Appendix to the paper "Determining the Number of Factors from Empirical Distribution of Eigenvalues"

Alexei Onatski

Economics Department, Columbia University

May 8, 2009

Abstract

This technical appendix contains proofs of Lemmas 1 and 3 of the paper.

1 Proof of Lemma 1

The original Theorem 1.2.1 in Zhang (2006) is as follows. Using Zhang's notation, let $B_n = (1/N)\, T_{2n}^{1/2} X_n T_{1n} X_n^* T_{2n}^{1/2}$. Then B_n is said to be a general sample covariance matrix if the following conditions are satisfied.

(i) $X_n = [x_{ij}]$ is $N \times n$ consisting of independent complex random variables with $Ex_{ij} = 0, E\left|x_{ij}\right|^2 = 1$ satisfying for each $\delta > 0$, as $n \to \infty : \frac{1}{\delta^2 nN} \sum_{ij} E\left(\left|x_{ij}\right|^2 I_{\left(|x_{ij}|>\delta\sqrt{n}\right)}\right) \to 0.$

(ii) T_{1n} is $n \times n$ Hermitian and T_{2n} is $N \times N$ Hermitian nonnegative definite.

(iii) With probability 1, as $n \to \infty$, the empirical spectral distributions of T_{1n} and T_{2n}, denoted by $F^{T_{1n}}$ and $F^{T_{2n}}$, converge weakly to two probability functions H_1 and H_2, respectively.

(iv) $N = N(n)$ with $n/N \to c > 0$.

(v) X_n, T_{1n}, T_{2n} are independent.

Zhang (2006) proves the following:

Theorem 1.2.1 (Zhang, 2006) *Let B_n be the general sample covariance matrices defined above. Then with probability 1, as n $\to \infty$, the empirical spectral distribution of B_n converges weakly to some non-random probability distribution function $\underline{F}$ for which if $H_1 \equiv I_{[0,\infty)}$ or $H_2 \equiv I_{[0,\infty)}$, then $\underline{F} \equiv I_{[0,\infty)}$; otherwise if for each $z \in \mathbb{C}^+$,*

$$\begin{cases} s(z) = -z^{-1}(1-c) - z^{-1}c \int \frac{1}{1+q(z)x} dH_1(x) \\ s(z) = -z^{-1} \int \frac{1}{1+p(z)y} dH_2(y) \\ s(z) = -z^{-1} - p(z)q(z) \end{cases} \tag{1}$$

is viewed as a system of equations for the complex vector $(s(z), p(z), q(z))$, then the Stieltjes transform of $\underline{F}$, denoted as $s_{\underline{F}}(z)$, together with two other functions, denoted by $g_1(z)$ and $g_2(z)$, both of which are analytic on $\mathbb{C}^+$, will satisfy that $(s_{\underline{F}}(z), g_1(z), g_2(z))$ is the unique solution to the above system in the set:

$$\overline{U} = \{(s(z), p(z), q(z)) : \operatorname{Im} s(z) > 0, \operatorname{Im}(zp(z)) > 0, \operatorname{Im} q(z)\} > 0.$$

Note the difference in notation between our paper and Zhang's. Our T is her N, our ε is her X_n^*, our $A^{(n,T)\prime}A^{(n,T)}$ is her T_{1n} and our BB' is her T_{2n}. Our Lemma 1 is about the limiting spectral distribution of $\frac{1}{T}A\varepsilon BB'\varepsilon' A'$, whereas Zhang's Theorem 1.2.1 is about the limiting spectral distribution of $\frac{1}{T}B'\varepsilon' A' A\varepsilon B$ (in our notation). Obviously, our Assumption 2 implies the assumptions of Theorem 1.2.1.

Let us define

$$m(z) = c^{-1}s(z) + \left(c^{-1} - 1\right) z^{-1}, v(z) = -\frac{1}{p(z)}, \text{ and } u(z) = -\frac{1}{q(z)} \tag{2}$$

so that $s(z) = cm(z) + (1-c)z^{-1}, p(z) = -v^{-1}(z)$ and $q(z) = -u^{-1}(z)$. Making a substitution in (1), we obtain:

$$\begin{cases} zm(z) + 1 = \int \frac{u(z)}{x-u(z)} dH_1(x) + 1 \\ zm(z) + 1 = c^{-1}\left[\int \frac{v(z)}{y-v(z)} dH_2(y) + 1\right] \\ zm(z) + 1 = -c^{-1}\frac{z}{u(z)v(z)} \end{cases}, \tag{3}$$

which is equivalent to the system in our Lemma 1.

Now, let $(s_{\underline{F}}(z), g_1(z), g_2(z)) \in \overline{U}$ be a solution to (1). Then,

$$m_F(z) \equiv c^{-1}s_{\underline{F}}(z) + \left(c^{-1} - 1\right)z^{-1}, f_1(z) \equiv -1/g_1(z) \text{ and } f_2(z) = -1/g_2(z) \tag{4}$$

constitute a solution (m, v, u) to (3). Let us show that

$$\operatorname{Im} m_F(z) > 0, \operatorname{Im} f_1(z) > 0 \text{ and } \operatorname{Im} f_2(z) > 0, \tag{5}$$

and that (4) is the only solution to (3) which satisfies (5).

