

[Testing hypotheses about the number of factors in
large factor models], Supplementary material:

[Supplementary Appendix]

Alexei Onatski

Economics Department, Columbia University

April 4, 2009

Detailed proof of Lemma 5

Lemma 5 *Let $\hat{e} \equiv [\hat{e}_1(n), \dots, \hat{e}_m(n)]$. Then, under the assumptions of Theorem 1, there exists an $n \times m$ matrix \tilde{e} with independent $N_n^{\mathbb{C}}(0, 2\pi S_n^e(\omega_0))$ columns, independent from \hat{F} , and such that $\sigma_1^2(\hat{e} - \tilde{e}) = o_p(n^{-1/3})$.*

Proof: First, suppose that Assumption 2ii holds and $n \sim m = o(T^{3/8})$. Define $\eta \equiv ((\text{Re } \hat{e}_1)', (\text{Im } \hat{e}_1)', \dots, (\text{Re } \hat{e}_m)', (\text{Im } \hat{e}_m)')$. First, let us show that $E\eta\eta' = V + R$ with a block-diagonal $V = \pi I_m \otimes \begin{pmatrix} \text{Re } S_n^e(\omega_0) & -\text{Im } S_n^e(\omega_0) \\ \text{Im } S_n^e(\omega_0) & \text{Re } S_n^e(\omega_0) \end{pmatrix}$ and $R_{ij} = \delta_{[i/2n], [j/2n]} O(m/T) + O(T^{-1})$, where δ_{st} is the Kronecker's delta, and $O(m/T)$ and $O(T^{-1})$ are uniform in i and j running from 1 to $2nm$.

By the definition of the discrete Fourier transform (dft), we have:

$$E\hat{e}_{js}\hat{e}_{rl} \equiv \frac{1}{T}E \left[\left(\sum_{t=1}^T e_{jt}e^{-i\omega_s t} \right) \left(\sum_{t=1}^T e_{rt}e^{-i\omega_l t} \right) \right].$$

Hence, we can write: $E\hat{e}_{js}\hat{e}_{rl} = \frac{1}{T} \sum_{u=1-T}^{T-1} e^{-i\omega_s u} c_{jr}(u) \sum_{t=1}^T h(t+u) e^{-i(\omega_s+\omega_l)t}$, where $h(\tau) = 1$ for $1 \leq \tau \leq T$ and $h(\tau) = 0$ otherwise. Denote $\sum_{t=1}^T h(t+u) e^{-i(\omega_s+\omega_l)t} - \sum_{t=1}^T e^{-i(\omega_s+\omega_l)t}$ as U_1 and denote $\sum_{u=1-T}^{T-1} e^{-i\omega_s u} c_{jr}(u) - 2\pi [S_n^e(\omega_s)]_{jr}$ as U_2 . Then,

$$E\hat{e}_{js}\hat{e}_{rl} - \frac{2\pi}{T} \sum_{t=1}^T e^{-i(\omega_s+\omega_l)t} [S_n^e(\omega_s)]_{jr} = \frac{1}{T} \sum_{u=1-T}^{T-1} e^{-i\omega_s u} c_{jr}(u) U_1 + U_2 \sum_{t=1}^T e^{-i(\omega_s+\omega_l)t}.$$

But $|U_1| \leq |u|$ and $|U_2| = \left| \sum_{|u| \geq T} e^{-i\omega_s u} c_{jr}(u) \right| \leq \sum_{|u| \geq T} \frac{|u|}{T} |c_{jr}(u)|$. Hence,

$$\left| E\hat{e}_{js}\hat{e}_{rl} - \frac{2\pi}{T} \sum_{t=1}^T e^{-i(\omega_s+\omega_l)t} [S_n^e(\omega_s)]_{jr} \right| \leq \frac{1}{T} \sum_{u=1-T}^{T-1} |u| |c_{jr}(u)| + \sum_{|u| \geq T} \frac{|u|}{T} |c_{jr}(u)| \left| \sum_{t=1}^T e^{-i(\omega_s+\omega_l)t} \right|.$$

Note that by the definition of ω_s and ω_l , $\omega_s + \omega_l = \frac{2\pi(s_s+s_l)}{T} \neq 0$ for all s and l .

