Market equilibrium and relative risk

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Portfolios and diversification

Investor choice

Models of market equilibrium: Capital Asset Pricing Model
Portfolios and diversification

Diversification
Efficient frontier

Investor choice

Models of market equilibrium: Capital Asset Pricing Model
Portfolios and investment choices

- Available investment choices can be expanded by mixing assets in **portfolios**
- Simple approach to identifying available investment choices
  - What combinations of *portfolio expected return* and *portfolio return variance* or *volatility*—representing risk—are available?
  - Are any of these combinations clearly superior or inferior to others?
- Based on expected returns, volatilities and correlations of constituent assets
- **Example:** Facebook Inc. (ticker FB) and Coca-Cola Co. (KO) 18May2012 to 24Sep2020

<table>
<thead>
<tr>
<th></th>
<th>FB</th>
<th>KO</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean daily logarithmic return (%)</td>
<td>0.090177</td>
<td>0.012446</td>
</tr>
<tr>
<td>Standard deviation of daily returns (%)</td>
<td>2.346570</td>
<td>1.141170</td>
</tr>
<tr>
<td>Correlation coefficient</td>
<td></td>
<td>0.21290</td>
</tr>
</tbody>
</table>

- Once we understand menu of available and reasonable choices clearly, we can analyze which ones investors *prefer*
Portfolio expected return

- The **portfolio expected return** is a simple weighted average of the constituent assets’ expected returns:
- In the case of just two constituents:

\[ \mu_p = w\mu_1 + (1 - w)\mu_2, \]

with \( w \equiv \text{asset 1 weight} \)
- Portfolio expected return changes proportionally to a change in constituent expected return
- **Example:** the expected return of a 50-50 FB-KO portfolio is

\[ \mu_p = 0.5 \cdot 0.000902 + 0.5 \cdot 0.000124 = 0.00051311 \]

or 5.13 basis points per day
Portfolio return variance and volatility

- The portfolio return variance is *not* a weighted average of the constituent variances:

\[ \sigma_p^2 = w^2 \sigma_1^2 + (1 - w)^2 \sigma_2^2 + 2w(1 - w)\sigma_1\sigma_2\rho_{12}, \]

- **Portfolio volatility** is its square root: \( \sigma_p = \sqrt{\sigma_p^2} \)
- Portfolio variance and volatility do *not* change proportionally to a change in constituent volatility
- And the portfolio variance can be strongly influenced up or down by return correlation
- **Example:** the return variance of a 50-50 FB-KO portfolio is

\[ \sigma_p^2 = 0.5^2 \cdot 0.000551 + 0.5^2 \cdot 0.000130 \\
+ 0.5 \cdot 0.5 \cdot 0.0235 \cdot 0.0114 \cdot 0.2129 = 0.000199 \]

and the portfolio volatility is \( \sqrt{0.000199} = 0.01409690 \) or 1.410 percent daily.
Diversification is powerful

- **Diversification**: combining assets can lead to a reduction of risk without sacrificing return
- Diversification expands investors’ opportunity set
  - Adding a small amount of even a high-volatility asset can reduce portfolio volatility
  - But effect more limited if return correlation strongly positive
- Lower correlation enables investor to achieve lower portfolio volatility for any given expected return
  - Negative correlation provides the strongest volatility reduction
  - Mixing risky assets can reduce portfolio return volatility even if correlation is positive
Impact of diversification on portfolio return volatility

Left panel: volatility (y-axis) of portfolios combining long positions in KO stock with long positions in FB and KB Home (KBH), assuming a daily return correlation of 0, both plotted as a function of the FB or KBH portfolio weight (x-axis). Right panel: volatility of portfolios combining long positions in KO stock with long positions in FB assuming different non-zero return correlations.
Feasible and efficient portfolios

- Not every **feasible** or **attainable** portfolio is **efficient**
  - For each portfolio, find return and volatility
  - Two portfolios may have same volatility but different returns (or v.v.)
  - Portfolio with same volatility but lower return—or same return but higher volatility—than some other is not efficient

- **Efficient frontier**: return and volatility points of efficient portfolios
  - Traces risk-return tradeoff in mean-variance framework

