarantees and policyholder behavior
  - Economically equivalent to sale of put option on bond market by insurer to policyholder
- Rising interest rates: early surrender optimal → duration falls rapidly
- Falling interest rates: minimum guaranteed rate in effect → duration rises rapidly
Interest rates

Interest rate risk

**Credit spreads and spread risk**
- Credit spreads
- Credit spread risk
- Credit spreads and hazard rates

Interest rate risk measurement
Credit spreads and credit quality

- Credit spreads vary across several dimensions: Differences in
  - Spreads of different obligors on bonds with similar credit quality
    - Third-party guarantees, collateral, position in (→)capital structure
  - Spreads of same obligor on bonds with different credit quality
    - →Credit spread of senior unsecured<subordinated debt
  - Spreads of same obligor on bonds with different maturities
- Credit spread compensates for several sources of risk
  - Credit risk, reflecting both default probability and expected loss given default
  - Market liquidity risk: many credit-risky bonds infrequently traded or have small issuance volume
Measuring credit spreads

**Spread to swaps or Treasuries** with similar maturity. Can be implemented in different ways, e.g.

- **z-spread**: spread between zero-coupon curves
- **i-spread**: spread over interpolated risk-free curve

**Option-adjusted spread** (OAS): spread to a benchmark curve after accounting for the value of options embedded in the security

- Corporate bonds often **callable**
- Callable bond is bundled with call option on bond sold by investor to issuer
ICE BofA Merrill Lynch U.S. indexes of option-adjusted spreads (OAS) to the Treasury curve for corporate securities rated AA (C0A2), BBB (C0A4) and BB (H0A1), in percent. The BBB-AAA quality spread is the difference between the BBB and AA (C0A1) indexes. Daily, 31Dec1996 to 14Feb2019, downloaded from FRED.
European credit spreads 1999—2019

Bloomberg Barclays Euro Corporate Bond Index option-adjusted spread (OAS) to the treasury curve for securities rated AA and BBB, in basis points. The BBB-AAA quality spread is the difference, in percent, between the BBB and AAA indexes. Daily, 30Apr1999 to 15Feb2019, downloaded from Barclays Live.
Credit derivatives

**Credit default swaps** (CDS): OTC contract, one counterparty pays regular premium and is made whole by the other if default of specific bond issuer occurs

- Economically similar to guarantee or insurance on firm’s debt, but without requirement of direct economic exposure
- Partial standardization: *single-name CDS* governed by master agreements
- Pre-crisis, large volumes of CDS on residential and commercial real-estate securitizations

**Credit default swap indexes** (CDX)

- Protection on a set of *reference entities*
- **Examples**: CDX.IG.NA series covering North America, iTraxx series covering Europe, Asia

**Credit default swap derivatives** based on underlying CDS or CDX

- Standard tranches (→structured credit products)
- OTC swaptions on CDS and CDX
Pricing of credit derivatives

CDS spread: market-clearing premium in basis points paid for default protection
- Investment-grade CDS quoted in terms of credit spread
- High-yield CDS quoted in points up front: spread is standardized to 100 or 500 basis points, an upfront fee equates present value to market-based credit spread

CDS basis: difference between the CDS and bond spread
U.S. credit default swap index 2011–2019

Credit spread risk

- **Credit spread risk**: risk of loss from change in credit spreads
  - Spread risk is a *market* risk, albeit credit-related
  - **Spread 01**: price/value impact of change in spreads, market risk of credit exposures
- Both the spread and spread risk a function of obligor credit quality
  - As well as of bond tenor, liquidity and other factors
Some special types of credit spread risk

**Basis risk**  Risk that two similar, but not identical, securities diverge or converge in price.

**Example:** *Bond-CDS basis*, difference between credit spreads implied by bonds and CDS, influenced by availability of funding

**Convexity risk**  Risk of loss from under-hedging a security with a highly non-linear payoff profile.

**Example:** *Structured credit products*, equity tranche may have highly non-linear response to change in default correlation (May 2005)
Difference between the z-spread over Libor of Citigroup Inc. bonds and the premium on Citigroup 10-year senior unsecured CDS. The bond spread is blended from spreads on two senior unsecured issues: the 4.7% maturing May 29, 2015 (CUSIP 172967CY5) and the 5.85% maturing Aug. 2, 2016 (CUSIP 172967DQ1). Source: Bloomberg Financial LP.
Estimating default probability from credit spreads

- Credit spread expresses **risk-neutral hazard rate** and **default probability**
- Simplest case:
  - 1-year risk-free with yield $r$ and credit-risky bonds with yield $r + z$ both exist
  - Recovery rate zero
- These present values should be equal:

\[
\frac{(1 - \tilde{\pi}_1) \times 1 + \tilde{\pi}_1 \times 0}{1 + r} = \frac{1}{1 + r + z}
\]