Therefore, $\frac{1}{T} \sum_{t=1}^T e^{-i(\omega_s+\omega_l)t} = \frac{e^{-i(\omega_s+\omega_l)} - 1}{T} \frac{e^{-i(\omega_s+\omega_l)T} - 1}{e^{-i(\omega_s+\omega_l)} - 1} = 0$ for all s and l , and we have:

$|E\hat{e}_{js}\hat{e}_{rl}| \leq \frac{1}{T} \sum_{u=1-T}^{T-1} |u| |c_{jr}(u)| = O(T^{-1})$ uniformly in s and l , but also in j and r

by Assumption 2ii. Similarly, $|E\hat{e}'_{js}\hat{e}'_{rl}| = O(T^{-1})$ uniformly in s, l, j and r .

Consider now $E\hat{e}_{js}\hat{e}'_{rl}$. Similar to above,

$$\left| E\hat{e}_{js}\hat{e}'_{rl} - \frac{2\pi}{T} \sum_{t=1}^T e^{-i(\omega_s-\omega_l)t} [S_n^e(\omega_s)]_{jr} \right| \leq \frac{1}{T} \sum_{u=1-T}^{T-1} |u| |c_{jr}(u)| + \sum_{|u| \geq T} \frac{|u|}{T} |c_{jr}(u)| \left| \sum_{t=1}^T e^{-i(\omega_s-\omega_l)t} \right|,$$

and if $s \neq l$, we have $|E\hat{e}_{js}\hat{e}'_{rl}| = O(T^{-1})$ uniformly in s, l, j and r . However, if $s = l$,

then $\omega_s - \omega_l = 0$ and we have:

$$\begin{aligned} E\hat{e}_{js}\hat{e}'_{rl} - 2\pi [S_n^e(\omega_s)]_{jr} &= \frac{1}{T} \sum_{u=1-T}^{T-1} e^{-i\omega_s u} c_{jr}(u) (T-u) - 2\pi [S_n^e(\omega_s)]_{jr} \\ &= -\frac{1}{T} \sum_{u=1-T}^{T-1} u e^{-i\omega_s u} c_{jr}(u) - \sum_{|u|\geq T} e^{-i\omega_s u} c_{jr}(u) \end{aligned}$$

so that $\left| E\hat{e}_{js}\hat{e}'_{rl} - 2\pi [S_n^e(\omega_s)]_{jr} \right| \leq \frac{1}{T} \sum |u| |c_{jr}(u)| = O(T^{-1})$ uniformly in s, l, j and r . To summarize:

$$E\hat{e}_{js}\hat{e}_{rl} = O(T^{-1}), E\hat{e}'_{js}\hat{e}'_{rl} = O(T^{-1}) \quad (1)$$

$$E\hat{e}_{js}\hat{e}'_{rl} = \delta_{sl} 2\pi [S_n^e(\omega_s)]_{jr} + O(T^{-1}), \quad (2)$$

where $O(T^{-1})$ is uniform in s, l, j and r .

This result is very similar to Theorem 4.3.2 of Brillinger (1981), which is more general in that it gets estimates for higher order cumulants of dft's in addition to the second order cumulants, but which is less general in that it only considers situation when j and r are bounded so that uniformity of $O(T^{-1})$ in j and r is trivial.

Note that

$$\begin{aligned} E\hat{e}_{js}\hat{e}_{rl} &= E(\operatorname{Re} \hat{e}_{js} + i \operatorname{Im} \hat{e}_{js})(\operatorname{Re} \hat{e}_{rl} + i \operatorname{Im} \hat{e}_{rl}) \\ &= E(\operatorname{Re} \hat{e}_{js} \operatorname{Re} \hat{e}_{rl} - \operatorname{Im} \hat{e}_{js} \operatorname{Im} \hat{e}_{rl}) + iE(\operatorname{Re} \hat{e}_{js} \operatorname{Im} \hat{e}_{rl} + \operatorname{Im} \hat{e}_{js} \operatorname{Re} \hat{e}_{rl}) \end{aligned}$$