- **Global minimum variance portfolio** has lowest return and volatility among efficient portfolios

- (→)**Risk-free assets** may also be available for inclusion
  - Have non-zero return (usually but not always positive) but zero volatility
Feasible and efficient portfolios: example

- Portfolio consisting of all or mostly low-return/low-volatility KO not efficient
- Adding some high-return/high-volatility FB lowers portfolio volatility and raises portfolio return
  - Unambiguously more desirable to investors than KO alone
- Portfolios with high share of FB have higher volatility and return
  - May be more desirable to some investors

<table>
<thead>
<tr>
<th>FB wt. (%)</th>
<th>KO wt. (%)</th>
<th>return (%)</th>
<th>volatility (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>100</td>
<td>0.012446</td>
<td>1.141170</td>
</tr>
<tr>
<td>10</td>
<td>90</td>
<td>0.020219</td>
<td>1.101140</td>
</tr>
<tr>
<td>12.92</td>
<td>87.08</td>
<td>0.022486</td>
<td>1.098950</td>
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<tr>
<td>20</td>
<td>80</td>
<td>0.027992</td>
<td>1.111820</td>
</tr>
<tr>
<td>50</td>
<td>50</td>
<td>0.051311</td>
<td>1.409690</td>
</tr>
<tr>
<td>90</td>
<td>10</td>
<td>0.082404</td>
<td>2.139120</td>
</tr>
<tr>
<td>100</td>
<td>0</td>
<td>0.090177</td>
<td>2.346570</td>
</tr>
</tbody>
</table>

In percent. The global minimum variance portfolio is highlighted.
Volatility (x-axis) and mean (y-axis) of portfolios combining long positions in KO and FB stock. Purple plot shows feasible portfolios estimated using the historical return correlation of 0.2129. The heavy part of the plot is the efficient frontier. Orange plot shows efficient frontier if the return correlation were -0.25.
Portfolios and diversification

**Investor choice**
- Investor choice and market outcomes
- Investor optimization

Models of market equilibrium: Capital Asset Pricing Model
Explaining equilibrium asset prices and returns

- Market-clearing process determines asset prices and prospective returns by finding **equilibrium price**, given supply and demand schedules for securities
  - Assumptions about investors determine demand schedules
- Steps in the explanation:
  1. Take investment choices/prospective returns as given, analyze from individual point of view:
    1.1 Identify **efficient portfolios**: portfolios that don’t waste opportunities
    1.2 Explain how individuals choose among efficient portfolios
  2. Once we know how individuals choose, how does market clear and establish the prospective returns individuals face?
- **Mean-variance framework**: payoffs on individual risky asset depend only on mean return and volatility
Investor preferences and risk

- Problems for quantitative definition of risk arise from preferences as well.
- Mathematical optimization requires unambiguous preference ranking of sets of choices, portfolios.
- Even with well-defined probability distribution of outcomes, difficulties in obtaining
  - Unambiguous preference ranking
  - Useful definition of risk aversion
- **Expected utility** axioms: require specification of utility function.
- Approaches include
  - **Mean-variance dominance**: provides limited ability to rank outcomes, doesn’t consider tail returns.
  - **Stochastic dominance**: looks at entire probability distribution.
    - Approaches may contradict one another and may fail to provide unambiguous ranking.
Choosing among portfolios

- Simple model: investor assumed to engage in mean-variance optimization
  - Happiness/wealth/utility increases with mean return and decreases with return volatility
- Modeled via utility function

\[ V(\mu_p, \sigma_p) = \mu_p - \frac{1}{2} k \sigma_p^2, \]

with \( k \) expressing strength of investor’s risk aversion

- **Indifference curves** express mean/volatility tradeoff
  - Defined by fixing utility at \( V^\circ \) and differentiating the utility function

\[ \frac{d\mu_p}{d\sigma_p} \bigg|_{V=V^\circ} = k\sigma_p \]

- The slope is positive: investor must be compensated with additional expected return if risk increases
- Convex to the origin: slope is increasing in \( \sigma_p \)

- Investor chooses efficient portfolio that just touches the highest indifference curve she can achieve
Indifference curves for utility function $V(\mu_p, \sigma_p) = \mu_p - \frac{1}{2} k \sigma_p^2$ with $k = 4$ and $k = 2$ and efficient frontier of portfolios combining long positions in KO stock and FB stock.
Investor choice if there is a risk-free asset