Denote the Stieltjes transform of the spectral distribution of $\frac{1}{T}B'\varepsilon'A'A\varepsilon B$ as $s_n(z)$. Then, according to Theorem 1.2.1, $s_n(z) \to s_{\underline{F}}(z)$ almost surely. Note that the Stieltjes transform of the spectral distribution of $\frac{1}{T}A\varepsilon BB'\varepsilon'A'$, denoted as $m_n(z)$, is related to $s_n(z)$ as follows:

$$\begin{aligned} s_n(z) &= \frac{1}{T}\sum\nolimits_{i=1}^{T}\left(\lambda_i\left(\frac{1}{T}B'\varepsilon'A'A\varepsilon B\right) - z\right)^{-1} \\ &= \frac{1}{T}\left(\sum\nolimits_{i=1}^{n} - \sum\nolimits_{i=T+1}^{n}\right)\left(\lambda_i\left(\frac{1}{T}A\varepsilon BB'\varepsilon'A'\right) - z\right)^{-1} \\ &= \frac{n}{T}m_n(z) - \frac{n-T}{T}(-z)^{-1}. \end{aligned}$$

Therefore, $m_n(z) \to c^{-1}s_{\underline{F}}(z) + (c^{-1}-1)z^{-1} \equiv m_F(z)$ almost surely, and hence, $m_F(z)$ is the Stieltjes transform of the limiting spectral distribution of $\frac{1}{T}A\varepsilon BB'\varepsilon'A'$.

Since the imaginary part of any Stieltjes transform evaluated at $z \in \mathbb{C}^+$ is positive, we have: $\operatorname{Im} m_F(z) > 0$. Next, since $\operatorname{Im} g_2(z) > 0$ for $z \in \mathbb{C}^+$, we have: $\operatorname{Im} f_2(z) > 0$. Finally, note that, since $\frac{1}{T}B'\varepsilon' A' A\varepsilon B$ is non-negative definite, $\operatorname{Im}\left[z s_{\underline{F}}(z)\right] \equiv \operatorname{Im}\int \frac{z}{x-z} d\underline{F}(x) = \operatorname{Im}\int \frac{x}{x-z} d\underline{F}(x) > 0$ for $z \in \mathbb{C}^+$. On the other hand, from the second equation of (1), $\operatorname{Im} z s_{\underline{F}}(z) = -\operatorname{Im}\int \frac{g_1^{-1}(z)}{g_1^{-1}(z)+y} dH_2(y) = \operatorname{Im}\int \frac{y}{g_1^{-1}(z)+y} dH_2(y)$, which must have the same sign as that of $\operatorname{Im}\left[-g_1^{-1}(z)\right]$. That is, $\operatorname{Im}\left[-g_1^{-1}(z)\right] > 0$ for $z \in \mathbb{C}^+$, and $\operatorname{Im} f_1(z) > 0$.

It remains to show that there is only one solution to (3) that satisfies inequalities (5). Suppose not, and $\left(\tilde{m}_F(z), \tilde{f}_1(z), \tilde{f}_2(z)\right)$ is another such solution. Then, $\left(\tilde{s}_{\underline{F}}(z), \tilde{g}_1(z), \tilde{g}_2(z)\right)$, where

$$\tilde{s}_{\underline{F}}(z) \equiv c\tilde{m}(z) - (1-c)\, z^{-1}, \tilde{g}_1(z) = -\tilde{f}_1^{-1}(z) \text{ and } \tilde{g}_2(z) = -\tilde{f}_2^{-1}(z)$$

must be another solution to (1). Moreover, for $z \in \mathbb{C}^+$, $\operatorname{Im}\tilde{g}_2(z) > 0$ because $\operatorname{Im}\tilde{f}_2(z) > 0$. Further, $\operatorname{Im}\left[z\tilde{g}_1(z)\right] > 0$ because, otherwise, we would have: $\operatorname{Im}\left[z/\tilde{f}_1(z)\right] > 0$. But from the third and the first equations of (3), we have:

$$\begin{aligned}
\operatorname{Im}\left[z/\tilde{f}_1(z)\right] &= -c\operatorname{Im}\left[\tilde{f}_2(z)\left(z\tilde{f}_2(z)+1\right)\right] = -c\operatorname{Im}\int \frac{x\tilde{f}_2(z)}{x-\tilde{f}_2(z)} dH_1(x) \\
&= -c\int \frac{x^2\operatorname{Im}\tilde{f}_2(z)}{\left|x-\tilde{f}_2(z)\right|^2} dH_1(x) < 0,
\end{aligned}$$

hence, a contradiction. Finally, $\operatorname{Im}\tilde{s}_{\underline{F}}(z) > 0$ because, by the second equation of (1), $\operatorname{Im}\tilde{s}_{\underline{F}}(z) = -\operatorname{Im}\int \frac{1}{z+z\tilde{g}_1(z)y} dH_2(y)$, which must have the same sign as that of $\operatorname{Im}\left(z + z\tilde{g}_1(z)y\right)$. But for $z \in \mathbb{C}^+$, $\operatorname{Im}\left(z + z\tilde{g}_1(z)y\right) > 0$. Since (1) cannot have two different solutions belonging to $\bar{U}$, we have: $\left(\tilde{s}_{\underline{F}}(z), \tilde{g}_1(z), \tilde{g}_2(z)\right) = \left(s_{\underline{F}}(z), g_1(z), g_2(z)\right)$ and hence, $\left(\tilde{m}_F(z), \tilde{f}_1(z), \tilde{f}_2(z)\right) = \left(m_F(z), f_1(z), f_2(z)\right)$ is the unique solution to (3) with positive imaginary parts for $z \in \mathbb{C}^+$.$\square$

2 Proof of Lemma 3

To prove Lemma 3, we will need the following Lemmas A1-A4.

