Lecture notes on risk management, public policy, and the financial system

Volatility behavior and forecasting

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Time variation in return volatility and correlation

Volatility forecasting
Time variation in return volatility and correlation

Time variation in return volatility
Time variation in return correlation

Volatility forecasting
Volatility forecasts

- A major departure from standard model: *risk or volatility changes over time*
- Volatility, unlike return, not directly observable, must be estimated
  - Challenge: method for estimating volatility that captures typical patterns of volatility
- Recent past and long-term volatility help predict future volatility
  - But: while estimators efficacious for forecasting near-term volatility, they often miss sharp changes in volatility
- **Second-moment efficiency**: option market does less-poor job forecasting return variance than forward markets of forecasting mean return
**Typical patterns of volatility behavior over time**

**Persistence:** volatility tends to stay near its current level
- Periods of high or low volatility tend to be enduring
- Once a large-magnitude return shocks volatility higher, volatility persists at its higher level
- Magnitude or square of return as well as return volatility display positive autocorrelation

**Abrupt changes** in volatility are not unusual
- Together with persistence, leads to volatility clustering or volatility regimes
- Shifts from low to high volatility are more abrupt, while shifts from high to low volatility are more gradual

**Long-term mean reversion:** volatility of an asset’s return tends to gravitate to a long-term level
- In turn implies a term structure of volatility: different current estimates of volatility for different time horizons
Volatility of oil prices 1986-2018

Volatility behavior and forecasting

Time variation in return volatility and correlation

Time variation in return volatility

Conditional volatility

- Volatility regimes suggest use of *conditional volatility*: estimate weighted toward more recent information
- Formally, volatility forecasts based on some information (“shocks” or “innovations”) up to present time $t$
  - $\sigma_t \equiv$ current estimate of future return volatility based on (a model and) information through time $t$
- What new information drives $\sigma_t$? In most models:
  - Magnitude (and possibly the sign) of recent *returns*
  - Recent estimates of *volatility*
- Term structure of volatility, e.g. weekly volatility higher or lower than daily
  - Typically, volatility expected to rise (fall) when low (high) relative to long-term average level
Impact of time-varying correlation

- Like volatility, correlations vary over time
- Correlations have strong impact on portfolio returns, hedged positions
- Abrupt changes in correlation during periods of financial stress →
  - Failure of hedging strategies
  - Diminution of diversification benefits
- “Risk-on risk-off” behavior: tendency for correlations across many assets to rise when risk appetites diminish in stress periods
- **Examples:**
  - Increase in return correlations among equities
  - Higher correlation between equity returns, Treasury yields
Time-varying correlation of stock and bond returns

- Persistent changes over time in general level of correlation between stock and bond returns
  - 1960s-1990s: generally negative
  - 1990s-: generally positive
- Experience during the inflation and disinflation from late 1960s
  - Rising rates driven by rising inflation expectations, associated with adverse impact on economic growth
  - “Fed model”: increase in discount rate for future earnings reduces present value
- Experience once low-inflation monetary policy fully credible
  - Rising rates driven by increases in anticipated real returns ($r^*$), associated with positive impact on economic growth
  - Risk-on risk-off: investors reduce allocations to risky in favor of safe assets
- N.B. positive correlation of equity returns and yield changes corresponds to negative correlation of stock and bond returns
Correlation of stock returns and rates 1962-2020

Time variation in return volatility and correlation

**Volatility forecasting**
- Simple approaches conditional volatility estimation
- GARCH
- The exponentially-weighted moving average model
Using conditional volatility estimators

- General approach: revise most recent estimate of volatility based on most recent return data
- Simplified notation when working with daily data: return from yesterday’s to today’s close

\[ r_t \equiv r_{t-1,t} \equiv \ln(S_t) - \ln(S_{t-1}) \]

- At close of each day \( t \), use \( r_t \) to update yesterday’s volatility estimate \( \sigma_{t-1} \)
- Use the new estimate \( \sigma_t \) to measure risk or forecast volatility over the next business day \( t + 1 \)
- Volatility forecast horizon includes \textit{trading days}, not calendar time
  - Price can change only when market open
  - \( \leftrightarrow \) Holding period and cash flows accrue every \textit{calendar} day
Volatility is easier to estimate than mean

