Lecture notes on risk management, public policy, and the financial system

Risk, expectations and asset prices

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Risk and uncertainty

Asset returns

Portfolios and diversification
Risk and uncertainty

Quantifying risk
Asset positions
Risk-free assets

Asset returns

Portfolios and diversification
The difficulty of quantifying risk

- Quantitative analysis of risk involves understanding asset prices in two dimensions
  - Configuration of asset prices at a point in time: equilibrium asset prices
  - Behavior of asset prices over time: asset price dynamics
- Financial modeling typically assumes outcomes or prices have well-defined probability distributions
  - Typically in finance, assumption of normal distribution
- Artificial and possibly misleading procedure, assumes
  - Outcomes can be summarized quantitatively
  - Ability to assign probabilities to outcomes
  - Low dimensionality, lack of richness: limited number of things about which information would be useful
- Two types of problem arise:
  - Ignores role of fine-grained, qualitative information
  - Ability to unambiguously rank outcomes by desirability
Importance of qualitative information

- **Knightian uncertainty**: after Frank H. Knight, American economist, *Risk, uncertainty and profit* (1921)
- Distinction between
  - **Risk**: pertains to situations or problems that are quantifiable, and which one can reasonably describe with a probability distribution.
  - **Uncertainty** refers to situations in which no quantitative estimates can be made
    - The outcomes are unknown or can only be guessed at, “because the situation dealt with is in a high degree unique.”
    - Entrepreneurial profit: compensation for bearing uncertainty, not risk
    - Knightian uncertainty, not risk, predominant in real-world situations
Arbitrage

- **Arbitrage**: profiting by buying and selling equivalent goods in two different markets
- What does “equivalent” mean
  - Identical goods
  - Goods with cash flows that are identical in size, risk, and timing
  - Weakest form of equivalence: same present value
- **Law of One Price** (*no-arbitrage principle*, Jevons’ *Law of Indifference*) states that arbitrage opportunities are exceptional
Risk-neutral pricing

- **Risk-neutral pricing**: approach to finding *fair value* when market prices of securities not available
- Based on market inputs together with pricing models
  - For derivatives, use underlying asset price
  - For illiquid securities, use spreads of comparable securities
- Application of no-arbitrage principle: market price, if equal to risk-neutral price, would be consistent with such market inputs as we observe
- **Risk premiums** already embedded in market inputs to pricing models
  - Market price is a *certainty equivalent*: lower than value if uncertainty were absent
  - Thus increasing return and generating risk premium
Agreement and disagreement

- Models may assume disagreement or agreement among market participants about expected means and volatilities
- One common form of agreement is the rational expectations hypothesis:
  - Market participants each have a correct model of the economy and financial system
  - Implies agreement among market participants on model
  - Subjectively-expected moments then equal to statistical moments
    - Expected value of asset $i$’s return $E[r_i]$ equal to statistical mean $\mu_i$
  - Distributions may then be time-varying, with expectations updated using most recent data
Profit and loss

- $S_t$ the time-$t$, $S_{t+1}$ next-period (1 day, week, month...) asset price
- A position in an asset is a holding of $x$ units of the asset is
  - $xS_t$ is the current (market or book) value of the position
- The profit and loss (P&L) on a position of $x$ units of an asset is the change in value $x(S_{t+1} - S_t)$
- P&L may be
  - Realized: based on actual purchase or sale that unwinds or adds to position
  - Mark-to-market (MTM) or fair value: based on observed market price or assessment of what market price would be if there were an active market
- $x < 0$ for a short position: loss if price rises
- Asset price floored at 0⇒
  - Short P&L has no lower limit: risk of a short position unlimited
  - Long P&L has no upper limit
Mechanics of short positions

- Establishing a short position:
  - Borrow units of the asset itself, sell in spot market, buy and return to lender at future date
- Cash flows of short positions
  - Lender of asset retains right to interest and dividends
  - Interest and dividends must consequently be paid by borrower to lender of asset
  - Short seller—borrower of asset—receives interest on cash proceeds from sale
- Very different legal structure and mechanics for stocks, bonds and across countries
- Can be expressed through derivatives trades: sell a forward, futures or option on the asset, or engage in a swap
- Short position may be used to hedge a long position, reducing rather than increasing exposure and risk
The convention of risk-free assets