- Suppose there really were a risk-free security with certain return $r^f$
  - Its mean would also be $r^f$ and its volatility zero
- Suppose investor able to lend or borrow freely at risk-free rate
  - Lending: invest in risk-free asset
  - Borrowing: finance additional risky assets (→leverage)
- We can then define

  **Expected excess return** of an asset: the difference $\mu_i - r^f$
  between its expected return and the risk-free rate

  **Sharpe ratio** of an asset: ratio $\frac{\mu_i - r^f}{\sigma_i}$ of excess return to volatility
  - Expected excess return per unit of risk
  - Reported Sharpe ratios usually *ex post*, based on realized/historical estimate of expected future return
Two-fund separation

- Also called **mutual fund theorem**
- All investors have same risky asset portfolio but different amounts of risk-free asset and risky portfolio
  - Risky asset portfolio has same constituents and same weights within the portfolio for everyone
- What is that risky asset portfolio?
- If there is a risk-free asset, efficient frontier $\rightarrow$ ray from $(0, r_f)$ through **tangency portfolio**
  - Tangency portfolio is risky asset portfolio common to all investors
  - Tangency $\Rightarrow$ attainable risky asset portfolio with highest Sharpe ratio
  - $\Rightarrow$ Slope of efficient frontier is highest attainable Sharpe ratio
- Investor mixes risk-free asset and risky portfolio
  - The mix depends on her risk preferences
Optimal investor choice with a risk-free asset

**Efficient frontier** of portfolios combining only long positions in KO stock and FB stock, efficient frontier of portfolios that also include a risk-free asset, with \( r^f = 0 \), and the indifference curve for \( V(\mu_p, \sigma_p) = \mu_p - \frac{1}{2} k\sigma_p^2 \) with \( k = 4 \) at the optimal portfolio of a risk-free as well as risky assets. The **tangency portfolio** is the point on the efficient frontier of portfolios combining risky assets only that is tangent to the efficient frontier, a line through \((0, r^f)\).
## Summary of optimal investor choice

<table>
<thead>
<tr>
<th>Weight</th>
<th>No risk-free asset</th>
<th>Including risk-free asset</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$k = 3$</td>
<td>$k = 4$</td>
</tr>
<tr>
<td>Risk-free</td>
<td>NA</td>
<td>NA</td>
</tr>
<tr>
<td>FB</td>
<td>0.586</td>
<td>0.472</td>
</tr>
<tr>
<td>KO</td>
<td>0.414</td>
<td>0.528</td>
</tr>
<tr>
<td></td>
<td>$k = 3$</td>
<td>$k = 4$</td>
</tr>
<tr>
<td></td>
<td>NA</td>
<td>0.379</td>
</tr>
<tr>
<td></td>
<td>0.866</td>
<td>0.537</td>
</tr>
<tr>
<td></td>
<td>0.134</td>
<td>0.083</td>
</tr>
<tr>
<td></td>
<td>0.063</td>
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</tr>
</tbody>
</table>

Weights in the optimal portfolio of a mean-variance investor with utility function $V(\mu_p, \sigma_p) = \mu_p - \frac{1}{2}k\sigma_p^2$ for $k = 3, 4$. Optimization is over portfolios combining only long positions in KO stock and FB stock, or portfolios that also include a risk-free asset.
Risk premiums and equilibrium market prices

- Risky asset prices embed **risk premiums**:
  - Expected excess return $\mu_i - r^f$ of risky ($r_i$) over risk-free security
  - Market discounts risky income streams at risk-free rate *plus* risk premium

- Investment $i$’s Sharpe ratio $\frac{\mu_i - r^f}{\sigma_i}$ is the ratio of risk premium to volatility

- Risk premiums not directly observable, must be estimated via model of how market finds equilibrium asset prices
  - Equilibrium prices then related to investor preferences as well as assets’ return characteristics
  - Leads also to explanation of relative prices of different risky assets
Portfolios and diversification