- Spread $z$ approximately equal to 1-year default probability $\tilde{\pi}_1$:

\[
\tilde{\pi}_1 \approx z
\]
Default probability and the hazard rate

- Credit risk models typically in continuous-time setting
- With $r$ the risk-free continuously-compounded rate
  - $z$ is the spread and $r + z$ is the continuously-compounded risky yield
  - Risk-neutral 1-year hazard rate $\tilde{\lambda}$:
    \[
    \tilde{\lambda} = z
    \]
- Via $e^a - 1 \approx a$
- Approximation reasonably accurate for coupon securities, longer maturities
Credit spreads and recovery rates

- With recovery rate $R > 0$, credit spread approximately equal to expected loss (EL)
  \[ z \approx \tilde{\pi}_t(1 - R) \]

- If $R > 0$, risk-neutral default probability higher:
  \[ \tilde{\pi}_1 \approx \frac{z}{1 - R} \]

- If you observe a given spread, then the lower the LGD, the higher must be the default probability

- **Example** (all in percent):
  
<table>
<thead>
<tr>
<th>Credit spread</th>
<th>$z$</th>
<th>2.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-yr. risk-neutral default probability ($R = 0$)</td>
<td>$\tilde{\pi}_1 \approx z$</td>
<td>2.00</td>
</tr>
<tr>
<td>1-yr. risk-neutral default probability ($R = 0.40$)</td>
<td>$\tilde{\pi}_1 \approx \frac{z}{1 - R}$</td>
<td>3.33</td>
</tr>
</tbody>
</table>
Interest rates

Interest rate risk

Credit spreads and spread risk

**Interest rate risk measurement**

- Measuring bond price sensitivity to rates
- Duration and convexity
- Measuring Value-at-Risk for a bond position
Bond and derivatives exposure to interest rates

- Interest rate risk can be measured as exposure to
  
  **Parallel shifts** of entire yield curve
  
  **Yield to maturity:** closed-form measures in some cases
  
  **Key rates:** impact of changes in rates for specific maturities, e.g. 3-month or 2-year rates
    
    - Useful for hedging, bond portfolio management

- Complexities introduced by need to measure impact of changes in rates on price or value
Scenario analysis

- Analysis in which yield curve assumed to change in specific ways
- Can be carried out using expression for price $p_t$ of a bond in terms of spot rates
  - Scenario result is price change $\tilde{p}_t - p_t$ resulting from specified changes in spot rates
- Some commonly-encountered scenario analyses include
  - **Parallel shift**: measure price change if all spot rates rise, say, 25 basis points
  - **Curve steepening**: longer-term risk-free rates rise, but short-term rates and credit spreads unchanged
  - **Credit spread widening**: credit spreads increase, but risk-free rates unchanged
    - Change in *price* is the same for equal change in risk-free rate or spread
  - **Roll-down return**: bond “ages” by, say, 1 year, but rates and spreads unchanged
Scenario analysis: parallel shift

- Price change $\tilde{p}_t - p_t$ if all spot rates rise 25 basis points:

$$\tilde{p}_t = c_t \left[ \frac{1}{1 + r_1 + 0.0025} + \cdots + \frac{1}{(1 + r_t + 0.0025)^t} \right]$$

$$+ \frac{1}{(1 + r_t + 0.0025)^t}$$

- All bond prices fall, but longer-term bond prices fall most
- In the example:

<table>
<thead>
<tr>
<th>term</th>
<th>initial $p_t$</th>
<th>shocked $\tilde{p}_t$</th>
<th>$\tilde{p}_t - p_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 year</td>
<td>98.692</td>
<td>98.449</td>
<td>-0.246</td>
</tr>
<tr>
<td>2 years</td>
<td>100.104</td>
<td>99.618</td>
<td>-0.486</td>
</tr>
<tr>
<td>3 years</td>
<td>100.237</td>
<td>99.517</td>
<td>-0.718</td>
</tr>
<tr>
<td>4 years</td>
<td>100.029</td>
<td>99.082</td>
<td>-0.946</td>
</tr>
</tbody>
</table>
Scenario analysis: curve steepening

- **Example:** 1- and 2-year spot rates unchanged, 3- and 4-year spot rates rise by 15 and 25 bps (0.0015 and 0.0025 percent)
- Short-term bond prices unchanged, but longer-term bond prices fall

