Market risk measurement in practice

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Outline

- Nonlinearity in market risk
- Market risk measurement for options
- Portfolio VaR
- Market risk in insurance
Nonlinearity in market risk
  Nonlinearity and convexity
  Delta-gamma and option risk

Market risk measurement for options

Portfolio VaR

Market risk in insurance
Nonlinearity in market risk

- **Nonlinearity**: P&L or payoff of a security doesn’t respond proportionally to risk factor returns
- **Examples** of securities with nonlinear payoffs:
  - Options: *gamma risk*
  - Bonds: *convexity risk* in non-callable as well as callable coupon and zero-coupon, mortgage-backed securities
- Security value $f(S_t)$ a function of risk factor $S_t$,
- $f(S_t)$ has second and nonzero higher derivatives $\rightarrow$
  - Large-magnitude returns have a proportionally larger or smaller P&L impact than small returns
  - $\rightarrow$ Price changes in one direction may have a larger P&L impact than changes in the opposite direction
VaR techniques for nonlinear positions

**Simulation with full repricing** using asset valuation model, e.g. Black-Scholes formula

- Can use Monte Carlo or historical simulation of underlying price or risk factor returns
- But revaluation of position at each simulated return may itself require “expensive” simulations

**Delta-gamma** using linear-quadratic approximation of P&L responses to risk factor returns

- Trades accuracy for speed
- Tractable and quite accurate in many cases
- But may be inaccurate for some portfolios
- Can be combined with Monte Carlo or historical simulation of risk factor returns
Option risk and the “greeks”

- Option risk stemming from underlying asset price risk is nonlinear
  - Price risk of the underlying asset (→ delta, gamma)
  - Sensitivity to underlying price greatest near strike, may fall off rapidly in- or out-of-the-money
  - → Apply delta-gamma, with \( f(S_t) \) representing option price or value

- Options are exposed to other risk factors, including
  - Interest-rate risk or rho, since an option matures at a future date
  - Implied volatility or vega risk

- Options have time value that decays over time at a rate theta
  - Theta is not a risk, but a deterministic quantity
  - Depends on interest rates, implied volatility, and terms of the option
  - Particularly high relative to option value for short-term options
Definitions of delta and gamma

**Delta** $\delta_t$: rate at which option value changes with underlying asset price

\[
\delta_t \equiv \frac{\partial f(S_t)}{\partial S_t}
\]

- $0 < \delta_t < 1$ for plain-vanilla call option
- $-1 < \delta_t < 0$, for plain-vanilla put

**Gamma** $\gamma_t$: rate at which delta changes with underlying asset price

\[
\gamma_t \equiv \frac{\partial}{\partial S_t} \delta_t = \frac{\partial^2 f(S_t)}{\partial S_t^2}
\]

- $\gamma_t \geq 0$ for a vanilla put or call option
The delta-gamma approximation

- Approximate change in value $f(S_t)$ of option on 1 unit of underlying asset—or any security—if $S_t$ changes by $\Delta S$:

$$f(S_t + \Delta S) - f(S_t) \approx \delta_t \Delta S + \frac{1}{2} \gamma_t \Delta S^2$$

- With other market variables—volatility, risk-free rate, cash flow rate—held constant
- For vanilla option, $f(S_t)$ can represent **Black-Scholes formula**
  - $S_t$ the underlying price
  - With implied volatility, risk-free rate and cash flow rate (dividends, foreign interest, etc.) held constant
- Many other securities have nonlinear responses to changes in a risk factor that can be described similarly
  - For example, bond value can be represented by first- and second-order sensitivities to interest rates
Nonlinearity in market risk

**Market risk measurement for options**
  Applying delta-gamma to the value of an option
  Nonlinearity and option risk

Portfolio VaR

Market risk in insurance
## Approximating the option return distribution

- Extension of parametric normal VaR approach
  - $\Delta S$—change in underlying price—same as in parametric normal VaR
  - But $f(S_t + \Delta S) - f(S_t)$ the change in option value
- Assume arithmetic returns normally distributed:

$$\frac{\Delta S}{S_t} \sim \mathcal{N}(0, \sigma^2 \tau)$$

- Estimate return volatility $\sigma$ of underlying price
- Quantile $z_p \sigma \sqrt{\tau}$ represents scenario for future underlying price return with probability $p$
- $(1 - p)$-quantile of change in option value approximated by

$$\delta_t z_{1-p} \sigma \sqrt{\tau} S_t + \frac{1}{2} \gamma_t (z_{1-p} \sigma \sqrt{\tau} S_t)^2 \quad \text{for long call option}$$

$$\delta_t z_p \sigma \sqrt{\tau} S_t + \frac{1}{2} \gamma_t (z_p \sigma \sqrt{\tau} S_t)^2 \quad \text{for long put option}$$
Delta-gamma VaR for option positions

