Lecture notes on risk management, public policy, and the financial system

Volatility behavior and forecasting

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Standard model of asset price dynamics

Time variation in return volatility and correlation

Volatility forecasting
Standard model of asset price dynamics

Forecasting asset return: random walks
The geometric Brownian motion model of asset price dynamics

Time variation in return volatility and correlation

Volatility forecasting
Basic characteristics of the standard model

- Log returns normally distributed and approximately follow a random walk
  - Many asset prices and factor returns empirically distributed close to normally
  - But just an approximation: possible positive drift and serial dependence in higher moments
- Tractable, easy to estimate, and few parameters
  - And what do you do if no alternative model is unambiguously better?
- Define behavior of price or return via a few properties, then derive many more
- Consistent with (→) efficient markets hypothesis
  - Zero or low return autocorrelation: correlation of successive returns
Defining properties of a random walk

- Function of time, starts at \((0, 0)\): time 0, position 0
- Discrete: adds increments at time steps of length \(\Delta t\)
- Two possible sizes of increments: \(\pm \sqrt{\Delta t}\)
- With probabilities:
  - Positive increments \(\pi\)
  - Negative increments \(1 - \pi\)
- It can’t stay in the same position two consecutive time steps
  - But it can return to a position after at least one more step
- With passing of time, random walk may arrive at several different values
  - But in general several sample paths lead to each value
  - Each sample path has same probability
One sample path of a random walk

Plot shows the first 16 positions of one simulation of a random walk over 1 time unit. The time interval between steps is $\Delta t = \frac{1}{16}$ and the magnitude of the increments is $\sqrt{\Delta t} = \frac{1}{\sqrt{16}} = \frac{1}{4}$. The dots show the value the position takes on at the start of each time interval.
Convergence of random walk to Brownian motion

- Let length of time step $\Delta t \to 0$
- Random walk $\to$ Brownian motion or Wiener process or diffusion process
  - $\to$ same Brownian motion regardless of probability $\pi$ of positive increment to random walk
- Defining properties of Brownian motion:
  - Position or level of asset price or risk factor $S_t$ a function of time $t$, starts at $S_0 = 0$
  - Every sample path is continuous (but awfully jagged)
  - Position after $t$ time units (but uncountably many steps) a standard normal variate
    
    $$S_t \sim \mathcal{N}(0, t)$$

- **Martingale property**: successive and non-overlapping increments to $S_t$ are independent of one another and of initial position
From Brownian to geometric Brownian motion

- There's only one Brownian motion (though infinitely many possible paths)
- By generalizing Brownian motion and applying it to \textit{logarithm} (rather than \textit{level}) of asset price $S_t$, arrive at \textbf{geometric Brownian motion}
- Assume tiny increments to logarithm of $S_t$ follow Brownian motion, and
  - Scale the variance of $S_t$ by a \textit{volatility} parameter $\sigma$
  - \textit{Drift} term: deterministic increase at rate $\mu + \frac{1}{2} \sigma^2$ per unit of time
    - Added to every sample path of $S_t$
- \textit{Itô's Lemma} $\Rightarrow$ \textit{level} of asset price $S_t$ follows \textbf{lognormal distribution}: log returns normally distributed:
  \[
  r_{t,t+\tau} \equiv \ln(S_{t+\tau}) - \ln(S_t) \sim \mathcal{N}(\mu\tau, \sigma^2\tau),
  \]
- And therefore
  \[
  \frac{r_{t,t+\tau} - \mu \tau}{\sigma \sqrt{\tau}} \sim \mathcal{N}(0, 1)
  \]
The Jensen’s Inequality term

- Itô’s Lemma—normality of log returns—implies
  \[ E[r_{t,t+\tau}] = E[\ln(S_{t+\tau})] - \ln(S_t) = \mu \tau \]

- (Statistical) expected value \( E[r_{t,t+\tau}] \) of logarithmic—continuously compounded—return