- Analysis may stipulate existence and availability of a **risk-free asset**
  - As part of the set of available investment choices
  - As a tool for valuation, e.g. discount factor used in measuring fair value of an asset
- Implied risk-free rate may be calculated within models, based on macroeconomic fundamentals, household time and risk preferences (→ **stochastic discount factor:**)
- Real-world investment choices include low-risk, but not truly risk-free assets
- Assets conventionally classified as riskless include
  - **Cash or money balances:** legal tender, or redeemable on very short notice at par value
  - **Government bonds:** generally understood to include only issues of advanced market economies
    - Sovereign credit risk near-zero for bonds issued in local currency
Government curves and swap curves

- Risk-free rates often used in financial contracts as base for setting prices of risky fixed-income securities
- Define **risk-free curve** or set of risk-free rates with different terms to maturity
- Two types of risk-free curves widely used as benchmark or reference rates:
  - **Government curves**: rates on central government issues
    - Assumed to be issuer of currency in which bonds are denominated
    - **Example**: U.S. Treasury curve
  - **Swap curves**: rates on interbank money market instruments and par swaps
    - **Example**: *London Interbank Offered Rate* (LIBOR) curve, with rates fixed privately daily
    - Contains credit, counterparty and other risk premiums likely absent from government curves
    - Generally higher than corresponding government curve, but anomalous behaviors since 2008 crisis
Choice of reference rates

- Wide agreement on choice of reference rates driven by network effects
- Government curves less frequently used as major-currency reference rates
  - Propensity for central government yields to decline sharply in financial stress (→flight to quality)
  - Possibility of dearth of issuance
- Wide agreement on choice of reference rates since mid-1980s driven by network effects
- **LIBOR manipulation scandal**: submitted rates manipulated to affect valuations or avoid stigma
  - Some reference rates published by central banks, e.g. fed funds, and used as policy tools or indicators
  - Central banks now heavily involved in identifying substitute reference rates for LIBOR
Problems in the use of risk-free asset conventions

- Conventionally-defined riskless assets subject to several risks:
  - Currency appreciation or depreciation
  - Inflation
  - Liquidity risk: risk-on bond selloff, suspension of convertibility
  - Credit risk: sovereign default, bank failure, decline in collateral value
  - Interest-rate risk

- Yields on government bonds may contain (possibly negative) compensation for risk

- Extremely low or negative yields since crisis
  - High bond values that fluctuate widely over time
Risks of cash balances

- Cash and cash substitutes includes
  - Currency and other forms of central bank or narrow money
  - Short-term government or commercial paper
  - Bank deposits
  - MMMF accounts

- Cash balances exposed to:
  - Theft, physical destruction
  - Disruption of payment or settlement systems
  - Currency reform: change in currency system
  - Demurrage: imposition of cost on holding currency

- Examples of recent risk events affecting central bank money:
  - Demonetisation of largest-denomination notes in India on 08Nov2016, intended to thwart tax evasion
  - Negative interest rates on some central bank deposits from 2009
Safe assets

- **Safe assets** term with different definitions employed in discussion of crisis and background
- Generally said to include debt with low credit, inflation and currency risk, high liquidity
  - Including issues not generally classified as “riskless,” senior tranches of mortgage-backed securities, other securitizations
  - Also characterized as **information-insensitive**: requires little or no credit analysis or research to be accepted
- Shortage of safe assets said to have played key role in depressing interest rates before crisis
  - **Global savings glut hypothesis**: supply of safe assets lagging behind demand as world wealth grows
  - Demand for dollar-denominated assets contributes to large gross international capital flows
- Shortage exacerbated by disappearance or “disqualification” of some previously “safe” assets during crisis
Risk and uncertainty

**Asset returns**
- Defining asset returns
- Return distributions

Portfolios and diversification
Arithmetic and logarithmic returns

- Risk analysis focus usually on asset returns rather than price levels
- **Price return or capital gain** can be defined as
  - **Arithmetic or simple or linear return:** based on price change in currency units
    \[
    r_{t,t+1}^{\text{arith}} = \frac{S_{t+1} - S_t}{S_t} = \frac{S_{t+1}}{S_t} - 1
    \]
    - \(1 + r_{t,t+1}^{\text{arith}}\) is called the **gross return**
    - P&L on a position of \(x\) units of an asset is \(xS_tr_{t,t+1}^{\text{arith}} = x(S_{t+1} - S_t)\)
  - **Logarithmic or geometric or continuously compounded return:** change in logarithm of price
    \[
    r_{t,t+1} = \ln\left(\frac{S_{t+1}}{S_t}\right) = \ln(S_{t+1}) - \ln(S_t)
    \]
    - \(t\) may be measured in any time unit, e.g. seconds, days, months, years,...
Comparing arithmetic and logarithmic returns