- Value of an option position: \( xf(S_t) \), with \( x \) the number of options
  - Apply \( x > 0 \) for a long option position, \( x < 0 \) for a short position
  - Use appropriate signs for \( \delta_t \) and \( \gamma_t \) for P&L of put and call
- \( \tau \)-period VaR at confidence level \( \alpha \) for a long option position:
  - Unhedged long call or short put suffers losses when \( S_t \) falls
    \[
    \text{VaR}_t(\alpha, \tau) = -x \left[ \delta_t z_{1-\alpha} \sigma \sqrt{\tau S_t} + \frac{1}{2} \gamma_t \left( z_{1-\alpha} \sigma \sqrt{\tau S_t} \right)^2 \right]
    \]
    \[0 < \delta_t < 1, \gamma_t \geq 0, x > 0 \quad \text{for a long call position}\]
  - Unhedged long put or short call suffers losses from higher \( S_t \)
    \[
    \text{VaR}_t(\alpha, \tau) = -x \left[ \delta_t z_\alpha \sigma \sqrt{\tau S_t} + \frac{1}{2} \gamma_t \left( z_\alpha \sigma \sqrt{\tau S_t} \right)^2 \right]
    \]
    \[-1 < \delta_t < 0, \gamma_t \geq 0, x > 0 \quad \text{for a long put position}\]
Nonlinearity and option risk

- Underlying price moves amplifies loss for long call or short put
- $\delta_t \geq 0$ for a long call, $\delta_t \leq 0$ for a long put, so
  - Unhedged long call and short put positions behave like long positions in underlying
  - Unhedged short call and long put positions behave like short positions in underlying
  - Large-magnitude $\delta_t$ increases VaR for a long option position
- $\gamma_t \geq 0$ for a long call or put, so
  - Gamma dampens P&L for long option positions and amplifies P&L for short option positions
  - High $\gamma_t$ reduces VaR for a long option position and increases VaR for a short option position
  - Difference between P&L results of large and very large underlying price changes is also greater for short positions
Nonlinearity and option risk

Each panel plots the P&L in currency units of an unhedged option position, using the Black-Scholes valuation formula.
Example of delta and gamma calculations

- Short position in at-the-money (ATM) put on one share of non-dividend paying stock with one month to expiry
  - Initial stock price $S_t = 100$, money market rate 1 percent, implied volatility 15 percent
  - Short position, so reverse signs of $\delta_t$ and $\gamma_t$
- Model of the underlying price: assume zero drift, lognormal returns
- Assume volatility estimate/forecast 15 percent per annum, equal to implied vol
  - But note historical volatility estimate generally somewhat lower than implied vol (volatility premium)
- To compute one-week VaR ($\tau = \frac{1}{52}$), compare option value at initial underlying price to value in VaR scenario
  - P&L: value of 3-week options with shock to underlying price minus value without shock
  - Excludes time decay—which is non-random—from revaluation
  - But retain zero-drift assumption on underlying price
Delta-gamma VaR results

- Black-Scholes delta of 3-week ATM put is -0.4857; gamma is 0.1050
  - Short put has delta equivalent of 48.50 worth of stock
  - And high gamma: e.g. delta declines to -0.3829 if price rises to 101
- Compute VaR scenarios—quantiles of $S_{t+\tau}$ for $\alpha = 0.95, 0.99$
  - With $\sigma = 0.15$ annually, $\sigma\sqrt{\tau} = 0.0208$
- Delta-gamma results in good approximation for non-extreme changes in $S_t$
- Compare VaR computed using Black-Scholes formula, changing only underlying price

<table>
<thead>
<tr>
<th>VaR scenario</th>
<th>delta-only</th>
<th>VaR estimates</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>delta-only</td>
</tr>
<tr>
<td>$\alpha = 0.95$</td>
<td>-3.422</td>
<td>1.662</td>
</tr>
<tr>
<td>$\alpha = 0.99$</td>
<td>-4.839</td>
<td>2.350</td>
</tr>
</tbody>
</table>
Delta and delta-gamma approximations

