Default correlation in the single-factor model

Portfolio credit VaR in the single-factor model
Default correlation in the single-factor model

Single-factor model for portfolios
Conditional independence in the single-factor model

Portfolio credit VaR in the single-factor model
Standard model for portfolios: overview

- Correlations of two individual firms’ asset returns key parameters of their default correlation
- Assume no correlation between idiosyncratic risk of different firms
- (Eventually,) assume all obligors identical:
  - Same default probability for all credits
  - Same default correlation for all pairs of credits
- Exploit conditional independence: once a realization of the market factor is stipulated, firms’ returns independent
- Law of Large Numbers $\Rightarrow$ idiosyncratic risk disappears
- Model distribution of portfolio credit loss as if it were probability distribution of single-obligor default
  - Correlation nonetheless affects default distribution, in conjunction with market shock
Asset return correlation in the single-factor model

- Firms $i = 1, 2, \ldots$, each with its own $\beta_i$ to the market factor $m$ and its own standard normal idiosyncratic shock $\epsilon_i$:

$$ r_i = \beta_i m + \sqrt{1 - \beta_i^2} \epsilon_i, \quad i = 1, 2, \ldots $$

- $\beta_i$ is correlation of firm $i$’s return to market return
- Assume no correlation between idiosyncratic risk of different firms: $\epsilon_i$ uncorrelated across firms:

$$ \mathbb{E}[\epsilon_i \epsilon_j] = 0, \quad i, j = 1, 2, \ldots $$

- $\Rightarrow$ Asset returns of firms $i$ and $j$ follow bivariate standard normal distribution
  - Mean of each firm’s return is 0, variance of each firm’s return is 1
  - Asset return correlation of firms $i$ and $j$ is $\beta_i \beta_j$
  - Example: $\beta_i = 0.25, \beta_j = 0.5 \Rightarrow$ asset return correlation 0.125
Asset return and default correlation

- Asset return correlation $\beta_i \beta_j$ related to—but not exactly the same concept as—firm $i$ and $j$ default correlation
- Default correlation $\rho_{ij}$ related to asset return correlation $\beta_i \beta_j$ via

$$\rho_{ij} = \frac{\pi_{ij} - \pi_i \pi_j}{\sqrt{\pi_i (1 - \pi_i)} \sqrt{\pi_j (1 - \pi_j)}}$$

- $\Phi(r_i, r_j; \beta_i \beta_j)$ denotes asset returns’ joint cumulative probability distribution function (CDF)
  - $\Phi(x_i, x_j; \rho_{ij})$ is the joint CDF of two standard normal variates $x_i$ and $x_j$ with a correlation of $\rho_{ij}$
  - $\pi_{ij} = \Phi(k_i, k_j; \beta_i \beta_j)$ is $i$th, $j$th firms’ joint default probability
    - $k_i, k_j$: firm $i$ and $j$ default thresholds
- Asset return correlation and default correlation thus related by

$$\Phi(k_i, k_j; \beta_i \beta_j) = \pi_i \pi_j + \rho_{ij} \sqrt{\pi_i (1 - \pi_i)} \sqrt{\pi_j (1 - \pi_j)}$$
Correlated and uncorrelated defaults

Simulation of defaults applying the single-factor model in a portfolio of two credits, both with \( \pi = 0.01 \). Left panel: correlation coefficient \( \rho = 0 \). Right panel: correlation coefficient \( \rho = 0.50 \). Orange grid lines are placed at default thresholds. Simulated return pairs marked by points if they result in default of at most one credit and by \( x \)'s if they result in default for both. Realizations of the asset return pair have a 99.5 percent probability of falling within the density contour.
Asset return and default correlation: example

- Identical firms with common default threshold \( k \) and probability \( \pi = 0.01 \)
- Asset return correlation and default correlation related by

\[
\Phi(k, k; \beta^2) = \pi^2 + \rho\pi(1 - \pi)
\]