- $r_{t,t+1}$ and $r_{t,t+1}^{\text{arith}}$ are related by
  
  $$r_{t,t+1}^{\text{arith}} = e^{r_{t,t+1}} - 1$$
  
  $$r_{t,t+1} = \ln \left(1 + r_{t,t+1}^{\text{arith}}\right)$$

- Apart from trivial case where both are zero,
  
  $$r_{t,t+1} < r_{t,t+1}^{\text{arith}}$$

- If $r_{t,t+1}$ and $r_{t,t+1}^{\text{arith}}$ relatively small,
  
  $$r_{t,t+1}^{\text{arith}} \approx r_{t,t+1}$$

In percent. The vertical distance is the difference between the two return definitions for a given initial level and change in asset prices.
When to use arithmetic and logarithmic returns?

- Log returns most useful when measuring returns over *multiple periods*
- Arithmetic returns most useful when measuring *portfolio returns*
- Risk models generally framed in terms of logarithmic rather than arithmetic returns
Multiple-period logarithmic returns

- Log returns aggregate nicely across time: **cumulative logarithmic rate of return**—return over multiple periods—is sum of shorter-term log returns.

- For example, with $t$ measured in months, annual related to monthly returns by

$$r_{t,t+12} = \ln \left( \frac{S_{t+12}}{S_t} \right) = \ln \left( \frac{S_{t+12}}{S_{t+11}} \cdot \frac{S_{t+11}}{S_{t+10}} \cdot \cdots \cdot \frac{S_{t+1}}{S_t} \right)$$

$$= \sum_{\theta=1}^{12} r_{t+\theta-1,t+\theta}$$

- With $t$ measured in years, **return at an annual rate** or annualized $\tau$-year return is simple average of yearly rates:

$$\frac{1}{\tau} r_{t,t+\tau} = \frac{1}{\tau} \sum_{\theta=1}^{\tau} r_{t+\theta-1,t+\theta}$$
Multiple-period arithmetic returns

- Arithmetic returns don’t aggregate so nicely across time
  - Inconvenient in modeling or working with return time series
- **Cumulative arithmetic return** related to single-period returns by

\[
r_{t', t+\tau}^{\text{arith}} = \frac{S_{t+\tau}}{S_t} - 1 = \frac{S_{t+\tau}}{S_{t+\tau-1}} \cdot \frac{S_{t+\tau-1}}{S_{t+\tau-2}} \cdots \frac{S_{t+1}}{S_t} - 1
\]

\[
= \prod_{\theta=1}^{\tau} \frac{S_t + \theta}{S_{t+\theta-1}} - 1 = \prod_{\theta=1}^{\tau} \left(1 + r_{t+\theta-1, t+\theta}^{\text{arith}}\right) - 1
\]

- $\rightarrow \tau$-year return at an annual rate a geometric average:

\[
(1 + r_{t', t+\tau}^{\text{arith}})^{\frac{1}{\tau}} - 1 = \left(\frac{S_{t+\tau}}{S_t}\right)^{\frac{1}{\tau}} - 1
\]

- Realized cumulative return can be expressed as index of current value, using historical short-term returns and setting initial $S_0 = 100$: 

\[
S_t = S_0(1 + r_0^{\text{arith}}) = 100 \prod_{\theta=1}^{t} (1 + r_{\theta-1, \theta}^{\text{arith}})
\]
Portfolios and constituent asset weights

- **Portfolios** are combinations of asset positions
- With $x_i$ the number of units of asset $i$ and $S_{i,t}$ its time-$t$ price,
  - The value of each position is $x_i S_{i,t}$
  - The value of the portfolio is $\sum_i x_i S_{i,t}$
- **Example:** with two positions, the portfolio value is $x_1 S_{1,t} + x_2 S_{2,t}$
- For a portfolio consisting of long positions only, $x_i > 0$, $\forall i$
  - Asset $i$’s **weight** is $w_i = \frac{x_i S_{i,t}}{\sum_i x_i S_{i,t}}$
  - $w_i > 0$, $\forall i$ and $\sum_i w_i = 1$
- For a long-short portfolio with zero net market value, $x_i \geq 0$ and $\sum_i x_i S_{i,t} = 0$
  - Portfolio weights may be defined in several ways to avoid division by 0, e.g. $w_i = \frac{|x_i S_{i,t}|}{\sum_i |x_i S_{i,t}|}$
  - Major issue in measuring portfolio (→)leverage
# Arithmetic and logarithmic portfolio returns