and

$$\begin{aligned} E\hat{e}_{js}\hat{e}'_{rl} &= E(\operatorname{Re} \hat{e}_{js} + i \operatorname{Im} \hat{e}_{js})(\operatorname{Re} \hat{e}_{rl} - i \operatorname{Im} \hat{e}_{rl}) \\ &= E(\operatorname{Re} \hat{e}_{js} \operatorname{Re} \hat{e}_{rl} + \operatorname{Im} \hat{e}_{js} \operatorname{Im} \hat{e}_{rl}) + iE(-\operatorname{Re} \hat{e}_{js} \operatorname{Im} \hat{e}_{rl} + \operatorname{Im} \hat{e}_{js} \operatorname{Re} \hat{e}_{rl}). \end{aligned}$$

Therefore,

$$\begin{aligned}
E(\operatorname{Re} \hat{e}_{js} \operatorname{Re} \hat{e}_{rl}) &= \frac{1}{4}(E\hat{e}_{js}\hat{e}_{rl} + E\hat{e}'_{js}\hat{e}'_{rl} + E\hat{e}_{js}\hat{e}'_{rl} + E\hat{e}'_{js}\hat{e}_{rl}), \\
E(\operatorname{Im} \hat{e}_{js} \operatorname{Im} \hat{e}_{rl}) &= \frac{1}{4}(E\hat{e}_{js}\hat{e}'_{rl} + E\hat{e}'_{js}\hat{e}_{rl} - E\hat{e}_{js}\hat{e}_{rl} - E\hat{e}'_{js}\hat{e}'_{rl}), \\
E(\operatorname{Re} \hat{e}_{js} \operatorname{Im} \hat{e}_{rl}) &= \frac{1}{4i}(E\hat{e}_{js}\hat{e}_{rl} - E\hat{e}'_{js}\hat{e}'_{rl} - E\hat{e}_{js}\hat{e}'_{rl} + E\hat{e}'_{js}\hat{e}_{rl}), \\
E(\operatorname{Im} \hat{e}_{js} \operatorname{Re} \hat{e}_{rl}) &= \frac{1}{2i}(E\hat{e}_{js}\hat{e}_{rl} - E\hat{e}'_{js}\hat{e}'_{rl} + E\hat{e}_{js}\hat{e}'_{rl} - E\hat{e}'_{js}\hat{e}_{rl}).
\end{aligned}$$

Using formulas (1) and (2), we finally get:

$$\begin{aligned}
E(\operatorname{Re} \hat{e}_{js} \operatorname{Re} \hat{e}_{rl}) &= \delta_{sl} \frac{\pi}{2} \left([S_n^e(\omega_s)]_{jr} + [S_n^e(\omega_s)]_{rj} \right) + O(T^{-1}), \\
E(\operatorname{Im} \hat{e}_{js} \operatorname{Im} \hat{e}_{rl}) &= \delta_{sl} \frac{\pi}{2} \left([S_n^e(\omega_s)]_{jr} + [S_n^e(\omega_s)]_{rj} \right) + O(T^{-1}), \\
E(\operatorname{Re} \hat{e}_{js} \operatorname{Im} \hat{e}_{rl}) &= \delta_{sl} \frac{\pi}{2i} \left(-[S_n^e(\omega_s)]_{jr} + [S_n^e(\omega_s)]_{rj} \right) + O(T^{-1}), \\
E(\operatorname{Im} \hat{e}_{js} \operatorname{Re} \hat{e}_{rl}) &= \delta_{sl} \frac{\pi}{2i} \left([S_n^e(\omega_s)]_{jr} - [S_n^e(\omega_s)]_{rj} \right) + O(T^{-1}).
\end{aligned}$$