- Use relationship to
  - Assume value for default correlation and solve joint default probability \( \Phi(k, k; \beta^2) \) for asset correlation \( \beta^2 \)
  - Assume value for \( \beta \) and calculate default correlation \( \rho \) via \( \Phi(k, k; \beta^2) \)

<table>
<thead>
<tr>
<th></th>
<th>( \beta )</th>
<th>( \sqrt{0.25} )</th>
<th>( \beta^2 )</th>
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<tr>
<td>Market return correlation</td>
<td>0.5251</td>
<td>0.5251</td>
<td>0.2757</td>
<td>0.25</td>
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<td>Asset return correlation</td>
<td>0.2757</td>
<td>0.2757</td>
<td>0.25</td>
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<td>Default correlation</td>
<td>0.04</td>
<td>0.04</td>
<td>0.0341</td>
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<tr>
<td>Joint default probability</td>
<td>( 4.9600 \times 10^{-4} )</td>
<td>( 4.9600 \times 10^{-4} )</td>
<td>( 4.3752 \times 10^{-4} )</td>
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Market factor and conditional independence

- Let $m$ take on a particular realization $\bar{m}$
- The standardized asset return—now has only one random driver, idiosyncratic factor $\epsilon_i$

\[ r_i = \beta_i \bar{m} + \sqrt{1 - \beta_i^2} \epsilon_i, \quad i = 1, 2, \ldots \]

- Distance-to-default—the default-triggering return—becomes $-k_i + \beta_i \bar{m}$
- $\epsilon_i$ independent $\Rightarrow$ conditional returns of two different obligors

\[ \sqrt{1 - \beta_i^2} \epsilon_i, \sqrt{1 - \beta_j^2} \epsilon_j, i \neq j \]

are independent

- $\Rightarrow$ **Conditional independence**: default outcomes for different firms independent
  - Conditioning is on realization of market risk factor
Conditional default distribution of a single obligor

- Once the market shock is known:
  - *Mean* of the return distribution *changes*: $0 \rightarrow \beta_i \bar{m}$
  - *Variance* of the return distribution *reduced*: $1 \rightarrow 1 - \beta_i^2$
    - Because we have eliminated market factor as source of variation
    - And *standard deviation* from $1 \rightarrow \sqrt{1 - \beta_i^2}$
  - *Distance-to-default* changes: $-k_i \rightarrow -(k_i - \beta_i \bar{m})$
    - In standard units: $-k_i \rightarrow -\frac{k_i - \beta_i \bar{m}}{\sqrt{1 - \beta_i^2}}$
  - *Default probability* $\rightarrow \Phi\left(\frac{k_i - \beta_i \bar{m}}{\sqrt{1 - \beta_i^2}}\right)$
    - With $\Phi(x)$ the CDF of a standard normal variate $x$

- $\Rightarrow$ **Conditional default probability distribution function**:

$$p_i(m) = P[r_i \leq k_i|m] = \Phi\left(\frac{k_i - \beta_i m}{\sqrt{1 - \beta_i^2}}\right), \quad i = 1, 2, \ldots$$
Conditional default probability: given market shock

Density and cumulative probability as a function of idiosyncratic shock. Graph assumes $\beta_i = 0.4, k_i = -2.33$ (⇔ $\pi_i = 0.01$), and $\bar{m} = -2.33$. The unconditional default distribution is a standard normal, while the conditional default distribution is $\mathcal{N}(\beta_i \bar{m}, \sqrt{1 - \beta_i^2}) = \mathcal{N}(-0.9305, 0.9165)$. The orange area in the density plot and horizontal grid line in the cumulative distribution plot identify $p(\bar{m})$, as in the example.
Conditional default distributions: example

- Firm: $\beta_i = 0.4, k_i = -2.33$ (so $\pi_i = 0.01$)
- Market shock: $\bar{m} = -2.33$ (sharp downturn)