- Period-end value is more simply related to arithmetic than to logarithmic return:

\[
S_{t+1} = (1 + r_{t,t+1}^{\text{arith}}) S_t = e^{r_{t,t+1}} S_t
\]

- Arithmetic returns aggregate nicely across assets:

\[
r_{p,t,t+1}^{\text{arith}} \equiv \frac{\sum_i x_i S_{i,t+1} - \sum_i x_i S_{i,t}}{\sum_i x_i S_{i,t}}
= \frac{\sum_i x_i (1 + r_{i,t,t+1}^{\text{arith}}) S_{i,t} - \sum_i x_i S_{i,t}}{\sum_i x_i S_{i,t}}
= \sum_i w_i r_{i,t,t+1}^{\text{arith}}
\]

- Portfolio P&L is equal to the arithmetic portfolio return times the initial portfolio value: \( r_{p,t,t+1}^{\text{arith}} \sum_i x_i S_{i,t} \)

- Logarithmic returns don’t aggregate so nicely across assets:

\[
r_{p,t,t+1} = \ln \left( \frac{\sum_i x_i S_{i,t+1}}{\sum_i x_i S_{i,t}} \right) = \ln \left( 1 + \sum_i w_i r_{i,t,t+1} \right)
\]
Total return

- **Total return** includes cash flows thrown off by assets, such as interest, dividends or rent, in addition to capital gain.
- With time-$t + 1$ cash flows $d_{t+1}$
  - Arithmetic total return is $\frac{S_{t+1} + d_{t+1}}{S_t} - 1$
  - Logarithmic total return is $\ln\left(\frac{S_{t+1} + d_{t+1}}{S_t}\right)$
- If $d_{t+1}$ is a coupon payment and the period is one year, then $\frac{d_{t+1}}{S_t}$ is the bond’s **current yield**
- For assets such as real estate and commodities
  - Estimated physical storage and maintenance costs can be represented as $d_{t+1} < 0$
  - Possibly offset by the **convenience yield** of readily available inventory or of housing services
- Over longer periods of time, with compounding, the differences between total and price return can be very large
Excess return

- **Excess return**: difference between the return on an asset and the so-called (→) risk-free or riskless return $r^f$
  - Risk-free rate typically proxied by yield or return on short-term central government debt denominated in same currency
  - For long-term investment, longer-term proxy, e.g. government bond, for risk-free rate may be appropriate for some purposes
- Risk-free return should be known at time $t$ → desirable to match term to return measurement period
  - Longer-term proxies subject to price fluctuations within return measurement period → “riskless return” uncertain
- Over longer periods of time, with compounding, the differences between excess return and other return measures can be very large
Cumulative total and excess return of S&P 500

Nominal and real returns

- Asset prices generally expressed in **nominal** terms, i.e. money units.
- Asset prices and returns can also be expressed in **real** terms, i.e. units of goods or purchasing power.
- Real price levels computed by applying a **price index** or **deflator**.
- If $P_t$ represents price index level at time $t$, real asset price is $\frac{S_t}{P_t}$.
- If $P_t$ rising over time, inflation rate $> 0$ and real return $< \text{nominal}$.
- In U.S., two widely-used deflators of consumer goods—not asset—prices, differ in what goods included and how computed:
  - **Consumer Price Index for All Urban Consumers (CPI-U)**, used for indexing Social Security.
  - **Price Index for Personal Consumption Expenditures (PCE)**, defines Federal Reserve’s 2 percent inflation goal.
- Price indexes have “core” variants excluding volatile food and energy.
Measures of real return

- Logarithmic real return:

\[
\ln \left[ \frac{S_{t+1}}{P_{t+1}} \left( \frac{S_t}{P_t} \right)^{-1} \right] = \ln \left[ \frac{S_{t+1}}{S_t} \left( \frac{P_{t+1}}{P_t} \right)^{-1} \right] = r_{t,t+1} - \pi_{t,t+1}
\]

and \( \pi_{t,t+1} \) is the logarithmic rate of inflation from \( t \) to \( t + 1 \)

- Definition can include asset cash flows \( d_{t+1} \rightarrow \text{total real return} \)