- But level of \( S_t \) wiggles up by **Jensen’s Inequality term** \( \frac{1}{2} \sigma^2 \tau \) over time:
  \[ E[S_{t+\tau}] = S_t e^{(\mu + \frac{1}{2} \sigma^2) \tau} \]

- Discrete changes “compound up” as the base rises, just due to noise
Sample paths of geometric Brownian motion

Fifteen simulations of the path of the price level over time of an asset following a geometric Brownian motion process with annualized volatility 25 percent, initial asset price 100. At each point in time, the levels of the paths are lognormally distributed. Orange rays and hyperbolas plot the mean paths and 95 percent confidence intervals over time.
Properties of geometric Brownian motion

- If zero drift ($\mu = 0$), asset price almost (Jensen’s Inequality) equally likely to move up or down
  - Like random walk, but always near 50-50
- The further into the future we look, the likelier it is that the asset price will be far from its current level
- The higher the volatility, the likelier it is that the asset price will be far from its current level within a short time
- The time series of daily logarithmic asset returns are independent draws from a normal distribution
- Variation properties of geometric Brownian motion:
  - **Total variation** The total distance traveled over even a tiny time interval is *infinite* in every sample path
  - **Quadratic variation** But the variance of the distance traveled in different sample paths is *finite*
Volatility behavior and forecasting

Standard model of asset price dynamics

The geometric Brownian motion model of asset price dynamics

Volatility is easier to estimate than mean

- Imagine asset return approximately follows diffusion with drift
  - Observed at regular intervals over a period of time
  - Drift and volatility may change over time, but slowly
- You only observe one sample path in real history
- The only information on mean/drift is return over entire period
- But finer intervals—every 5 min. instead of daily—provide more information on volatility
  - Finer intervals provide more information on tendency to wander
  - Confidence interval of volatility estimate $\to 0$
  - But not confidence interval of mean estimate
- Tail risk very hard to estimate
Square-root-of-time rule

- In simple Brownian motion model, variance (vol squared) of return proportional to time elapsed
  - Position after $t$ time units
    \[ S_t \sim \mathcal{N}(0, t) \]
  - Together with martingale property
    \[ S_{t+\tau} - S_t \sim \mathcal{N}(0, \tau) \]
- Carries over to standard lognormal model: variance increases in proportion to time elapsed
- Vol increases in proportion to square root of time elapsed
- Useful rule-of-thumb even if returns only approximately lognormal
- Volatility forecast horizon includes trading days, not calendar time
  - Interest, accrues every calendar day
  - Typical year includes about 250-255 trading days
  - Assume 256 trading days, $\sqrt{256} = 16$
  - Annualized volatility $\approx 16 \times$ daily volatility
- Example: Swaption normal volatility 80 bps $\Rightarrow$ vol per day 5 bps
Standard model of asset price dynamics

**Time variation in return volatility and correlation**
- Time variation in return volatility
- Time variation in return correlation

Volatility forecasting
Volatility forecasts

- A major departure from standard model: risk or volatility changes over time
- Volatility, unlike return, not directly observable, must be estimated
  - Challenge: method for estimating volatility that captures typical patterns of volatility
- Recent past and long-term volatility help predict future volatility
  - But: while estimators efficacious for forecasting near-term volatility, they often miss sharp changes in volatility
- Second-moment efficiency: option market does less-poor job forecasting return variance than forward markets of forecasting mean return
Typical patterns of volatility behavior over time

**Persistence:** volatility tends to stay near its current level
- Periods of high or low volatility tend to be enduring
- Once a large-magnitude return shocks volatility higher, volatility persists at its higher level
- Magnitude or square of return as well as return volatility display positive **autocorrelation**

**Abrupt changes** in volatility are not unusual
- Together with persistence, leads to **volatility clustering** or **volatility regimes**
- Shifts from low to high volatility are more abrupt, while shifts from high to low volatility are more gradual