- Imagine asset return approximately follows diffusion with drift
  - Observed at regular intervals over a period of time
  - Drift and volatility may change over time, but slowly
- You only observe one sample path in real history
- The only information on mean/drift is return over entire period
- But finer intervals—every 5 min. instead of daily—provide more information on volatility
  - Finer intervals provide more information on tendency to wander
  - Confidence interval of volatility estimate $\to 0$
  - But not confidence interval of mean estimate
- Tail risk very hard to estimate
Zero-mean assumption

- A typical risk-measurement modeling choice:
  - *Estimate* return volatility
  - But *assume* mean return = 0
- In lognormal model, assume drift $\mu = 0 \Rightarrow$:
  - Mean logarithmic return $\mu = 0$

\[ r_{t,t+\tau} = \ln(S_{t+\tau}) - \ln(S_t) \sim \mathcal{N}(0, \sigma^2\tau) \]

- But discrete returns have non-zero mean due to Jensen’s Inequality term:

\[ \mathbb{E}[S_{t+\tau}] = S_t e^{\frac{1}{2}\sigma^2\tau} \]
Why assume zero-mean returns?

- Because we can:
  - Small impact of mean on volatility over short intervals
  - But: mean return increases linearly with time, return volatility increases as square root of time
  - ⇒ Over longer periods, mean has larger impact than volatility

- Because we must:
  - Expected return very hard to measure
  - Estimation of mean introduces additional source of statistical error into variance estimate
  - Bad enough to assume return normality, let’s not also invent mean
Square-root-of-time rule

- In standard (\(\rightarrow\))geometric Brownian motion model, *variance* (vol squared) of price change proportional to time elapsed
  - Position after \(t\) time units
    \[
    S_t \sim \mathcal{N}(0, t)
    \]
  - Together with martingale property
    \[
    S_{t+\tau} - S_t \sim \mathcal{N}(0, \tau)
    \]
- Carries over to standard lognormal/geometric Brownian motion model: variance increases in proportion to time elapsed
  - \(\Rightarrow\) Vol increases in proportion to square root of time elapsed
- Useful rule-of-thumb even if returns only approximately lognormal
  - But assumes constant return volatility, i.e. flat term structure of volatility
  - At odds with changes in volatility over time and with long-term mean reversion
Volatility behavior and forecasting

Volatility forecasting

Simple approaches conditional volatility estimation

Applying the square-root-of-time rule

- Volatility forecast horizon includes *trading days*, not calendar time
  - Typical year includes about 250-255 trading days
  - Assume 256 trading days, $\sqrt{256} = 16$
  - Annualized volatility $\approx 16 \times \text{daily volatility}$

- **Examples:**
  - Long-term average annual volatility of U.S. stock indexes $\approx 16 - 20$ percent $\Rightarrow$ daily vol $\approx 1 - 1.25$ percent
  - Swaption *normal volatility* 80 bps $\Rightarrow$ daily vol 5 bps
Simple conditional volatility estimators

Use moving window incorporating past $m$ trading days’ returns

**Root mean square:** square root of the sum of squared returns (deviations from zero) divided by the number of observations

\[
\sigma_t = \sqrt{\frac{1}{m} \sum_{\tau=1}^{m} r_{t-m+\tau}^2}
\]

- Incorporates assumption of zero mean return

**Standard deviation:** the square root of the sum of squared deviations from the mean return $\bar{r}_t = \frac{1}{m} \sum_{\tau=1}^{m} r_{t-m+\tau}$ divided by the number of observations minus 1

\[
\sigma_t = \sqrt{\frac{1}{m-1} \sum_{\tau=1}^{m} (r_{t-m+\tau} - \bar{r}_t)^2},
\]

- Bias-corrected for 1 degree of freedom lost due to use of $\bar{r}_t$
GARCH model of volatility