- Arithmetic real return is a slightly more complicated expression
  - But the approximation is generally just fine

\[
\frac{S_{t+1}}{P_{t+1}} \frac{P_t}{S_t} - 1 = \frac{S_{t+1}}{P_{t+1}} - 1 = \frac{P_t}{S_t} - 1 = 1 + r_{t,t+1} - \pi_{t,t+1}
\]

\[
\approx r_{t,t+1} - \pi_{t,t+1}
\]

\[
S_{t+1} \approx S_t + r_{t,t+1} S_t + \pi_{t,t+1} S_t + \ldots
\]

\[
\approx S_t \left( 1 + r_{t,t+1} - \pi_{t,t+1} + \ldots \right)
\]
Nominal and real return of S&P 500 1970–2018

Index of cumulative total nominal and real returns, monthly, logarithmic scale, Dec. 1970 (=100) to to Nov. 2018. Real returns computed by subtracting monthly year-over-year change in CPI-U, treated as geometric average of monthly rates, slightly smoothing the month-to-month effect of inflation. *Data source:* Bloomberg LP.
Return experience of the S&P 500 1970–2018

<table>
<thead>
<tr>
<th></th>
<th>Annual</th>
<th>Cumulative</th>
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</thead>
<tbody>
<tr>
<td>Price return</td>
<td>7.35</td>
<td>2995</td>
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<tr>
<td>Total return</td>
<td>10.43</td>
<td>11619</td>
</tr>
<tr>
<td>Excess return</td>
<td>5.52</td>
<td>1311</td>
</tr>
<tr>
<td>Real total return</td>
<td>6.22</td>
<td>1800</td>
</tr>
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</table>

Hypotheses about return distributions

- Estimates of the probability distribution of future returns (or prices) a fundamental component of risk measurement
- Normal distribution usually a good first approximation to return behavior
  - **Central limit theorem:** averages behave like normals, even if the underlying data non-normal
  - But normal-based may be poor estimate of tail events
  - And assumes symmetry between positive and negative returns (→tail risk, higher moments)
- Distributional hypotheses are often needed, but highly restrictive
  - E.g. underestimate of risk due to normality assumption
- Payoffs and risk of asset captured by characteristics (usually parameters or moments) of return distribution
- Models may assume distribution and its moments invariant over time, or time-varying moments
What we forecast when we forecast asset returns

**Expected return** or **mean**
- Average or likeliest or “central tendency” of future return/asset value
- Fundamentals-oriented, trade-ideas seeking analysts: focus on expected return

**Volatility** or **standard deviation**
- How far might future return stray from recent or expected?
- Model-oriented risk managers: focus on variance

**Extreme events** or **tail risks**
- Likelihood of extremely large returns, clusters of defaults

**Correlation** of returns
- Do values of different assets tend to move in the same or in opposite direction?
Where do distributional hypotheses come from?

**Statistical estimates** of the return distribution, sometimes called the *actuarial, physical*, or simply *real-world* distribution

**Subjectively-expected** distribution: that expected by market participants
  - Unobservable, can only be inferred using models and surveys

**Risk-neutral** return distribution is that implied by market prices
  - Can be estimated if prices of derivatives—futures, forwards, and options—in addition to cash assets are observed
Expected values and expectations

- “Standard” models, e.g. normal, typically focus on
  
  **Expected return** ($\mu_i$): expected value of the rise in asset $i$’s value (plus cash flows it yields over time)
  
  **Return variance** or its square root ($\sigma_i$)—the return **standard deviation** or **volatility**—measures how much asset $i$’s value “bounces around” over time

- Higher moments shed light on → tail risk, extreme events
Comparison of more and less volatile stocks

Daily logarithmic stock price returns of Google Class A (Alphabet Inc., ticker GOOGL) and AT&T Inc. (ticker T), Q1 2011-Q2 2016. Volatility: daily return standard deviation, measured at an annual rate.
Multivariate return distributions

- Portfolio payoffs depend on combined or joint behavior of returns of constituents.
- In simple normal model, joint behavior of returns on several assets described by multivariate normal distribution.
- Parameters include return correlations in addition to means and volatilities.

Density of a bivariate standard normal distribution with correlation coefficient $\rho = 0.5$. 
Visualizing joint distributions via density contours

- **Constant probability density contours:** set of points with constant density
- Contours are computed by equating standardized realizations to a constant:
  \[
  c = (x - \mu)'\Sigma^{-1}(x - \mu),
  \]
  - \(x\) represents realizations of the jointly normal pair
  - \(x\) are the pairs on the “floor” of the bivariate distribution graph
  - \(\mu\) the mean and \(\Sigma\) the covariance matrix of the distribution
- To generate contours such that realizations of the pair of random variables fall within each contour with probability \(\alpha\)
  - Set the constant \(c = \chi_2^2(1 - \alpha)\)
  - \(\chi_2^2(1 - \alpha)\) is the \(1 - \alpha\) quantile of the **chi-square distribution** with 2 degrees of freedom,
Correlated and uncorrelated returns