<table>
<thead>
<tr>
<th>term</th>
<th>initial $p_t$</th>
<th>shocked $\tilde{p}_t$</th>
<th>$\tilde{p}_t - p_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 year</td>
<td>98.692</td>
<td>98.692</td>
<td>0.000</td>
</tr>
<tr>
<td>2 years</td>
<td>100.104</td>
<td>100.104</td>
<td>0.000</td>
</tr>
<tr>
<td>3 years</td>
<td>100.237</td>
<td>99.813</td>
<td>-0.423</td>
</tr>
<tr>
<td>4 years</td>
<td>100.029</td>
<td>99.102</td>
<td>-0.926</td>
</tr>
</tbody>
</table>
Scenario analysis: roll-down

- Assume spot curve unchanged 1 year hence
- Short-term bonds will return exactly their initial yield to maturity
- If yield curve upward-sloping, all bonds will experience positive return
  - As maturities shorten, bonds’ cash flows discounted at lower spot rates
- In the example:

<table>
<thead>
<tr>
<th>term</th>
<th>initial</th>
<th>1 year hence</th>
<th>Δ price</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initially 1 year</td>
<td>98.692</td>
<td>100.000</td>
<td>1.325</td>
</tr>
<tr>
<td>Initially 2 years</td>
<td>100.104</td>
<td>100.419</td>
<td>0.315</td>
</tr>
<tr>
<td>Initially 3 years</td>
<td>100.237</td>
<td>100.592</td>
<td>0.355</td>
</tr>
<tr>
<td>Initially 4 years</td>
<td>100.029</td>
<td>100.237</td>
<td>0.208</td>
</tr>
</tbody>
</table>
Rate sensitivity using duration and convexity

- Approximate measures of impact of small changes in yield $y_t$ (or of small parallel shifts) on bond price $p_t$:

**Modified duration**: denoted $\text{mdur}_t \equiv -\frac{1}{p_t} \frac{dp_t}{dy_t}$

- Percent change in bond price as yield or level of curve rises 1 percent
- Negative relation between price and yield $\rightarrow$ convention of multiplying by $-1$

**Convexity**: denoted $\text{conv}_t \equiv \frac{1}{p_t} \frac{d^2p_t}{dy^2_t}$

- Change in duration as yield or level of curve changes

- Effect on price/value of a $\Delta y$ increase in yield in bps is Taylor-approximated by

$$\frac{\Delta p_t}{p_t} \approx -\text{mdur}_t \Delta y + \frac{1}{2} \text{conv}_t \Delta y^2,$$

with $\text{mdur}_t$ the delta, $\text{conv}_t$ the gamma of a bond
Calculating duration

- **Effective duration**: approximate $\frac{dp_t}{dy_t}$ by shifting curve up and down by 1 basis point:

$$\frac{dp_t}{dy_t} \approx \frac{p_t(\text{shifted up}) - p_t(\text{shifted down})}{2 \times 0.0001}$$

- Divide by initial bond price (and express as a positive number)

<table>
<thead>
<tr>
<th>term</th>
<th>initial</th>
<th>+1bp shock</th>
<th>−1bp shock</th>
<th>duration</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 year</td>
<td>98.692</td>
<td>98.683</td>
<td>98.702</td>
<td>0.987</td>
</tr>
<tr>
<td>2 years</td>
<td>100.104</td>
<td>100.084</td>
<td>100.123</td>
<td>1.950</td>
</tr>
<tr>
<td>3 years</td>
<td>100.237</td>
<td>100.208</td>
<td>100.266</td>
<td>2.886</td>
</tr>
<tr>
<td>4 years</td>
<td>100.029</td>
<td>99.991</td>
<td>100.067</td>
<td>3.807</td>
</tr>
</tbody>
</table>

- Why use effective duration?
  - Often easier to implement
  - Needed for securities with cash flows that depend on interest rates, e.g. mortgage-backed securities (prepayments), bonds with embedded options

- **Effective convexity** estimated analogously
Duration and convexity of U.S. 10-year note

Example: rate sensitivity of U.S. 10-year note

- 2 percent coupon, expiration 15Nov26 (term 9 year 10 months)
- Price quoted in dollars, 32nds and 64ths of a dollar per $100 of par value:

\[
97-01+ \equiv 97 + \frac{1}{32} + \frac{1}{64} = 97.0468750
\]

- This quote includes 01 32nds; the + indicates an additional 64th
- Bloomberg estimate of modified duration \( \text{mdur}_t = 8.8160 \)
  - If yield falls by 1 bp, price rises by 0.088160 percent

\[
97.0468750 \times (1 + 0.088160) = 97.1324
\]

- Bloomberg estimate of convexity \( \text{conv}_t = 0.8713 \)
  - **Positive convexity**: duration declines slightly as yield rises, attenuates price decline
  - Note: Bloomberg divides conventional convexity measure by 100
  - Makes it easier to work with percent rather than decimal changes
- Some types of bonds, e.g. mortgage-backed securities, display large-magnitude **negative convexity**
  - Behaves like negative option gamma, increases risk
Example: response of U.S. 10-year note to shocks