- Generalized autoregressive conditionally heteroscedastic model
- Volatility driven by
  - Recent volatility
  - Recent returns
  - Long-term “point of rest” of volatility or “forever vol” \( \bar{\sigma} \)
- Estimate \( \sigma_t \) made at today’s close updates yesterday’s estimate \( \sigma_{t-1} \) with latest return \( r_t \)
- Look back one period \( \rightarrow \) GARCH(1,1):
  \[
  \sigma_t^2 = \alpha r_t^2 + \beta \sigma_{t-1}^2 + \gamma \bar{\sigma}^2
  \]
- Feedback to returns via “shock” or “innovation” \( \epsilon_t \)
  \[
  r_t = \epsilon_t \sigma_{t-1},
  \]
- Today’s return \( r_t \) the only pertinent new information on date \( t \)
  - \( \epsilon_t \) assumed i.i.d. with mean 0 and variance 1
  - \( \epsilon_t \) together with current volatility \( \sigma_{t-1} \) determines new return \( r_t \)
  - \( r_t \) random but not “free,” set by current vol and random shock \( \epsilon_t \)
- The weights satisfy \( \alpha, \beta, \gamma > 0 \) and \( \alpha + \beta + \gamma = 1 \)
Role of parameters in the GARCH model

- Impact of $\alpha$: high $\alpha \Rightarrow$
  - Large $r_t$ causes large, immediate change in estimated return volatility $\sigma_t$
  - Wider range of variation of $\sigma_t$ over time
- Impact of $\beta$: high $\beta \Rightarrow$
  - $\sigma_t$ and deviations from $\bar{\sigma}^2$ very persistent
  - Less variation of $\sigma_t$ over time
- Long-term variance $\bar{\sigma}^2 > 0$
  - Presence of $\bar{\sigma}^2$ generates a term structure of volatility
  - **Example:** $\bar{\sigma}^2$ approximately 1.0–1.15 percent for U.S. equity market (at daily rate)
- Low $\gamma \Rightarrow$ little mean reversion
- Estimates of $\beta$ generally not very far from 1, $\alpha + \beta$ quite close to 1
- Estimated parameter values lead to (hopefully realistic) behavior of volatility
Estimating GARCH(1,1) model parameters

- **Maximum likelihood method** a standard approach
  - Assume **conditional normality**, shocks $\epsilon_t$ normally distributed, a stronger assumption than i.i.d.:
    \[
    \epsilon_t \sim \mathcal{N}(0, 1) \quad \forall t
    \]
  - Joint normal density of $m$ return observations $\Rightarrow$ log likelihood function
    \[
    \sum_{\tau=1}^{m} \left[ -\ln(\sigma_{t-\tau}^2) - \frac{r_t^2}{\sigma_{t-\tau}^2} \right],
    \]
    with initial volatility value $\sigma_0$
  - Use numerical search procedure to find parameters that maximize log likelihood function
    - Numerical search procedure can be sensitive to initial trial guess
    - $\omega \equiv \gamma\overline{\sigma}^2$ treated as a single parameter
    - $\gamma$ can then be recovered as $1 - \alpha - \beta$ and
    \[
    \overline{\sigma} = \sqrt{\frac{\omega}{1 - \alpha - \beta}}
    \]
Influence of past returns in GARCH model

- GARCH(1,1) formula can be recast in terms of most recent and past squared returns (setting $m = t$):

$$\sigma_1^2 = \alpha r_1^2 + \beta \sigma_0^2 + \omega$$

$$\sigma_2^2 = \alpha r_2^2 + \beta \sigma_1^2 + \omega = \alpha r_2^2 + \beta (\alpha r_1^2 + \beta \sigma_0^2 + \omega) + \omega$$

$$= \alpha r_2^2 + \alpha \beta r_1^2 + \beta^2 \sigma_0^2 + (1 + \beta) \omega$$

$$\vdots$$

$$\sigma_t^2 = \alpha \sum_{\tau=1}^{t} \beta^{t-\tau} r_\tau^2 + \sum_{\tau=1}^{t} \beta^{t-\tau} \omega + \beta^t \sigma_0^2$$

$$\approx \alpha \sum_{\tau=1}^{t} \beta^{t-\tau} r_\tau^2 + \frac{1}{1 - \beta} \omega$$

- $\alpha < 1, \beta < 1 \Rightarrow$ small influence of more remote past returns, starting value $\sigma_0$

- Tradeoff bet influence of long-term variance and that of most recent volatility estimate
Example of GARCH(1,1) model estimation