**Long-term mean reversion:** volatility of an asset’s return tends to gravitate to a long-term level
- In turn implies there is a **term structure of volatility**, e.g. weekly volatility higher or lower than daily
Volatility of oil prices 1986-2018

Conditional volatility

- Volatility regimes suggest use of *conditional volatility*: estimate weighted toward more recent information.
- Formally, volatility forecasts based on some information ("shocks" or "innovations") up to present time $t$:
  - $\sigma_t \equiv$ current estimate of future return volatility based on (a model and) information through time $t$.
- What new information drives $\sigma_t$? In most models:
  - Magnitude (and possibly the sign) of recent *returns*.
  - Recent estimates of *volatility*. 
Impact of time-varying correlation

- Like volatility, correlations vary over time
- Correlations have strong impact on portfolio returns, hedged positions
- Abrupt changes in correlation during periods of financial stress →
  - Failure of hedging strategies
  - Evaporation of diversification benefits
- Risk-on risk-off behavior
  - Tendency for correlations across many assets to rise in stress periods
  - High positive correlation between equity returns, Treasury yields
Correlation of stock returns and rates 1962-2015

Standard model of asset price dynamics

Time variation in return volatility and correlation

**Volatility forecasting**
- Basic approach of conditional volatility estimation
- GARCH
- The exponentially-weighted moving average model
Using conditional volatility estimators

- General approach: revise most recent estimate of volatility based on most recent return data
- Simplified notation when working with daily data: return from yesterday’s to today’s close
  \[ r_t \equiv r_{t-1,t} \equiv \ln(S_t) - \ln(S_{t-1}) \]
- At close of each day \( t \), use \( r_t \) to update yesterday’s volatility estimate \( \sigma_{t-1} \)
- Use the new estimate \( \sigma_t \) to measure risk or forecast volatility over the next business day \( t + 1 \)
Zero-mean assumption

- A typical risk-measurement modeling choice:
  - Estimate return volatility
  - But assume mean return $= 0$
- In lognormal model, assume drift $\mu = 0 \Rightarrow$:
  - Mean logarithmic return $\mu = 0$
    \[ r_{t,t+\tau} = \ln(S_{t+\tau}) - \ln(S_t) \sim \mathcal{N}(0, \sigma^2 \tau) \]
  - But discrete returns have non-zero mean due to Jensen’s Inequality term:
    \[ \mathbb{E}[S_{t+\tau}] = S_t e^{\frac{1}{2} \sigma^2 \tau} \]
Why assume zero-mean returns?

- Because we can:
  - Small impact of mean on volatility over short intervals
  - But: mean return increases linearly with time, return volatility increases as square root of time
  - ⇒ Over longer periods, mean has larger impact than volatility
- Because we must:
  - Expected return very hard to measure
  - Estimation of mean introduces additional source of statistical error into variance estimate
  - Bad enough to assume return normality, let’s not also invent mean
Simplest conditional volatility estimator

- Estimate vol via root mean square, not standard deviation

\[ \sigma_t = \sqrt{\frac{1}{m} \sum_{\tau=1}^{m} r_{t-m+\tau}^2} \]

- Moving window incorporates past \( m \) business days’ returns 
  \( r_{t-m+1}, \ldots, r_t \)
- Tantamount to assuming zero mean return
GARCH model of volatility

- **Generalized autoregressive conditionally heteroscedastic model**
- Volatility driven by:
  - Recent volatility
  - Recent returns
  - Long-term “point of rest” of volatility or “forever vol” $\bar{\sigma}$
- Estimate $\sigma_t$ made at today’s close updates yesterday’s estimate $\sigma_{t-1}$ with latest return $r_t$
- Look back one period $\rightarrow$ **GARCH(1,1):**
  \[
  \sigma_t^2 = \alpha r_t^2 + \beta \sigma_{t-1}^2 + \gamma \bar{\sigma}^2
  \]
- Feedback to returns via “shock” or “innovation” $\epsilon_t$, together with current volatility
  \[
  r_t = \epsilon_t \sigma_{t-1},
  \]
  - $\epsilon_t$ assumed i.i.d. with mean 0 and variance 1
  - Today’s return the only pertinent new information on date $t$
  - The weights satisfy $\alpha, \beta, \gamma > 0$ and $\alpha + \beta + \gamma = 1$
Role of parameters in the GARCH model