Lemma A1. *Let F be the cdf of a non-negative random variable with a finite upper boundary of support $u(F)$ and a finite positive expectation E_F. Let $m(z)$ be the Stieltjes transform of F. Then, for any $z \in \mathbb{C}^+$ such that $|z| > u(F)$ we have: $|zm(z)+1| \leq \frac{u(F)}{|z|-u(F)}$; and for any $z \in \mathbb{C}^+$ such that $z = x+iy$, where $|x| > u(F) + 3y$ we have: $|zm(z)+1| \geq \frac{E_F}{4(|z|+u(F))}$.*

Proof: For $|z| > u(F)$, we have: $|zm(z)+1| = \left|\int \left(\frac{z}{\lambda - z}+1\right) dF(\lambda)\right| = \left|\int \frac{\lambda}{\lambda-z} dF(\lambda)\right| \leq \int \left|\frac{\lambda}{\lambda - z}\right| dF(\lambda) \leq \int \frac{\lambda}{|z|-\lambda} dF(\lambda) \leq \frac{u(F)}{|z|-u(F)}$, which proves one of the lemma's inequalities. Further, write:

$$zm(z)+1 = \int \frac{\lambda}{\lambda - z} dF(\lambda) = \int \frac{\lambda(\lambda - x)}{|\lambda - z|^2} dF(\lambda) + i \int \frac{\lambda y}{|\lambda - z|^2} dF(\lambda). \tag{6}$$

Since for any real a and b, $|a+ib|^2 = a^2 + b^2 \geq \frac{1}{2}(a+b)^2$, we have: $|zm(z)+1| \geq \frac{1}{\sqrt{2}}\left|\int \frac{\lambda(\lambda - x+y)}{|\lambda - z|^2} dF(\lambda)\right|$. If $x < 0$, then, since for any real and positive a and b, $\frac{a+b}{a^2+b^2} \geq (a^2+b^2)^{-1/2}$, we have $\frac{\lambda(\lambda-x+y)}{|\lambda-z|^2} \geq \frac{\lambda}{|\lambda - z|}$, and therefore: $|zm(z)+1| \geq \frac{1}{\sqrt{2}}\int \frac{\lambda}{|\lambda-z|} dF(\lambda) \geq \frac{1}{\sqrt{2}}\int \frac{\lambda}{|z|+\lambda} dF(\lambda) \geq \frac{1}{\sqrt{2}}\int \frac{\lambda}{|z|+u(F)} dF(\lambda) = \frac{E_F}{\sqrt{2}(|z|+u(F))}$. If $x > 0$, then by assumption of the lemma, $x > u(F) + 3y$, which implies that $\frac{1}{2}(x - \lambda - y) > y$ and hence, $x - \lambda > y + \frac{1}{2}(x - \lambda + y) \geq y + \frac{1}{2}|\lambda - z|$ so that $x - \lambda - y \geq \frac{1}{2}|\lambda - z|$. Therefore: $|zm(z)+1| \geq \frac{1}{\sqrt{2}}\int \frac{\lambda(x-\lambda-y)}{|\lambda - z|^2} dF(\lambda) \geq \frac{1}{2\sqrt{2}}\int \frac{\lambda}{|\lambda-z|} dF(\lambda) \geq \frac{E_F}{2\sqrt{2}(|z|+u(F))}$.□

Lemma A2. *Suppose that Assumption 2 in the main text of the paper holds and that $u(z), v(z) \in \mathbb{C}^+$ are analytic functions satisfying the system of equations from Lemma 1:*

$$\begin{cases} zm(z)+1 = u(z)\, m_A(u(z)) + 1 \\ zm(z)+1 = c^{-1}\left[v(z)\, m_B(v(z)) + 1\right] \\ zm(z)+1 = -c^{-1}\frac{z}{u(z)v(z)} \end{cases} \tag{7}$$

Let $U = \{z = x+iy: \ x > \underline{x} \text{ and } 0 < y < \bar{y}\}$. Then, for any $\underline{x} > u\left(\mathcal{F}^{c,A,B}\right)$, there exists $\bar{y} > 0$ such that for any z from U : $\operatorname{Re} u(z) > u(\mathcal{F}_A)$ and $\operatorname{Re} v(z) > u(\mathcal{F}_B)$.

Proof: The idea of the proof is as follows. First, using Lemma A1 we prove that for any $\underline{x} > u\left(\mathcal{F}^{c,A,B}\right)$ and $\bar{y} > 0$, there exists $z_1 \in U$ such that $\operatorname{Re} u\left(z_1\right) > u\left(\mathcal{F}_A\right)$ and $\operatorname{Re} v\left(z_1\right) > u\left(\mathcal{F}_B\right)$. Then, we assume that Lemma A2 does not hold so that for some $\underline{x} > u\left(\mathcal{F}^{c,A,B}\right)$ and any $\bar{y} > 0$ there exists $z_2 \in U$ such that $\operatorname{Re} u\left(z_2\right) \leq u\left(\mathcal{F}_A\right)$ or/and $\operatorname{Re} v\left(z_2\right) \leq u\left(\mathcal{F}_B\right)$. Connecting z_1 and z_2 by a continuous path $z(t) \in U$, we establish the existence of $z_3 \in U$ such that $\operatorname{Re} u\left(z_3\right) = u\left(\mathcal{F}_A\right)$ or/and $\operatorname{Re} v\left(z_3\right) = u\left(\mathcal{F}_B\right)$. Then, we show that for small enough $\bar{y}$, $\operatorname{Im}\left(z_3 m\left(z_3\right)+1\right)$ must be smaller than $\operatorname{Im}\left(u\left(z_3\right) m_A\left(u(z_3)\right)+1\right)$ or than $c^{-1}\operatorname{Im}\left(v\left(z_3\right) m_B\left(v(z_3)\right)+1\right)$, which contradicts the assumption that $u(z), v(z)$ satisfy system (7).