<table>
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<th>Conditional</th>
<th>Change</th>
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<tr>
<td>Mean return</td>
<td>0</td>
<td>-0.9305</td>
<td>-0.9305</td>
</tr>
<tr>
<td>Return variance</td>
<td>1</td>
<td>0.8400</td>
<td>-0.1600</td>
</tr>
<tr>
<td>Return std. deviation</td>
<td>1</td>
<td>0.9165</td>
<td>-0.0835</td>
</tr>
<tr>
<td>Distance-to-default</td>
<td>2.33</td>
<td>1.3958</td>
<td>-0.9305</td>
</tr>
<tr>
<td>(standardized)</td>
<td>2.33</td>
<td>1.5230</td>
<td>-0.8034</td>
</tr>
<tr>
<td>Default probability</td>
<td>0.01</td>
<td>0.0639</td>
<td>0.0539</td>
</tr>
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</table>
Properties of the conditional distribution

- Once the market factor is realized, the default distributions of individual loans/obligors are independent.
- But the market factor continues to be a random variable—together with idiosyncratic risk—driving default.
- Both parameters $\beta_i$ and $k_i$ continue to influence the shape of the distribution function.
Conditional default distributions

Probability of default of a single obligor, conditional on the realization of $m$ (x axis). Default probability 1 percent ($k = -2.33$). conditional cumulative distribution function of default $p(m)$. Values of the distribution function run from 1 to 0 because it is plotted against $m$ rather than $\frac{k - \beta m}{\sqrt{1 - \beta^2}}$. 
Default correlation in the single-factor model

Portfolio credit VaR in the single-factor model
  Derivation of the credit loss distribution function
  Portfolio credit loss distribution
  Portfolio credit VaR
From conditional default probability to portfolio loss

- Additional assumptions on credit portfolio:
  - Identical obligors: market risk factor loading $\beta$, pairwise correlation $\beta^2$, default probability $\pi = \Phi^{-1}(k)$
  - Granularity: homogeneous and completely diversified portfolio
  - Zero recovery
- $\Rightarrow$ Conditional default probability common to all obligors:

  $$p(m) = \Phi \left( \frac{k - \beta m}{\sqrt{1 - \beta^2}} \right) = \Phi \left( \frac{\Phi^{-1}(\pi) - \beta m}{\sqrt{1 - \beta^2}} \right) \quad \forall i = 1, 2, \ldots$$

- Law of Large Numbers $\Rightarrow$
  - Granularity $\Rightarrow$ idiosyncratic risk disappears
  - Portfolio loss a function only of market shock
- Fraction $x$ of loans defaulting—portfolio loss rate—equals single-firm default probability, conditional on market shock:

  $$x = p(m) = \Phi \left( \frac{\Phi^{-1}(\pi) - \beta m}{\sqrt{1 - \beta^2}} \right)$$
Probability distribution of the credit loss rate

- Loss rate $x = p(m)$ is random, because it is a function of latent random factor, market shock $m$
- What is probability distribution of $x$?
- We’ve posited a standard normal distribution for $m$, from which we can derive distribution of $x$
  1. Find market shock $m$ that leads to a given loss rate $x$
  2. Probability of loss rate $x$ equals probability of market shock $m$ that leads to it
Market factor and loss rate

- Step 1: solve for $m$ as a function of $x$:

$$m = \frac{\Phi^{-1}(\pi) - \sqrt{1 - \beta^2 \Phi^{-1}(x)}}{\beta}$$

- Sharply negative market factor $m$ corresponds to high loss rate $x$

Market factor as a function of loss rate. Default probability $\pi = 0.01$ (1 percent, $k = -2.33$), $\beta = 0.25$. 
Credit loss distribution

- Step 2: associate probability of loss rate $x$ with that of corresponding market shock $m$

- Recall $m$ a standard normal variate:

$$P[\tilde{m} \leq m] = \Phi[m]$$

- $\Rightarrow$ Cumulative probability distribution function of credit loss is

$$P[\tilde{x} \leq x] = P[\tilde{m} \geq m] = 1 - P[\tilde{m} \leq m] = 1 - \Phi[m] = \Phi[-m]$$