- **Impact of $\alpha$: high $\alpha \Rightarrow$**
  - Large $r_t$ causes large, immediate change in estimated return volatility $\sigma_t$
  - Wider range of variation of $\sigma_t$ over time

- **Impact of $\beta$: high $\beta \Rightarrow$**
  - $\sigma_t$ and deviations from $\bar{\sigma}^2$ very persistent
  - Less variation of $\sigma_t$ over time

- **Long-term variance $\bar{\sigma}^2 > 0$**
  - Presence of $\bar{\sigma}^2$ generates a term structure of volatility
  - **Example:** $\bar{\sigma}^2$ approximately 1.0–1.15 percent for U.S. equity market (at daily rate)

- **Low $\gamma \Rightarrow$** little mean reversion

- Estimates of $\beta$ generally not very far from 1, $\alpha + \beta$ quite close to 1

- Estimated parameter values lead to (hopefully realistic) behavior of volatility
Estimating GARCH(1,1) model parameters

- **Maximum likelihood method** a standard approach
  - Assume **conditional normality**, shocks $\epsilon_t$ normally distributed, a stronger assumption than i.i.d.:

    $$\epsilon_t \sim \mathcal{N}(0, 1) \quad \forall t$$

- Joint normal density of $m$ return observations $\Rightarrow$ **log likelihood function**

    $$\sum_{\tau=1}^{m} \left[ -\ln(\sigma_{t-\tau}^2) - \frac{r_t^2}{\sigma_{t-\tau}^2} \right],$$

    with initial volatility value $\sigma_0$

- Use numerical search procedure to find parameters that maximize log likelihood function
  - Numerical search procedure can be sensitive to initial trial guess
  - $\omega \equiv \gamma \bar{\sigma}^2$ treated as a single parameter
  - $\gamma$ can then be recovered as $1 - \alpha - \beta$ and

    $$\bar{\sigma} = \sqrt{\frac{\omega}{1 - \alpha - \beta}}$$
Influence of past returns in GARCH model

- GARCH(1,1) formula can be recast in terms of most recent and past squared returns (setting $m = t$):

\[
\sigma^2_1 = \alpha r^2_1 + \beta \sigma^2_0 + \omega
\]

\[
\sigma^2_2 = \alpha r^2_2 + \beta \sigma^2_1 + \omega = \alpha r^2_2 + \beta (\alpha r^2_1 + \beta \sigma^2_0 + \omega) + \omega
\]

\[
= \alpha r^2_2 + \alpha \beta r^2_1 + \beta^2 \sigma^2_0 + (1 + \beta) \omega
\]

\[
\vdots
\]

\[
\sigma^2_t = \alpha \sum_{\tau=1}^{t} \beta^{t-\tau} r^2_\tau + \sum_{\tau=1}^{t} \beta^{t-\tau} \omega + \beta^t \sigma^2_0
\]

\[
\approx \alpha \sum_{\tau=1}^{t} \beta^{t-\tau} r^2_\tau + \frac{1}{1 - \beta} \omega
\]

- $\alpha < 1$, $\beta < 1 \Rightarrow$ small influence of more remote past returns, starting value $\sigma_0$

- Influence of long-term variance is at the expense of most recent volatility estimate
Example of GARCH(1,1) model estimation