First, we prove the existence of z_1. The last equation of (7) and the first inequality of Lemma A1 imply that $\left|\frac{z}{uv}\right| \to 0$ as $|z| \to \infty$. Hence, as $|z| \to \infty$, $\max\{|u|,|v|\} \to \infty$. Suppose without loss of generality that $|u| \to \infty$. Let us show that also $|v| \to \infty$. Indeed, from the first equation of system (7), $|zm(z)+1| = |um_A(u)+1|$. Therefore, for $z \in U$ with large enough $|z|$:

$$\frac{E_{\mathcal{F}^{c,A,B}}}{4\left(|z| + u(\mathcal{F}^{c,A,B})\right)} \leq |zm(z)+1| = |um_A(u)+1| \leq \frac{u(\mathcal{F}_A)}{|u| - u\left(\mathcal{F}_A\right)}. \tag{8}$$

where the latter inequality is obtained from Lemma A1 applied to $um_A(u)+1$. This implies that $\liminf_{|z|\to\infty}\left|\frac{z}{u}\right| > 0$, for $z \in U$. However, since $\left|\frac{z}{uv}\right| = c\,|zm(z)+1| \leq \frac{cu\left(\mathcal{F}^{c,A,B}\right)}{|z| - u(\mathcal{F}^{c,A,B})} \to 0$, we must have $|v| \to \infty$. Hence, as $|z| \to \infty$ so that z remains in U, both $|u| \to \infty$ and $|v| \to \infty$. Let us prove that, in addition, $\operatorname{Re} u \to \infty$ and $\operatorname{Re} v \to \infty$.

First, notice that for $z \in U$:

$$\operatorname{Im}\left(zm(z)+1\right) < \bar{y}\int \frac{\lambda}{|\lambda - z|^2} d\mathcal{F}^{c,A,B}(\lambda) \leq \frac{\bar{y}E_{\mathcal{F}^{c,A,B}}}{\left(x - u\left(\mathcal{F}^{c,A,B}\right)\right)^2}. \tag{9}$$

Further, $\operatorname{Im}\left(-\frac{z}{uv}\right) = \frac{x\operatorname{Im}(uv) - y\operatorname{Re}(uv)}{|uv|^2}$. Therefore, from the third equation of (7) and (9), we have: $\frac{x\operatorname{Im}(uv)-y\operatorname{Re}(uv)}{|uv|^2}\leq c\frac{\bar{y}E_{\mathcal{F}^{c,A,B}}}{(x-u(\mathcal{F}^{c,A,B}))^2}$. Hence, for $z\in U$, where $\bar{y}\leq\frac{\underline{x}-u\left(\mathcal{F}^{c,A,B}\right)}{3}$, we have:

$$\frac{\operatorname{Im}(uv)}{|uv|}\leq\frac{|uv|}{x}\frac{c\bar{y}E_{\mathcal{F}^{c,A,B}}}{(x-u(\mathcal{F}^{c,A,B}))^2}+\frac{y\operatorname{Re}(uv)}{x\,|uv|}\leq\frac{\bar{y}}{u(\mathcal{F}^{c,A,B})}\left(\left|\frac{cuv}{z^2}\right|\frac{E_{\mathcal{F}^{c,A,B}}\left(x^2+\bar{y}^2\right)}{(x-u(\mathcal{F}^{c,A,B}))^2}+1\right).$$

Now, the third equation of (7) and the second inequality of Lemma A1 imply that

$$\left|\frac{cuv}{z^2}\right|=\left|\frac{1}{z(zm(z)+1)}\right|\leq\frac{4\left(|z|+u(\mathcal{F}^{c,A,B})\right)}{|z|\,E_{\mathcal{F}^{c,A,B}}}\leq\frac{8}{E_{\mathcal{F}^{c,A,B}}}.$$

Therefore,

$$\frac{\operatorname{Im}(uv)}{|uv|}\leq\frac{\bar{y}}{u(\mathcal{F}^{c,A,B})}\left(\frac{8\left(x^2+\bar{y}^2\right)}{(x-u(\mathcal{F}^{c,A,B}))^2}+1\right)\leq\frac{\bar{y}}{u(\mathcal{F}^{c,A,B})}\left(\frac{8\left(\underline{x}^2+\bar{y}^2\right)}{(\underline{x}-u(\mathcal{F}^{c,A,B}))^2}+1\right).$$

Noting that $\frac{\operatorname{Im}u}{|u|}\leq\frac{\operatorname{Im}(uv)}{|uv|}$, we have:

$$\frac{\operatorname{Im}u}{|u|}\leq\frac{\bar{y}}{u(\mathcal{F}^{c,A,B})}\left(\frac{8\left(\underline{x}^2+\bar{y}^2\right)}{(\underline{x}-u(\mathcal{F}^{c,A,B}))^2}+1\right)\tag{10}$$

for $z\in U$, where $\bar{y}\leq\frac{\underline{x}-u\left(\mathcal{F}^{c,A,B}\right)}{3}$. The same inequality also holds for $\frac{\operatorname{Im}v}{|v|}$.