Short put option struck at 100, initial underlying asset price 100, money market rate 1 percent, valued using Black-Scholes formula.
Nonlinearity in market risk

Market risk measurement for options

**Portfolio VaR**
- Algebra of portfolio VaR
- Example of portfolio VaR
- Delta-normal approach to VaR computation

Market risk in insurance
Most VaR applications involve portfolios

- Multiple risk factors and/or multiple positions, e.g.
  - Hedged positions
  - Relative value trades such as spread trades
  - More general portfolios of long and short positions
  - Portfolio products such as structured credit
- Introduces additional complications to convexity:
  - Need to take account of *correlations* of risk factor returns
- May have P&L that is **nonmonotone** with respect to a risk factor’s returns
  - Sign of \( \frac{\partial f(S_t)}{\partial S_t} \) may change with \( S_t \)
- Example of nonmonotonicity: delta-hedged options, exposed to gamma
  - Long gamma: largest losses for smallest underlying returns
- **Delta-normal**: simple approach to computing portfolio VaR for market risk
  - But may be drastically inaccurate for some portfolios, e.g. delta-hedged options
Parametric computation of portfolio VaR

- Apply algebra of portfolio returns to sequence of computations of parametric single-position VaR
- Assume logarithmic risk factor returns jointly normal
  \[ r_t = (r_{1,t}, r_{2,t}, \ldots, r_{n,t})' \]
- Risk factor returns have time-varying variance-covariance matrix \( \Sigma_t \)
- Portfolio volatility with portfolio weights on risk factors an \( n \)-dimensional vector \( w \):
  \[ \sigma_t = w' \Sigma_t w \]
- VaR in return terms at confidence level \( \alpha \) equal to \( z_\alpha \sigma_t \sqrt{\tau} \)
Estimating the covariance matrix

- Compute volatilities and correlations of the \( n \) risk factors from the variances and covariances constituting \( \Sigma_t \)
- Can be estimated via EWMA, with a decay factor \( \lambda \), via

\[
\Sigma_t = \frac{1 - \lambda}{1 - \lambda^m} \sum_{\tau=1}^{m} \lambda^{m-\tau} r_t' r_t
\]

\[
\approx \lambda \Sigma_{t-1} + (1 - \lambda) r_t' r_t
\]

- \( r_t' r_t \) an outer product of return vector on date \( t \)
- Square matrix with same dimension as \( \Sigma_t \)
- VaR in return terms at confidence level \( \alpha \) equal to \( z_\alpha \sigma_t \sqrt{\tau} \)
Two-position portfolio

- Two positions or risk factors: 3 parameters to estimate

\[
\Sigma_t = \begin{pmatrix}
\sigma_{1,t}^2 & \sigma_{1,t}\sigma_{2,t}\rho_{12,t} \\
\sigma_{1,t}\sigma_{2,t}\rho_{12,t} & \sigma_{2,t}^2
\end{pmatrix}
\]

- Return volatility of a two-position portfolio

\[
\sigma_t^2 = w_1^2 \sigma_{1,t}^2 + w_2^2 \sigma_{2,t}^2 + 2w_1w_2\sigma_{1,t}\sigma_{2,t}\rho_{12,t}
\]
Long-short currency trade

- Long EUR and short CHF against USD, potentially motivated by view that
  - Extremely sharp safe-haven appreciation of CHF relative to EUR since beginning of global financial crisis economically unsustainable
  - “Risk-on” strategy: global recovery, decrease in uncertainty and risk aversion will reverse CHF appreciation

- Weights are 1 and -1

- Measure of risk at time $t$ is
  \[
  (1, -1) \Sigma_t \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \sigma_{1,t}^2 + \sigma_{2,t}^2 - 2\sigma_{1,t}\sigma_{2,t}\rho_{12,t}
  \]

- VaR expressed as quantile of USD portfolio loss relative to market value of one side of the trade
EUR-USD and USD-CHF risk parameters 2015-2016

EUR-USD and USD-CHF exchange rates, daily, 30Sep2015 to 30Sep2016. USD-CHF rates on an inverted scale.