- Applied to S&P 500 index, using $m + 1 = 1570$ closing-price observations 04Jan2010 to 30Mar2016
- $r_1^2$ used as starting value $\sigma_0$
  - Can also use sample variance of entire time series
- For each pass of the search procedure, successively apply GARCH(1,1) formula to calculate trial values $\sigma_1, \sigma_2, \ldots, \sigma_t$:

<table>
<thead>
<tr>
<th>Parameter estimates</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$ 0.13926</td>
</tr>
<tr>
<td>$\omega$ $4.08969 \times 10^{-6}$</td>
</tr>
<tr>
<td>$\gamma$ 0.04040</td>
</tr>
</tbody>
</table>
Exponentially-weighted moving average model

- **Exponentially-weighted moving average** (EWMA)
  - A.k.a. **RiskMetrics model**
  - Variance a weighted average of past returns
  - Weights smaller for more-remote past returns
- Single parameter: **decay factor** $\lambda$
  - Low $\lambda$: rapid adaptation to recent returns
  - High $\lambda$: slow adaptation to recent returns
- EWMA implies a flat term structure of volatility
  - Volatility follows square-root-of-time rule
  - Volatility behaves as a random walk, subject to shocks
- Decay factor estimation: $\lambda$ that minimizes forecast errors, e.g. RMS criterion
- Decay factor may also be chosen judgmentally
Estimating volatility with the EWMA model

- Current volatility estimate $\sigma_t$ uses $m$ most recent observed returns $r_{t-m+1}, \ldots, r_t$
  - Treat $\lambda$ as known parameter
  - Weight on each squared return $\frac{1-\lambda}{1-\lambda^m} \lambda^{m-\tau}$, $\tau = 1, \ldots, m$
    - Apply $\lambda^{m-m} = 1$ for $\tau = m$, most recent (time $t$) return
    - Apply $\lambda^{m-1} \approx 0$ for $\tau = 1$, most remote (time $t-m+1$) return

\[
\sigma_t^2 = \frac{1-\lambda}{1-\lambda^m} \sum_{\tau=1}^{m} \lambda^{m-\tau} r_{t-m+\tau}^2
\]

- $1 - \lambda^m \approx 1 \Rightarrow$

\[
\sigma_t^2 \approx (1 - \lambda) \sum_{\tau=1}^{m} \lambda^{m-\tau} r_{t-m+\tau}^2
\]

- $m$ doesn't have to be large
  - $m \approx 100$ more than adequate unless $\lambda$ quite close to 1
The EWMA model weighting scheme

The graph displays the values of the last 100 of $m = 250$ EWMA weights $\frac{1-\lambda}{1-\lambda^m} \lambda^{m-t}$ for $\lambda = 0.94$ and $\lambda = 0.97$. 

The exponentially-weighted moving average model

Volatility behavior and forecasting
Choosing the decay factor

- Low $\lambda \Rightarrow$ recent observations have greater weight:
  - Volatility changes rapidly
  - Recent observations have most information useful for short-term conditional volatility forecasting
- Low $\lambda$ effectively shortens historical sample size compared to high $\lambda$
- Estimates using low $\lambda$ much more variable than those using high $\lambda$
- Estimates using low $\lambda$ respond more rapidly to new information
- Estimates using low $\lambda$ may move in the opposite direction from those using high $\lambda$
  - Estimates using low $\lambda$ decline after a sequence of high-magnitude returns, while those using high $\lambda$ still rising in response
- No agreed method for estimating $\lambda$
- Widely adopted standard settings for decay factor:
  - $\lambda = 0.94$ for short-term (e.g. one-day, one-week) forecasts
  - $\lambda = 0.97$ for medium-term (e.g. one-month) forecasts
  - Minimizes RMS of forecast errors for range of assets in original 1994 RiskMetrics study
Effect of the decay factor on the volatility forecast

EWMA estimates of the volatility of daily S&P 500 index returns, at a daily rate in percent, from 02Jan2006 through 02Sep2016, using decay factors of $\lambda = 0.94$ and $\lambda = 0.99$. Points represent the absolute value of daily return observations.
Estimating volatility with the EWMA model