Choosing $\bar{y}$ so small that the right hand side of (10) is less than 1, we see that the fact that $|u|\to\infty$ implies that $|\operatorname{Re}u|\to\infty$ as $|z|\to\infty$ while z remains in U. Similarly, $|\operatorname{Re}v|\to\infty$. But $\operatorname{Re}u$ and $\operatorname{Re}v$ must be positive for $z\in U$ when $|z|$ is large enough. Indeed, (6) implies that $\operatorname{Re}(zm(z)+1)<0$. Hence, from the first equation of (7), $\operatorname{Re}(um_A(u)+1)<0$. But from (6) applied to $um_A(u)+1$ and the fact that $|\operatorname{Re}u|\to\infty$, $\operatorname{Re}(um_A(u)+1)$ must be of the same sign as $-\operatorname{Re}u$ for $|z|$ large enough. Hence, $\operatorname{Re}u\to+\infty$ as $|z|\to\infty$ while z remains in U. Similarly, $\operatorname{Re}v\to+\infty$ as $|z|\to\infty$ while z remains in U. This proves the existence of $z_1\in U$ such that $\operatorname{Re}u\left(z_1\right)>u\left(\mathcal{F}_A\right)$ and $\operatorname{Re}v\left(z_1\right)>u\left(\mathcal{F}_B\right)$.

Now, let us assume that Lemma A2 does not hold. Then, for some $\underline{x}>u\left(\mathcal{F}^{c,A,B}\right)$ and any $\bar{y}>0$, there exists $z_2\in U$ such that $\operatorname{Re}u\left(z_2\right)\leq u\left(\mathcal{F}_A\right)$ or/and $\operatorname{Re}v\left(z_2\right)\leq u\left(\mathcal{F}_B\right)$. Since

$u(z)$ and $v(z)$ are analytic, and hence continuous, functions of z, and since $\operatorname{Re} u(z_1) > u(\mathcal{F}_A)$ and $\operatorname{Re} v(z_1) > u(\mathcal{F}_B)$, there exists $z_3 \in U$ such that $\operatorname{Re} u(z_3) = u(\mathcal{F}_A)$ or/and $\operatorname{Re} v(z_3) = u(\mathcal{F}_B)$. Suppose without loss of generality that $\operatorname{Re} u(z_3) = u(\mathcal{F}_A)$.

Let us finish the proof of the lemma by comparing $\operatorname{Im}(z_3 m(z_3) + 1)$ with $\operatorname{Im}(u(z_3) m_A(u(z_3)) + 1)$ when $\bar{y}$ is small. By Assumption 2 iii, $\liminf_{\delta\to 0} \frac{1}{\delta} \int_{|\lambda - u(\mathcal{F}_A)| \le \delta} d\mathcal{F}_A(\lambda) = k^A > 0$. Therefore, for $u(z)$ such that $\operatorname{Re} u = u(\mathcal{F}_A)$ and $\operatorname{Im} u$ is small enough, we have:

$$\begin{aligned}
&\operatorname{Im}(u m_A(u) + 1) = \int \frac{\lambda \operatorname{Im} u}{(\lambda - u(\mathcal{F}_A))^2 + (\operatorname{Im} u)^2} d\mathcal{F}_A(\lambda) \\
\ge\ & \frac{1}{2 \operatorname{Im} u} \int_{|\lambda - u(\mathcal{F}_A)| \le \operatorname{Im} u} \lambda d\mathcal{F}_A(\lambda) \ge \frac{u(\mathcal{F}_A) - \operatorname{Im} u}{2 \operatorname{Im} u} \int_{|\lambda - u(\mathcal{F}_A)| \le \operatorname{Im} u} d\mathcal{F}_A(\lambda) \qquad (11) \\
\ge\ & \frac{u(\mathcal{F}_A) - \operatorname{Im} u}{2} k^A > \frac{k^A}{4} > 0.
\end{aligned}$$

From (11) and (10), we can choose $\bar{y}$ small enough so that $\operatorname{Im}(u(z_3) m_A(u(z_3)) + 1) \ge \frac{k^A}{4}$. On the other hand, from the first equation of (7), $u(z_3) m_A(u(z_3)) + 1 = z_3 m(z_3) + 1$ and hence $\operatorname{Im}(z_3 m(z_3) + 1) \ge \frac{k^A}{4}$. But from (9) we know that for small enough $\bar{y}$, $\operatorname{Im}(z_3 m(z_3) + 1)$ must be smaller than $\frac{k^A}{4}$. We have got a contradiction, which implies that the statement of Lemma A2 holds.□

Lemma A3. *Under the conditions of Lemma A2, for any real $x > u(\mathcal{F}^{c,A,B})$, there exist real limits $u(x) \equiv \lim_{z \in \mathbb{C}^+, z\to x} u(z)$ and $v(x) \equiv \lim_{z\in\mathbb{C}^+, z\to x} v(z)$ which satisfy the limit version of (7):*

$$\begin{cases} x m(x) + 1 = u m_A(u) + 1 \\ x m(x) + 1 = c^{-1}(v m_B(v) + 1) \\ x m(x) + 1 = -c^{-1} \frac{x}{uv} \end{cases}, \qquad (12)$$

The functions $u(x)$ and $v(x)$ are analytic for $x > u(\mathcal{F}^{c,A,B})$ and such that $\lim_{x\to\infty} u(x) = \lim_{x\to\infty} v(x) = \infty$.