- Therefore

$$P[\tilde{x} \leq x] = \Phi\left[\frac{\sqrt{1 - \beta^2}\Phi^{-1}(x) - \Phi^{-1}(\pi)}{\beta}\right]$$

- The complicated term “inside” is the market factor realization $m$ corresponding to any given loss rate $x$

- And $m$ is a standard normal, the standard normal CDF “outside” is that of $m$
Market factor and portfolio loss distribution

- The probability of a loss in excess of any stipulated level \( x \) is then

\[
P[\tilde{x} \geq x] = 1 - P[\tilde{x} \leq x] = P\left[ \tilde{m} \leq \frac{\Phi^{-1}(\pi) - \sqrt{1 - \beta^2\Phi^{-1}(x)}}{\beta} \right]
\]

\[
= \Phi\left[ \frac{\Phi^{-1}(\pi) - \sqrt{1 - \beta^2\Phi^{-1}(x)}}{\beta} \right]
\]

- A *high* loss rate \( x \) corresponds to a market factor realization with a *low* probability
- Probability of realizing a loss rate no higher than \( x \) is therefore *high*
- Random loss rate \( \tilde{x} \) *below* level \( x \) ⇔ realized value of market factor \( \tilde{m} \) *higher* than associated level \( m \)
Factor and portfolio loss distribution

- Although treating portfolio “as if” a single credit, correlation still affects default distribution
  - Correlation operates through market shock
- Expected loss (EL) rate equals default probability $\pi$, constant across the many small obligors
Impact of default probability

- For realistic default probabilities below 50 percent, median portfolio loss rate is below the loan default rate
- Low default probability:
  - For moderate correlation, low default probability induces Bernoulli-like, “binary” loss behavior in the portfolio
  - Loss density very skewed to low loss levels
  - High likelihood that portfolio losses low
- High default probability:
  - Higher likelihood of higher portfolio losses
  - Loss density more spread out over range of loss levels
- For realistic default probabilities below 50 percent, median portfolio loss rate is below the loan default rate
Single-factor model portfolio loss distribution

Granular portfolio, $\beta = \sqrt{0.3} = 0.5477$ for all obligors. Losses expressed as a fraction of portfolio par value.
Impact of correlation on credit loss distribution

- Correlation near 0: median loss close to default probability $\pi$
  - $P[\tilde{x} \leq \pi] \rightarrow 0.5$ for $\beta \rightarrow 0$
  - High probability that realized portfolio loss rate close to typical firm’s default probability
  - Intuition: with low default rates and low correlation, default clusters are close to impossible

- Correlation near 1: loss distribution, close to binary
  - Portfolio behaves as if single loan/obligor
  - $P[\tilde{x} = 0]$ almost $1 - \pi$: high probability of no loss
  - $P[\tilde{x} = 1]$ (complete loss) almost $\pi$: material probability of complete loss

- Correlation “in the middle” and typical ($\pi \leq 0.5$) default rates
  - $P[\tilde{x} \leq \pi] > 0.5$
  - Correlation benefit: probability that portfolio loss below typical firm’s default probability greater than 50%
  - Intuition: with low default rates and intermediate correlation, default clusters are rare
Granular portfolio; default probability 5 percent. Losses expressed as a fraction of portfolio par value.
Portfolio credit VaR

- Loss distribution function $\rightarrow$ quantiles of $P[\tilde{x} \leq x]$
- Quantiles of $P[\tilde{x} \leq x]$ (minus EL) $\rightarrow$ credit VaR
- Higher correlation leads to higher VaR
  - By increasing likelihood of default clusters
Granular portfolio; default probability 0.05 percent. Losses expressed as a rate or fraction of portfolio par value. Color-coded vertical grid lines indicate credit VaR at 99-percent confidence level for each default correlation assumption. Color-coded points mark quantiles of portfolio credit losses for each default correlation assumption.