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>Date</th>
<th>$S_{t-m+\tau}$</th>
<th>$r_{t-m+\tau}$</th>
<th>$\frac{1-\lambda}{1-\lambda^m} \lambda^{m-\tau}$</th>
<th>$\frac{1-\lambda}{1-\lambda^m} \lambda^{m-\tau} r_{t-m+\tau}^2$</th>
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<tr>
<td>0</td>
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<td>1973.63</td>
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<td>0.00000</td>
<td>0.00000×10^{-6}</td>
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</tr>
<tr>
<td>173</td>
<td>27Mar2015</td>
<td>2061.02</td>
<td>0.00237</td>
<td>0.00051</td>
<td>0.00286×10^{-6}</td>
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<tr>
<td>174</td>
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<td>0.08052×10^{-6}</td>
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<tr>
<td>175</td>
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<td>176</td>
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<tr>
<td>177</td>
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<td>0.00066</td>
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<td></td>
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<tr>
<td>246</td>
<td>13Jul2015</td>
<td>2099.60</td>
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<tr>
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</tbody>
</table>

Return vol of the S&P 500 index, estimated after the close on 17Jul2015 (date $t$), with $m = 250$, $\lambda = 0.94$. Return (4th column) expressed as a decimal. Add the last 250 values in the last column to get the estimated variance $\sigma_t^2$. 
The exponentially-weighted moving average model

Recursive formula for EWMA volatility estimates

- Recursive formula updates most recent volatility estimate with new data on return magnitude

\[ \sigma_t^2 = \lambda \sigma_{t-1}^2 + (1 - \lambda) r_t^2 \]

- Easy computation technique, very close to result of full EWMA weighting scheme

- Shows similarity of EWMA to “one-parameter” GARCH
  - But with long-term volatility term \( \gamma = 0, \alpha + \beta = 1 \)
  - \( \lambda \) analogous to \( \beta \), \( 1 - \lambda \) analogous to \( \alpha \)
  - Shocks to volatility permanent, no long-term “forever” vol
  - Also known as integrated GARCH or IGARCH(1,1)

- EWMA estimate usually close to unrestricted GARCH(1,1) estimate

- “Starter value” (orange in example on next slide):
  - Root mean square, using 6 months of data preceding sample
  - Starter value method not crucial, converges quickly (esp. for low \( \lambda \))
### Recursive formula for EWMA volatility estimates

<table>
<thead>
<tr>
<th>$t$</th>
<th>Date</th>
<th>$S_t$</th>
<th>$r_t$ (%)</th>
<th>$\lambda \sigma_{t-1}^2$</th>
<th>$(1 - \lambda) r_t^2$</th>
<th>$\sigma_t$ (%)</th>
</tr>
</thead>
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<tr>
<td>0</td>
<td>30Dec2005</td>
<td>1248.29</td>
<td>NA</td>
<td>NA</td>
<td>NA</td>
<td>0.61411</td>
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<td>1.6297</td>
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<td>0.72471 $\times 10^{-6}$</td>
<td>0.64685</td>
</tr>
</tbody>
</table>

Return vol of the S&P 500 index, estimated daily using the recursive formula, with $\lambda = 0.94$. Initial estimate uses the root mean square of the 127 daily returns 30Jun2005 to 30Dec2005. The return from 29Dec2005 to 30Dec2005 is used in the initial estimate, but not the recursive formula.
GARCH(1,1) and EWMA volatility estimates

Daily estimates of S&P 500 index’s annualized return volatility, 04Jan2010 to 30Mar2016. **EWMA estimates** with $\lambda = 0.94$ **GARCH(1,1) estimates** use parameters $\alpha = 0.139$, $\beta = 0.820$, $\gamma \hat{\sigma}^2 = 4.08969 \times 10^{-6}$. 