Proof: Let $G = \left\{ z \in \mathbb{C}^+ : u(\mathcal{F}^{c,A,B}) < \underline{x} \le \operatorname{Re} z \le \bar{x} < \infty, 0 < \operatorname{Im} z < \bar{y} < \infty \right\}$. Then there exists $\bar{y}$ such that, for the corresponding G, we have: $\sup_{z\in G} \max(|u(z)|, |v(z)|) < \infty$.

Had this been not true, there would have existed a sequence $\{z_n\} \in G$ such that $|u(z_n)| \to \infty$ or $|v(z_n)| \to \infty$. Without loss of generality, let $|u(z_n)| \to \infty$. Lemmas A1 and A2 then would have implied that $|u(z_n) m_A(u(z_n)) + 1| \to 0$, and hence, from the first equation of (7), $|z_n m(z_n) + 1| \to 0$. But, as follows from (6), $|\mathrm{Re}(z_n m(z_n) + 1)| \geq \frac{E_F(x - u(F))}{\bar{x}^2 + \bar{y}^2} > 0$, which gives a contradiction.

Since $\sup_{z \in G} \max(|u(z)|, |v(z)|) < \infty$, inequality (10) and a similar inequality for $\frac{\mathrm{Im}\, v}{|v|}$ imply that for any sequence $\{z_n\} \in G$ such that $z_n \to x \in \mathbb{R}$ the concentration points of $\{u(z_n)\}$ and $\{v(z_n)\}$ must be real. Suppose that there exist subsequences of $z_n, \{z_i\}$ and $\{z_j\}$, such that $u(z_i) \to u_1 \in \mathbb{R}$ and $u(z_j) \to u_2 \in \mathbb{R}$ and $u_1 \neq u_2$. By Lemma A2, $u_1 \geq u(\mathcal{F}_A)$ and $u_2 \geq u(\mathcal{F}_A)$. If $u_1 = u(\mathcal{F}_A)$, then using inequalities similar to (11), we find that $\mathrm{Im}(u(z_i) m_A(u(z_i)) + 1) \geq \frac{k^A}{4}$ for large enough i, which cannot be the case because $\mathrm{Im}(u(z_i) m_A(u(z_i)) + 1) = \mathrm{Im}(z_i m(z_i) + 1) \to 0$ as $i \to \infty$. Hence, $u_1 > u(\mathcal{F}_A)$. Similarly, $u_2 > u(\mathcal{F}_A)$.

Since $m(x)$ exists and is continuous for $x > u(\mathcal{F}^{c,A,B})$, we have: $\lim_{z_n \to x}(z_n m(z_n) + 1) = xm(x) + 1$. Since $m_A(u)$ exists and is continuous for $u > u(\mathcal{F}_A)$, we have: $\lim_{z_i \to x}(u(z_i) m_A(u(z_i)) + 1) = u_1 m_A(u_1) + 1$ and $\lim_{z_j \to x}(u(z_j) m_A(u(z_j)) + 1) = u_2 m_A(u_2) + 1$. The first equation of system (7) implies that we must have:

$$xm(x) + 1 = u_1 m_A(u_1) + 1 = u_2 m_A(u_2) + 1.$$

But this is not possible with $u_1 \neq u_2$ such that $u_1 > u(\mathcal{F}_A)$ and $u_2 > u(\mathcal{F}_A)$ because function $u m_A(u) + 1$ is strictly increasing for $u > u(\mathcal{F}_A)$. Hence, there exists only one concentration point of $\{u(z_n)\}$, that is there exists a real limit $u(x) \equiv \lim_{z \in \mathbb{C}^+, z \to x} u(z)$. Similarly, there exists a real limit $v(x) \equiv \lim_{z \in \mathbb{C}^+, z \to x} v(z)$.

That $u(x)$ and $v(x)$ satisfy the limit version of (7) follows from the existence and continuity of $m_A(u)$ for $|u| > u(\mathcal{F}_A)$ and from the existence and continuity of $m_B(v)$ for $|v| > u(\mathcal{F}_B)$. The analyticity of $u(x)$ follows from the analyticity of $F(x, u) \equiv xm(x) + 1 - (u m_A(u) + 1)$

for $x > u\left(\mathcal{F}^{c,A,B}\right)$ and $u > u\left(\mathcal{F}_A\right)$ and from the implicit function theorem. Similarly, the analyticity of $v(x)$ follows from the analyticity of $F_1(x,v) \equiv xm(x) + 1 - c^{-1}(vm_B(v) + 1)$ for $x > u\left(\mathcal{F}^{c,A,B}\right)$ and $v > u\left(\mathcal{F}_B\right)$ and from the implicit function theorem. Finally, (12) implies that as $x \to \infty$, $um_A(u) + 1 \to 0$ and $vm_B(v) + 1 \to 0$, which can be the case only when $\lim_{x\to\infty} u(x) = \lim_{x\to\infty} v(x) = \infty$.□