Constant probability density contours of a bivariate standard normal distribution and random sample of 500 realizations. Realizations of the joint normal pair $x$ fall within each contour with the indicated probability. Left panel: correlation coefficient $\rho = 0$. Right panel: correlation coefficient $\rho = 0.50$. 
Impact of correlation on joint return distributions

- For a bivariate normal distribution, a positive (negative) return correlation increases the probability of pairs of large-magnitude returns with the same (opposite) sign.
- For a bivariate normal constant density contours are ellipsoids:
  - Multivariate normal belongs to family of **elliptical distributions**
  - Multivariate normal with pairwise zero correlation and identical variances belong to family of **spherical distributions**
Risk and uncertainty

Asset returns

Portfolios and diversification

Diversification
Efficient frontier
Portfolios and investment choices

- Simple approach to identifying available investment choices
  - What combinations of expected return and volatility—representing risk—are available?
  - Are any of these combinations clearly superior or inferior to others?
- Available investment choices can be expanded by mixing assets in portfolios
- Once we understand menu of available choices clearly, we can analyze which ones investors prefer
Diversification and return correlation

- The expected return of a portfolio is a simple weighted average of the constituent assets’ expected returns:

\[ \mu_p = w\mu_1 + (1 - w)\mu_2, \]

with \( w \equiv \text{asset 1 weight} \)

- In the case of just two constituents:

\[ \sigma_p^2 = w^2\sigma_1^2 + (1 - w)^2\sigma_2^2 + 2w(1 - w)\sigma_1\sigma_2\rho_{12}, \]

- Portfolio variance and volatility rise more than proportionally to a rise in constituent volatility
- And the portfolio variance can be strongly influenced up or down by return correlation
Diversification is powerful

- **Diversification:** combining assets can lead to a reduction of risk without sacrificing return
  - Adding a small amount of even a high-volatility asset can reduce portfolio volatility
  - Unless return correlation strongly positive
- Lower correlation enables investor to achieve lower portfolio volatility for any given expected return
- Mixing risky assets reduces return volatility of portfolio even if correlation is positive
- And there may also be (→) **risk-free assets**
  - Have non-zero return (usually positive) but zero volatility
Impact of diversification on portfolio return volatility

Volatility (y-axis) of portfolios combining long positions in T stock with long positions in GOOGL and Bank of America Corp. (BAC), assuming a return correlation of zero. Plotted as a function of the GOOGL or BAC portfolio weight (x-axis).

Volatility (y-axis) of portfolios combining long positions in T stock with long positions in GOOGL under different return correlation assumptions. The orange plot uses the realized daily return correlation of 0.329 for Q1 2011-Q2 2016. Plotted as a function of the GOOGL portfolio weight (x-axis).
Feasible and efficient portfolios

- Not every **feasible** or **attainable** portfolio is **efficient**
  - For each portfolio, find return and volatility
  - Two portfolios may have same volatility but different returns (or v.v.)
  - Portfolio with same volatility but lower return—or same return but higher volatility—than some other is not efficient

- **Efficient frontier** is set of pairs of return and volatility of efficient portfolios
  - Describes risk-return tradeoff in mean-variance framework

- **Global minimum variance portfolio** has lowest return and volatility among efficient portfolios
Example of diversification

<table>
<thead>
<tr>
<th>GOOGL wt. (%)</th>
<th>T wt. (%)</th>
<th>return</th>
<th>volatility</th>
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<tbody>
<tr>
<td>10</td>
<td>90</td>
<td>7.896</td>
<td>15.158</td>
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<tr>
<td>19.29</td>
<td>80.71</td>
<td>8.701</td>
<td>14.983</td>
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<td>20</td>
<td>80</td>
<td>8.763</td>
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<tr>
<td>50</td>
<td>50</td>
<td>11.363</td>
<td>16.799</td>
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<tr>
<td>90</td>
<td>10</td>
<td>14.830</td>
<td>23.035</td>
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The global minimum variance portfolio is highlighted in cyan.
The risk-return tradeoff

Volatility (x-axis) and mean (y-axis) of portfolios combining long positions in GOOGL and T stock. Purple plot shows feasible portfolios estimated using the historical return correlation of 0.329. The heavy part of the plot is the efficient frontier. Orange plot shows efficient frontier if the return correlation were -0.25.