- Applied to S&P 500 index, using $m + 1 = 3651$ closing-price observations 30Jun2005 to 31Dec2019
- $r_1^2$ used as starting value $\sigma_0$
  - Can also use sample variance of entire time series
- For each pass of the search procedure, successively apply GARCH(1,1) formula to calculate trial values $\sigma_1, \sigma_2, \ldots, \sigma_t$:

<table>
<thead>
<tr>
<th>Parameter estimates</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$ 0.12195</td>
</tr>
<tr>
<td>$\omega$ $2.40805 \times 10^{-6}$</td>
</tr>
<tr>
<td>$\gamma$ 0.02196</td>
</tr>
</tbody>
</table>

- Practical application: (re-)estimate parameters infrequently, but use estimated model regularly to forecast volatility
Exponentially-weighted moving average model

- **Exponentially-weighted moving average** (EWMA)
  - A.k.a. **RiskMetrics model**
  - Variance a weighted average of past returns
  - Weights smaller for more-remote past returns
- Single parameter: **decay factor** $\lambda$
  - Low $\lambda$: rapid adaptation to recent returns
  - High $\lambda$: slow adaptation to recent returns
- EWMA implies a flat term structure of volatility
  - Volatility follows square-root-of-time rule
  - Volatility behaves as a random walk, subject to shocks
- Decay factor estimation: $\lambda$ that minimizes forecast errors, e.g. RMS criterion
- Decay factor may also be chosen judgmentally
Estimating volatility with the EWMA model

- Typically, assume a value for parameter $\lambda$ rather than estimate it, and apply a formula

- Current volatility estimate $\sigma_t$ uses $m$ most recent observed returns $r_{t-m+1}, \ldots, r_t$
  - Treat $\lambda$ as known parameter
  - Weight on each squared return $\frac{1-\lambda}{1-\lambda^m} \lambda^{m-\tau}, \tau = 1, \ldots, m$
    - Apply $\lambda^{m-m} = 1$ for $\tau = m$, most recent (time $t$) return
    - Apply $\lambda^{m-1} \approx 0$ for $\tau = 1$, most remote (time $t - m + 1$) return

$$
\sigma_t^2 = \frac{1-\lambda}{1-\lambda^m} \sum_{\tau=1}^{m} \lambda^{m-\tau} r_{t-m+\tau}^2
$$

- $1 - \lambda^m \approx 1 \Rightarrow$

$$
\sigma_t^2 \approx (1 - \lambda) \sum_{\tau=1}^{m} \lambda^{m-\tau} r_{t-m+\tau}^2
$$

- $m$ doesn’t have to be large
  - $m \approx 100$ more than adequate unless $\lambda$ quite close to 1
The EWMA model weighting scheme

The graph displays the values of the last 100 of $m = 250$ EWMA weights $\frac{1 - \lambda}{1 - \lambda^m} \lambda^{m - \tau}$ for $\lambda = 0.94$ and $\lambda = 0.97$. 

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Volatility behavior and forecasting

Volatility forecasting

The exponentially-weighted moving average model
Choosing the decay factor

- Low $\lambda \Rightarrow$ recent observations have greater weight:
  - Volatility changes rapidly
  - Recent observations have most information useful for short-term conditional volatility forecasting
- Low $\lambda$ effectively shortens historical sample size compared to high $\lambda$
- Estimates using low $\lambda$ much more variable than those using high $\lambda$
- Estimates using low $\lambda$ respond more rapidly to new information
- Estimates using low $\lambda$ may move in the opposite direction from those using high $\lambda$
  - Estimates using low $\lambda$ decline after a sequence of high-magnitude returns, while those using high $\lambda$ still rising in response
- No agreed method for estimating $\lambda$
- Widely adopted standard settings for decay factor:
  - $\lambda = 0.94$ for short-term (e.g. one-day) forecasts
  - $\lambda = 0.97$ for medium-term (e.g. one-month) forecasts
  - Minimizes RMS of forecast errors for range of assets in original 1994 RiskMetrics study
Effect of the decay factor on the volatility forecast

EWMA estimates of the volatility of daily S&P 500 index returns 01Jul2005 to 31Dec2019, at a daily rate in percent, using decay factors of $\lambda = 0.94$ and $\lambda = 0.99$. Points represent the absolute value of daily return observations.
Estimating volatility with the EWMA model

\[ \tau \quad \text{Date} \quad S_{t+\tau-m} \quad r_{t+\tau-m} \quad \frac{1-\lambda}{1-\lambda^m} \lambda^{m-\tau} \quad \frac{1-\lambda}{1-\lambda^m} \lambda^{m-\tau} r_{t+\tau-m}^2 \]