Lemma A4. *For $x > u(\mathcal{F}^{c,A,B})$, the following system*

$$\begin{cases} v = x\left(c\int \frac{\lambda u}{u-\lambda} d\mathcal{F}_A(\lambda)\right)^{-1} \\ u = x\left(\int \frac{\lambda v}{v-\lambda} d\mathcal{F}_B(\lambda)\right)^{-1} \end{cases} \tag{13}$$

has exactly two solutions (u_1, v_1) and (u_2, v_2) such that $u_i > u\left(\mathcal{F}_A\right)$ and $v_i > u\left(\mathcal{F}_B\right)$ for $i = 1, 2$. For $x = u\left(\mathcal{F}^{c,A,B}\right)$ and for $x < u(\mathcal{F}^{c,A,B})$, the system has only one such solution and no such solutions, respectively

Proof: For any $x > u\left(\mathcal{F}^{c,A,B}\right)$, one solution to (13) satisfying $u(x) > u\left(\mathcal{F}_A\right)$ *and* $v(x) > u\left(\mathcal{F}_B\right)$ is given by $u(x)$ and $v(x)$ defined in Lemma A3. That such $u(x)$ and $v(x)$ indeed provide a solution to (13) follows from the fact that (13) can be obtained from (12) by substituting the third equation into the first two. Let us now show that for $x > u(\mathcal{F}^{c,A,B})$, there exists another solution to (13).

First, note that $x\left(c\int \frac{\lambda u}{u-\lambda} d\mathcal{F}_A(\lambda)\right)^{-1}$ as a function of $u > u\left(\mathcal{F}_A\right)$ is concave, tends to zero as $u \downarrow u\left(\mathcal{F}_A\right)$ and to $x\left(cE_{\mathcal{F}_A}\right)^{-1}$ as $u \to \infty$. The concavity follows from the expression

$$\frac{d^2}{du^2} x\left(c\int \frac{\lambda u}{u-\lambda} d\mathcal{F}_A(\lambda)\right)^{-1} = 2xc^{-1}\left(\int \frac{\lambda u}{u-\lambda} d\mathcal{F}_A(\lambda)\right)^{-3} \times$$
$$\left(\left(\int \frac{\lambda^2}{(u-\lambda)^2} d\mathcal{F}_A(\lambda)\right)^2 - \left(\int \frac{\lambda^2}{(u-\lambda)^3} d\mathcal{F}_A(\lambda)\right)\left(E_{\mathcal{F}_A} + \int \frac{\lambda^2}{u-\lambda} d\mathcal{F}_A(\lambda)\right)\right)$$

and from the Cauchy inequality

$$\int \frac{\lambda}{(u-\lambda)^{3/2}} \frac{\lambda}{(u-\lambda)^{1/2}} d\mathcal{F}_A(\lambda) \le \left(\int \frac{\lambda^2}{(u-\lambda)^3} d\mathcal{F}_A(\lambda)\right)^{1/2} \left(\int \frac{\lambda^2}{u-\lambda} d\mathcal{F}_A(\lambda)\right)^{1/2}.$$

The tendency to zero follows from the fact that $c\int \frac{\lambda u}{u-\lambda} d\mathcal{F}_A(\lambda) \to \infty$ as $u \downarrow u(\mathcal{F}_A)$, which is easy to show using the monotone convergence theorem and the assumption that $\liminf_{\delta\to 0} \frac{1}{\delta}\int_{|\lambda-u(\mathcal{F}_A)|\leq\delta} d\mathcal{F}_A(\lambda) = k^A > 0$. Finally, the convergence to $x\left(cE_{\mathcal{F}_A}\right)^{-1}$ as $u \to \infty$ is obvious. Similarly, $x\left(\int \frac{\lambda v}{v-\lambda} d\mathcal{F}_B(\lambda)\right)^{-1}$ as a function of $v > u(\mathcal{F}_B)$ is concave, tends to zero as $v \downarrow u(\mathcal{F}_B)$ and to $x\left(cE_{\mathcal{F}_B}\right)^{-1}$ as $v \to \infty$.

The above properties of $x\left(c\int \frac{\lambda u}{u-\lambda} d\mathcal{F}_A(\lambda)\right)^{-1}$ and $x\left(\int \frac{\lambda v}{v-\lambda} d\mathcal{F}_B(\lambda)\right)^{-1}$ imply that the curves in the $\{u > u(\mathcal{F}_A), v > v(\mathcal{F}_B)\}$ subset of the (u,v)-plane defined by (13) are either intersecting at two points, touching at a single point, or having no common points. Since there exists a solution to (13) for any $x > u\left(\mathcal{F}^{c,A,B}\right)$ and since $x\left(c\int \frac{\lambda u}{u-\lambda} d\mathcal{F}_A(\lambda)\right)^{-1}$ and $x\left(\int \frac{\lambda v}{v-\lambda} d\mathcal{F}_B(\lambda)\right)^{-1}$ are monotone increasing in x, the curves must intersect at two points for any $x > u\left(\mathcal{F}^{c,A,B}\right)$. Let us show that the curves are touching at a single point when $x = u\left(\mathcal{F}^{c,A,B}\right)$.