Investor choice

Models of market equilibrium: Capital Asset Pricing Model
  Assumptions and conclusions of the CAPM
  The CAPM beta and systematic risk
  Multifactor models of market equilibrium
Overview

- **Capital asset pricing model (CAPM):**
  - All risk premiums driven by risk appetites and common source of risk
  
- Risk premium of any security is compensation for **systematic risk**
  - Related to risk of **value-weighted market portfolio** of all risky assets
  - Obviates need for or validity of security-specific analysis (→ efficient markets)

- Diversification → shedding **uncompensated nonsystematic** or **idiosyncratic risk**

- CAPM a model of relative rather than absolute risk
  - CAPM does not itself provide measure of risk of market portfolio, i.e. systemic risk
  - Volatility estimation
Market equilibrium and relative risk

Models of market equilibrium: Capital Asset Pricing Model

Assumptions and conclusions of the CAPM

CAPM assumptions

- Agents are all mean-variance optimizers, and don’t care about other distributional characteristics
- Complete information and agreement on means and variances of uncertain security returns
- Agents are not, however, identical
  - But they may have different risk preferences/aversion: differ in their pricing of risk
- Market portfolio well-defined, has identifiable observable counterpart, non-traded assets unimportant
  - Conventionally proxied by broad stock index, e.g. S&P 500, the observable market factor
- Market clearing with no frictions
- There is a risk-free asset at which all agents can freely borrow or lend
  - Risk-free rate typically proxied by U.S. Treasury bill yield or return
Key results of the CAPM

- The market portfolio is an efficient portfolio
  - Since all investors agree on expected return and volatility of each asset, each chooses the same portfolio of risky assets
  - All markets clear, so all investors must be choosing market portfolio to combine with risk-free asset
- Mutual fund theorem: all investors will engage in two-fund separation
  - Each will choose a mix of the market portfolio and the risk-free asset depending on her own risk preferences
- The market portfolio is the only source of risk
  - CAPM consistent with single risk premium “risk-on/risk-off” world
The CAPM beta

- CAPM a model of prices and risk relative to market portfolio
  - Any specific asset $i$’s risk premium related via **beta** to co-movement with that of market portfolio $\mu_m - r^f$

\[ \mu_i - r^f = \beta_i (\mu_m - r^f) \]

- Specific securities thereby priced relative to one another
- Where’s $\alpha$? It’s **zero** in the CAPM
- An asset $i$’s beta can be calculated from its excess return volatility $\sigma_i$, that of the market portfolio $\sigma_m$, and their correlation $\rho_{i,m}$:

\[ \beta_i = \rho_{i,m} \frac{\sigma_i}{\sigma_m} \]

- Beta increasing in correlation **and** asset volatility
The CAPM and risk premiums in equilibrium

- Any asset’s Sharpe ratio related to that of the market portfolio by
  \[
  \frac{\mu_i - r^f}{\sigma_i} = \rho_{i,m} \frac{\mu_m - r^f}{\sigma_m}
  \]

- Since \( \rho \leq 1 \), no asset can have a higher Sharpe ratio in equilibrium than the market portfolio
Systematic and nonsystematic risk

- Asset $i$’s excess return variance $\sigma_i^2$ a measure of its risk
- Market portfolio’s excess return variance $\sigma_m^2$ a measure of market risk
  - Beta captures comovement with/risk sensitivity to market factor
- $\sigma_i^2$ can be decomposed into
  
  **Systematic risk:** $\beta_i^2 \sigma_m^2$, the part of $\sigma_i^2$ (or $\sigma_i$) related to fluctuations in market returns
  
  **Idiosyncratic or nonsystematic risk:** the remainder $\sigma_i^2 - \beta_i^2 \sigma_m^2$, due to vagaries of individual firm’s returns alone

- Can be expressed as shares of total

  $$\frac{\beta_i^2 \sigma_m^2}{\sigma_i^2} + \frac{\sigma_i^2 - \beta_i^2 \sigma_m^2}{\sigma_i^2} = 1$$

- Systematic risk share in terms of excess return correlation:

  $$\frac{\beta_i^2 \sigma_m^2}{\sigma_i^2} = \rho_{i,m}^2 \frac{\sigma_i^2}{\sigma_m^2} \frac{\sigma_m^2}{\sigma_i^2} = \rho_{i,m}^2$$
Computing the CAPM beta