<table>
<thead>
<tr>
<th>( \tau )</th>
<th>Date</th>
<th>( S_{t+\tau-m} )</th>
<th>( r_{t+\tau-m} )</th>
<th>( \frac{1-\lambda}{1-\lambda^m} \lambda^{m-\tau} )</th>
<th>( \frac{1-\lambda}{1-\lambda^m} \lambda^{m-\tau} r_{t+\tau-m}^2 )</th>
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<td>NA</td>
<td>NA</td>
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<tr>
<td>1</td>
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<td>0.00000×10^{-6}</td>
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<tr>
<td>173</td>
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<td>2061.02</td>
<td>0.00237</td>
<td>0.00051</td>
<td>0.00286×10^{-6}</td>
</tr>
<tr>
<td>174</td>
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<td>0.08052×10^{-6}</td>
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<tr>
<td>175</td>
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<tr>
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<tr>
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<td>0.00111</td>
<td>0.06000</td>
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</tr>
</tbody>
</table>

Return vol of the S&P 500 index, estimated after the close on 17Jul2015 (date \( t \)), with \( m = 250 \), \( \lambda = 0.94 \). Return (4th column) expressed as a decimal. Add the 250 values in the last column to get the estimated variance \( \sigma_t^2 \).
Recursive formula for EWMA volatility estimates

- Recursive formula updates most recent volatility estimate with new data on return magnitude

\[ \sigma_t^2 = \lambda \sigma_{t-1}^2 + (1 - \lambda) r_t^2 \]

- Easy computation technique, very close to result of full EWMA weighting scheme
- Shows similarity of EWMA to “one-parameter” GARCH
  - But with long-term volatility term \( \gamma = 0, \alpha + \beta = 1 \)
  - \( \lambda \) analogous to \( \beta \), \( 1 - \lambda \) analogous to \( \alpha \)
  - Shocks to volatility permanent, no long-term “forever” vol
  - Also known as integrated GARCH or IGARCH(1,1)
- EWMA estimate usually close to unrestricted GARCH(1,1) estimate
- “Starter value” (orange in example on next slide):
  - Root mean square, using first 21 days of data
  - Starter value method not crucial, converges quickly (esp. for low \( \lambda \))
## Recursive formula for EWMA volatility estimates

<table>
<thead>
<tr>
<th>$t$</th>
<th>Date</th>
<th>$S_t$</th>
<th>$r_t$ (%)</th>
<th>$\lambda\sigma^2_{t-1}$</th>
<th>$(1 - \lambda)r_t^2$</th>
<th>$\sigma_t$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>30Jun2005</td>
<td>1191.33</td>
<td>NA</td>
<td>NA</td>
<td>NA</td>
<td>0.55583</td>
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<tr>
<td>2</td>
<td>01Jul2005</td>
<td>1194.44</td>
<td>0.2607</td>
<td>$0.29041 \times 10^{-4}$</td>
<td>$0.40783 \times 10^{-6}$</td>
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<td>$8.08946 \times 10^{-6}$</td>
<td>0.62444</td>
</tr>
<tr>
<td>7</td>
<td>11Jul2005</td>
<td>1219.44</td>
<td>0.6235</td>
<td>$0.36653 \times 10^{-4}$</td>
<td>$2.33279 \times 10^{-6}$</td>
<td>0.62439</td>
</tr>
</tbody>
</table>

Return vol of the S&P 500 index, estimated daily using the recursive formula, with $\lambda = 0.94$. Initial vol estimate: RMS of the 20 daily returns 01Jul2005–29Jul2005.
GARCH(1,1) and EWMA volatility estimates

Daily estimates of S&P 500 index’s annualized return volatility, 30Jun2005 to 31Dec2019. EWMA estimates with $\lambda = 0.94$ GARCH(1,1) estimates use parameters $\alpha = 0.12195$, $\beta = 0.85609$, $\gamma \hat{\sigma}^2 = 2.40805 \times 10^{-6}$. The annualized realized return volatility was 15.69 percent over the period.