Return volatilities and correlation of EUR-USD and USD-CHF exchange rates, daily, 28Oct2015 to 30Sep2016. Estimated via EWMA with decay factor $\lambda = 0.94$. 
Long EUR-USD versus short USD-CHF risk and returns 2015-2016

Cumulative returns on a portfolio consisting of a long position in EUR and position in CHF against USD, daily, 30Sep2015 to 30Sep2016.
Delta-normal VaR

- Delta-normal VaR: form of parametric VaR
- Simplification of VaR by means of two approximations:
  - Linearize exposures to risk factors
  - Treat arithmetic, not log returns, as normally distributed
- Letting $f(S_t)$ now represent the value of a security not necessarily an option, delta $\delta_t$ defined as the derivative or value w.r.t. risk factor:
  $$\delta_t = \frac{\partial f(S_t)}{\partial S_t}$$
  - $\delta_t$ may be positive or negative, $>1$ in magnitude
  - How many deltas and how they are measured depend on modeling choices: $S_t$ may be a vector
- Limitations: doesn’t capture convexity, other non-linearities
Delta equivalents

- **Delta equivalent** $x\delta_t S_t$ of a position
  - Or $\delta_t S_t$ per unit

- Measure of exposure, states how position affected by unit underlying risk factor return
  - Delta equivalent plays crucial role in hedging option risk
  - At underlying price $S_t$, position of $x$ options with $\delta_t$ has same response to small price change as underlying position $x\delta_t S_t$
Delta-normal VaR for a single position

- In many cases $\delta_t = \pm 1$
  - If risk factor identical to the security
    - Often the case for major foreign currencies, equity indexes
    - $\delta_t = -1$ for short position
  - Value of a security varies one-for-one with risk factor
    - E.g. local currency value of foreign stock as function of exchange rate
- Delta-normal VaR for a single position exposed to single risk factor at confidence level $\alpha$:

$$\text{VaR}_t(\alpha, \tau)(x) = -z_{1-\alpha} \sigma \sqrt{\tau} x \delta_t S_t$$

- Identical to approximation for single long position parametric VaR
- For short position, uses $z_{1-\alpha}$ rather than $z_\alpha$, offset by $\delta_t = -1$
- Normality rather than lognormality of returns $\Rightarrow$ long and short positions have identical VaR
- Single position exposed to several risk factors (→ portfolio VaR)
Nonlinearity in market risk

Market risk measurement for options

Portfolio VaR

Market risk in insurance
  Annuities and market risk
  Inflation risk
Types of annuities

- **Annuities** are contracts for exchange of a specified sequence of payments between an annuitant and intermediary, generally an insurance company.
- Very wide variety of types.
- Payments by annuitant may be a lump sum or periodic over a future time interval.
  - Annuities with periodic future payments may **lapse** or include **early surrender penalties**.
- Payments by insurance company may be fixed or vary:
  - **Fixed annuity**: payments or interest rate fixed over time.
  - **Variable annuity**: payments vary with return on a specified portfolio, generally equity-focused.
- Annuities may include guarantees by insurance company, such as **guaranteed minimum benefits**.
Risks of annuity issuance

- Market risks interact with risks arising from guarantees and policyholder behavior
- Variable annuities generally provide guaranteed minimum return
  - Economically equivalent to sale of put option on equity market by insurer to policyholder
  - Annuity is underpriced if value of put not fully incorporated
- Large losses to U.S. insurers in 2008
  - Hartford Life became a Troubled Asset Relief Program (TARP) recipient
- Fixed annuity issuance exposed to **convexity risk**
  - Assets generally duration-matched to liabilities
  - But liabilities exhibit greater convexity due to guarantees and policyholder behavior
  - Economically equivalent to sale of put option on bond market by insurer to policyholder
- Rising interest rates: early surrender optimal → duration falls rapidly
- Falling interest rates: minimum guaranteed rate in effect → duration rises rapidly
Inflation risk

- **Inflation rate risk** is the risk of loss from a rise in the general price level
  - Directly affects securities with payoffs defined in nominal terms
  - Indirectly affects real assets by affecting macroeconomic conditions
- Inflation difficult to hedge
  - **Inflation-indexed bonds** have yields defined in real terms
  - **Inflation swaps** and other derivatives
Insurance company exposure to inflation

- Insurers may benefit from inflation
- Long-term liabilities generally defined in nominal terms
  - Generally not fully hedged against changes in interest rates
  - And substantial allocation to real assets: real estate, equities
- Permanent risk in inflation rate reduces real value of liabilities