- Based on simple **linear regression** model of security $i$’s excess returns on market portfolio’s excess return

\[ r_{it} - r_{ft} = \alpha_i + \beta_i (r_{mt} - r_{ft}) + u_{it}, \quad t = 1, \ldots, T \]

- $u_{it}$ assumed i.i.d. or normal, and independent of $r_{mt} - r_{ft}$
- Daily, weekly or monthly observations
- Leads to estimates $\hat{\alpha}_i$ and $\hat{\beta}_i$
- The CAPM model predicts $\alpha_i = 0$
- $\hat{\sigma}_{u,i}$ is the **residual mean square** or **standard error of the regression**
  - $\hat{\sigma}_{u,i}^2$ an unbiased estimate of variance $\sigma_{u,i}^2$ of the model error term $u_i$
Computing beta: an example

Estimating systematic and nonsystematic risk

- Interpret standard regression properties in context of CAPM
- Decompose stock $i$’s observed excess return variance
  \[
  \frac{\sum_t (r_{it} - r_{ft})^2}{T-1} = \hat{\sigma}_i^2
  \]
  - Explained and residual variance
    - → Systematic and nonsystematic risk
  - Explained or systematic variance can be expressed as
  \[
  \hat{\beta}_i^2 \frac{\sum_t (r_{mt} - r_{ft})^2}{T-1} = \beta_i^2 \hat{\sigma}_m^2 = R^2 \hat{\sigma}_i^2
  \]
  - The estimated $\hat{\beta}_i^2$ times the variation in market excess returns
  - The estimated coefficient of determination (unadjusted) $R^2$ times the variation in stock $i$’s excess returns
  - $R^2$ equals sample excess return correlation
- Residual or nonsystematic variance is the difference:
  \[
  \hat{\sigma}_i^2 - \hat{\beta}_i^2 \hat{\sigma}_m^2 = \frac{T - 2}{T - 1} \hat{\sigma}_{u,i}^2
  \]
Example: systematic and nonsystematic risk

<table>
<thead>
<tr>
<th></th>
<th>FB</th>
<th>KO</th>
<th>KBH</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameter estimates:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\hat{\alpha}_i$</td>
<td>0.0004435</td>
<td>$-$0.0001763</td>
<td>0.0001699</td>
</tr>
<tr>
<td>$\hat{\beta}_i$</td>
<td>1.051</td>
<td>0.674</td>
<td>1.456</td>
</tr>
<tr>
<td>Goodness-of-fit and correlation:</td>
<td></td>
<td></td>
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<tr>
<td>$R^2$ (unadjusted)</td>
<td>0.23068</td>
<td>0.40096</td>
<td>0.28040</td>
</tr>
<tr>
<td>Adjusted $R^2$</td>
<td>0.23032</td>
<td>0.40067</td>
<td>0.28006</td>
</tr>
<tr>
<td>Excess return correlation to S&amp;P $\hat{\rho}_{i,m}$</td>
<td>0.21293</td>
<td>0.22568</td>
<td>0.48030</td>
</tr>
<tr>
<td>Risk decomposition:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Variance of excess returns $\hat{\sigma}_i^2$</td>
<td>0.0005507</td>
<td>0.0001302</td>
<td>0.0008689</td>
</tr>
<tr>
<td>Systematic variance $\hat{\beta}_i^2 \hat{\sigma}_m^2$</td>
<td>0.0001270</td>
<td>0.0000522</td>
<td>0.0002436</td>
</tr>
<tr>
<td>Nonsystematic variance $\hat{\sigma}_i^2 - \hat{\beta}_i^2 \hat{\sigma}_m^2$</td>
<td>0.0004236</td>
<td>0.0000780</td>
<td>0.0006253</td>
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<td>Risk decomposition (share of total variance):</td>
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<td></td>
<td></td>
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<tr>
<td>Systematic variance</td>
<td>0.23068</td>
<td>0.40096</td>
<td>0.28040</td>
</tr>
<tr>
<td>Nonsystematic variance</td>
<td>0.76932</td>
<td>0.59904</td>
<td>0.71960</td>
</tr>
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Excess returns relative to 3-month U.S. T-bill yield.
Beta, correlation, and volatility