Suppose the curves intersect at two points (u_1, v_1) and (u_2, v_2) when $x = u\left(\mathcal{F}^{c,A,B}\right)$. Let $u_2 > u_1$ and $v_2 > v_1$. Define $f_1(x,u,v) = x + cuv\left(um_A(u) + 1\right)$ and $f_2(x,u,v) = x + uv\left(vm_B(v) + 1\right)$. Note that system (13) is equivalent to $f_i(x,u,v) = 0$ for $i = 1, 2$. It is straightforward to check that the assumption of the proper intersection of the curves (not just a tangency at one point) is equivalent to $\det\begin{pmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{f_2}{\partial v} \end{pmatrix} \neq 0$ at any of the two intersection points. Then the implicit function theorem (see Krantz (1992), Theorem 1.4.11) implies that there exist holomorphic functions $u(z), v(z)$ defined in an open neighborhood of $z = u\left(\mathcal{F}^{c,A,B}\right)$ in $\mathbb{C}$, which satisfy $f_i(z,u,v) = 0$ for $i = 1, 2$. To each of the two intersection points, there will correspond its own set of holomorphic functions $u(z), v(z)$. We will consider the functions $u(z)$ and $v(z)$ corresponding to (u_2, v_2). For such a choice, it is straightforward to check that $\frac{d}{d(\operatorname{Re} z)} \operatorname{Re} u(z) > 0$ and $\frac{d}{d(\operatorname{Re} z)} \operatorname{Re} v(z) > 0$ at $z = u\left(\mathcal{F}^{c,A,B}\right)$.

Furthermore, using identities $f_i(z, u(z), v(z)) = 0$ for $i = 1, 2$ it is straightforward to check that in a small enough neighborhood of $z = u\left(\mathcal{F}^{c,A,B}\right)$ in $\mathbb{C}$, $\operatorname{Im} z > 0$ implies that $\operatorname{Im} u(z)$ and $\operatorname{Im} v(z)$ are of the same sign and are not equal to zero. Cauchy-Riemann equations for holomorphic functions imply that $\frac{d}{d(\operatorname{Im} z)} \operatorname{Im} u(z) = \frac{d}{d(\operatorname{Re} z)} \operatorname{Re} u(z) > 0$ and $\frac{d}{d(\operatorname{Im} z)} \operatorname{Im} v(z) = \frac{d}{d(\operatorname{Re} z)} \operatorname{Re} v(z) > 0$ at $z = u\left(\mathcal{F}^{c,A,B}\right)$. Hence, $\operatorname{Im} u(z)$ and $\operatorname{Im} v(z)$ are positive when $\operatorname{Im} z$ is positive and z lies in a small enough complex neighborhood of $u\left(\mathcal{F}^{c,A,B}\right)$. Let us define $m(z) = -\frac{c^{-1}}{u(z)v(z)} - \frac{1}{z}$. Clearly, for z in a small complex neighborhood of $u\left(\mathcal{F}^{c,A,B}\right)$, we have: $\operatorname{Im} m(z) > 0$ if $\operatorname{Im} z > 0$.

By Lemma 1, for any $z \in \mathbb{C}^+$, there is only one solution to (7) such that m, u and v belong to $\mathbb{C}^+$. Hence, $u(z), v(z)$, and $m(z)$ defined above constitute the solution to system (7) for z in a small neighborhood of $u\left(\mathcal{F}^{c,A,B}\right)$ and such that $\operatorname{Im} z > 0$. Finally, for any real x which belongs to the neighborhood of $u(\mathcal{F}^{c,A,B})$, we have: $\lim_{z \to x} \operatorname{Im} m(z) = \lim_{z \to x} \operatorname{Im} \left(-\frac{c^{-1}}{u(z)v(z)} - \frac{1}{z}\right) = 0$. Thus, using the Frobenius-Perron inversion formula, we get $\int_{u(\mathcal{F}^{c,A,B})-\delta}^{u(\mathcal{F}^{c,A,B})} d\mathcal{F}^{c,A,B}(\lambda) = 0$ for small positive δ, which is impossible by definition of $u(\mathcal{F}^{c,A,B})$. Hence, the curves are touching at a single point when $x = u\left(\mathcal{F}^{c,A,B}\right)$. This implies that they do not intersect when $x < u(\mathcal{F}^{c,A,B})$.□

Now we are ready to prove Lemma 3.

Proof of Lemma 3: Recall that by assumption, $\mathcal{F}^{AA'}$ almost surely weakly converges to $\mathcal{F}_A$ and $u\left(\mathcal{F}^{AA'}\right) \to u\left(\mathcal{F}_A\right)$. Similarly, $\mathcal{F}^{BB'}$ almost surely weakly converges to $\mathcal{F}_B$ and $u\left(\mathcal{F}^{BB'}\right) \to u\left(\mathcal{F}_B\right)$. These facts imply that if the curves in the $\{u > u\left(\mathcal{F}_A\right), v > u\left(\mathcal{F}_B\right)\}$ subset of the (u, v)-plane defined by (13) intersect at zero, or at two points, then the curves in the $\left\{u > u\left(\mathcal{F}^{AA'}\right), v > u\left(\mathcal{F}^{BB'}\right)\right\}$ subset of the (u, v)-plane defined by

$$\begin{cases} v = x\left(c_n \int \frac{\lambda u}{u-\lambda} d\mathcal{F}^{AA'}(\lambda)\right)^{-1} \\ u = x\left(\int \frac{\lambda v}{v-\lambda} d\mathcal{F}^{BB'}(\lambda)\right)^{-1} \end{cases}$$

also intersect at zero, or at two points for large enough n. Therefore, by Lemma A4, $u\left(\mathcal{F}^{c_n,A_n,B_n}\right)$ converges to $u\left(\mathcal{F}^{c,A,B}\right)$ and Lemma 3 follows from Lemma 2.□

References

[1] Krantz, S. G. (1992) Function theory of several complex variables. Second Edition. AMS Chelsea Publishing. Providence, Rhode Island.

[2] Zhang, L. (2006). Spectral Analysis of Large Dimensional Random Matrices. Ph. D. Thesis. National University of Singapore.