- Impact of 25 bps increase in rates
  - Duration: \(-\text{mdur}_t \Delta y = -8.816 \cdot 0.0025 = -0.0220 \text{ (-2.20 percent)}\)
  - Convexity: \(\frac{1}{2} \text{conv}_t \Delta y^2 = 87.13 \cdot 0.0025^2 = 0.00027 \text{ (0.027 percent)}\)
  - Total: \(-8.816 \cdot 0.0025 + 87.13 \cdot 0.0025^2 = -0.0217 \text{ (-2.17 percent)}\)
Interest rate volatility

- Value-at-Risk (VaR) measure requires interest rate volatility estimate
- Two standard conventions in fixed-income option markets for implied volatility

**Normal volatility**: standard deviation of changes in yield $\sigma_{n,t}$ in basis points

**Yield or Black volatility**: standard deviation of proportional changes in yield $\sigma_{y,t}$ in percent

- Divide normal vol by yield:

$$\sigma_{y,t} = \frac{\sigma_{n,t}}{y_t}$$

- Measures yield changes—oddly—in percent rather than bps
- But avoids—perhaps unnecessarily—negative yield scenarios

- Implied volatilities are (→)risk-neutral estimates
  - Reflect market information, but embed risk premiums embedded in option prices
Applying interest rate volatility

- **Example:** typical (as of late 2016) at-the-money short- to medium-term swaption on 10-year USD swap
  - Normal volatility of about 90bps at annual rate
  - Swap rate of 2.25 percent
  - ⇒ Yield volatility of about $\frac{90}{2.25} = 40$ percent at annual rate
  - Or $\frac{0.4}{\sqrt{256}} = 0.025$ (2.5 bps) per trading day applying (→) square-root-of-time rule
Quantiles of the distribution of yield

- Assume yield lognormally distributed
  - ⇒ Proportional yield changes normally distributed
- Compute return quantiles using yield volatility estimate, appropriate normal quantile, and time horizon $\tau$:
  - $z_\alpha$: $\alpha$-quantile of standard normal distribution
  - $\sigma_{y,t}$: statistical or risk-neutral estimate of itemize volatility
- Estimated $\alpha$-quantile of proportional change in yield from $t$ to $t + \tau$ is then $e^{z_\alpha \sigma_{t} \sqrt{\tau}} - 1$
  - Arithmetic approximation: $z_\alpha \sigma_{y,t}$
- Multiply VaR quantile by yield to obtain yield changes in basis points $y_t \left(e^{z_\alpha \sigma_{t} \sqrt{\tau}} - 1\right)$
  - Equivalent to using normal vol, hence the name

$$z_\alpha \sigma_{y,t} \sqrt{\tau} y_t = z_\alpha \sigma_{n,t} \sqrt{\tau}$$
Computing VaR for a bond

- If the model is accurate, the probability is $\alpha$ that yield will rise (fall) no more than $100z_{\alpha}\sigma_{y,t} (100z_{1-\alpha}\sigma_{y,t})$ percent
- Use upper (lower) tail of normal distribution for long (short) bond position
  - Since bond value declines as rates rise
- Apply (→)delta-gamma approach, with duration and convexity in roles of delta and gamma
  - Set $\Delta y$ to the VaR quantile of the change in yield in bps
- VaR—in percent of value—of a long position is then (using normal vol)
  \[
  mdur_t \, z_{\alpha}\sigma_{n,t} \sqrt{\tau} - \frac{1}{2} \, \text{conv}_t (z_{\alpha}\sigma_{n,t} \sqrt{\tau})^2
  \]
  - The signs are reversed to convert the negative impact of a rise in yield to a positive number
Example: VaR of long U.S. 10-year note position

- Assume yield (0.023381 in example) lognormally distributed, proportional yield changes then normal
- Use ballpark estimate of normal vol drawn from option markets—90 bps per annum—to compute yield volatility and VaR shock
  - ⇒Yield volatility $0.0090 \over 0.023381 = 0.384928$ or 38.5 percent at annual rate
- One-day return quantile of order 0.01 (99 percent confidence level):
  - Divide annual vol by $\sqrt{256} = 16$, multiply by $z_\alpha = 2.32635$
  - ⇒Yield volatility $2.32635 \cdot 0.384928 \over 16 = 0.0559673$ or a 5.6 percent change in yield
  - VaR quantile in basis points is $2.32635 \cdot 0.384928 \over 16 - 0.023381 = 2.32635 \cdot 0.0090 \over 16 = 0.00130857$
- VaR in dollars: initial bond value times proportional change in value using pertinent VaR yield change quantile

\[
\left(8.816 \cdot 0.00131 - \frac{1}{2} 87.13 \cdot 0.00131^2\right) \times 970468.75 = 11123.30
\]