- Excess returns of high-beta stocks typically, but not always strongly correlated with market’s.
- Relationships among beta, correlation, and market and asset volatilities constrained by

\[-1 \leq \beta_i \frac{\sigma_m}{\sigma_i} = \rho_{i,m} \leq 1\]

- If high-beta returns much more volatile than market returns, correlation may be weak
  - \(\Leftrightarrow\) Systematic risk share low, in spite of high beta
- High beta associated with higher return volatility than market, but tracking overall behavior of market return volatility
- Examples:
  - BAC has beta nearly triple that of T, but only moderately higher excess return correlation to the market
  - BAC has high beta, but systematic risk just a bit over \(\frac{1}{2}\) of total
Volatility behavior of high- and low-beta stocks

Empirical validation of the CAPM

- CAPM implies **cross-sectional variation** across individual stocks in expected returns at a point in time fully explained by beta
  - Expected returns measured empirically via realized excess returns
- In tests, CAPM does not fully capture systematic influences on individual stock prices
- CAPM a **single-factor** or **single-index** model
  - \( \Rightarrow \) Search for additional explanatory factors
- **Fama-French three-factor model** includes in addition to market factor
  - **Small Minus Big** (SMB): average return on small-cap minus return on large-cap portfolios
  - **High Minus Low** (HML): average return on value (high book-to-market/low price-to-book ratio) minus return on growth (low book-to-market) portfolios
- **Momentum** factor: stocks with high recent returns
Limitations of the market proxy

- **Roll critique** or **market proxy problem**
- Validation of CAPM requires accurate identification of market portfolio
- Conventional proxies, e.g. S&P 500 index omit important elements of wealth, esp. human capital, non-U.S. assets
- Analogous to the (→) **joint hypothesis problem** in testing market efficiency
  - Is the model or the proxy wrong?
Consumption CAPM

- Simplicity of the mean-variance optimization model contributes to empirical shortcomings of CAPM
- Investors care about many things, e.g.
  - Do high payoffs occur in good times or in bad, when they are more valuable?
  - Do high payoffs occur when investment opportunities are good, or capital goods cheap relative to consumption goods?
  - Tail risks, *rare consumption disasters*, e.g. financial crises and wars
- **Consumption CAPM**: asset prices driven by risk appetite, covariance of return with utility of consumption across states
  - Multiple periods, not just “now” and “future”
  - Declining *marginal utility of consumption*: additional consumption less valuable at higher consumption level
  - \( \Rightarrow \) Low asset payoffs in bad times less valuable to risk-averse agents (“anti-insurance”), lead to lower asset price/higher risk premium
  - \( \Leftrightarrow \) High payoffs in bad times \( \rightarrow \) higher asset price/lower risk premium
Stochastic discount factor and asset prices

- **Stochastic discount factor:** (SDF) discounted value of marginal utility of consumption, captures
  - **Time preference:** near-term more valuable than future consumption
  - **Risk preference:** how fast does marginal utility of consumption decline?
- SDF the same (for a particular or representative agent) for all assets
  - A different value of the SDF for each possible future state
- But state-contingent payoffs different for each asset
- Assets have positive risk premiums—are cheaper—if covariance of payoffs with SDF high
Risk factor approach to asset pricing

- Reduce dimensionality of covariance matrix of asset returns
  - Economic/market data that explain variance of returns
  - Possibly unobservable or latent characteristics
  - Definition of risk factors depends on model and available data
  - Asset returns as risk factors for other securities

- In efficient capital markets, risk premiums reflect priced factors
  - \(\rightarrow\) Arbitrage pricing theory (APT) introduces multiple risk factors
    - Asset or portfolio returns accurately predicted by returns on a set of factors \(\Rightarrow\) portfolio can be replicated by the factors
    - Example: Fama-French model prices stocks more accurately than CAPM
      - Each factor carries with it a risk premium that compensates for low return just when you can least afford it

- Two-fold motivation of risk factor approach:
  - **Reality**: many securities, far fewer meaningfully independent influences on them
  - **Parsimony**: make high-dimensional problem tractable and intuitive