Since $S_n^e(\omega_s)$ is a Hermitian matrix, we have:

$$\begin{aligned}
E(\operatorname{Re} \hat{e}_{js} \operatorname{Re} \hat{e}_{rl}) &= \delta_{sl} \pi \operatorname{Re} [S_n^e(\omega_s)]_{jr} + O(T^{-1}), \\
E(\operatorname{Im} \hat{e}_{js} \operatorname{Im} \hat{e}_{rl}) &= \delta_{sl} \pi \operatorname{Re} [S_n^e(\omega_s)]_{jr} + O(T^{-1}), \\
E(\operatorname{Re} \hat{e}_{js} \operatorname{Im} \hat{e}_{rl}) &= -\delta_{sl} \pi \operatorname{Im} [S_n^e(\omega_s)]_{jr} + O(T^{-1}), \\
E(\operatorname{Im} \hat{e}_{js} \operatorname{Re} \hat{e}_{rl}) &= \delta_{sl} \pi \operatorname{Im} [S_n^e(\omega_s)]_{jr} + O(T^{-1}).
\end{aligned}$$

Further, by the definition of the spectrum and by Assumption 2ii, $[S_n^e(\omega_s)]_{jr} - [S_n^e(\omega_0)]_{jr} = O(m/T)$ uniformly in j, r and s . Hence, the above covariance formulas for the real and imaginary parts of \hat{e}_{js} and \hat{e}_{rl} imply that the i, j -th entries of R equal $\delta_{[i/2n], [j/2n]} O(m/T) + O(T^{-1})$, where $O(m/T)$ and $O(T^{-1})$ are uniform in i and j

running from 1 to $2nm$.

Construct $\tilde{\eta} = V^{1/2}(V + R)^{-1/2}\eta$ and define an $n \times m$ matrix \tilde{e} with the s -th columns \tilde{e}_s so that $((\operatorname{Re} \tilde{e}_1)', (\operatorname{Im} \tilde{e}_1)', \dots, (\operatorname{Re} \tilde{e}_m)', (\operatorname{Im} \tilde{e}_m)')' = \tilde{\eta}$. Note that \tilde{e} has independent $N_n^{\mathbb{C}}(0, 2\pi S_n^e(\omega_0))$ columns by construction.

Using inequalities $\|BA\|_2 \leq \|B\| \|A\|_2$ and $\|AB\|_2 \leq \|A\|_2 \|B\|$ (see, for example, Horn and Johnson's (1985) problem 20, p.313), we obtain: $E \|\eta - \tilde{\eta}\|^2 = \left\| (V + R)^{1/2} - V^{1/2} \right\|_2^2 \leq \|V^{1/4}\|^4 \left\| (I + V^{-1/2}RV^{-1/2})^{1/2} - I \right\|_2^2$. Denote the i -th largest eigenvalue of $V^{-1/2}RV^{-1/2}$ as μ_i and note that $|\mu_i| \leq 1$ for large enough T . Since $\left| (1 + \mu_i)^{1/2} - 1 \right| \leq |\mu_i|$ for any $|\mu_i| \leq 1$, the i -th eigenvalue of $(I + V^{-1/2}RV^{-1/2})^{1/2} - I$ is no larger by absolute value than the i -th eigenvalue of $V^{-1/2}RV^{-1/2}$ for large enough T . Therefore, $E \|\eta - \tilde{\eta}\|^2 \leq \|V^{1/4}\|^4 \|V^{-1/2}RV^{-1/2}\|_2^2 \leq \|V^{1/4}\|^4 \|V^{-1/2}\|^4 \|R\|_2^2$. But $\|V^{1/4}\| = (\pi l_{1n})^{1/4}$ and $\|V^{-1/2}\| = (\pi l_{nn})^{-1/2}$ by construction, and $\|R\|_2^2 = \sum_{i,j=1}^{2nm} (\delta_{[i/2n],[j/2n]} O(m/T) + O(T^{-1}))^2 = m(2n)^2 O(m^2/T^2) + (2mn)^2 O(T^{-2}) = o(n^{-1/3})$ because $n \sim m = o(T^{3/8})$. Hence, $E \|\eta - \tilde{\eta}\|^2 \leq (\pi l_{1n}) (\pi l_{nn})^{-2} o(n^{-1/3}) = o(n^{-1/3})$, where the last equality holds because l_{1n} and l_{nn}^{-1} remain bounded as $n, m \rightarrow \infty$ by Assumption 3. Finally, Lemma 2 and Markov's inequality imply that $\sigma_1^2(\hat{e} - \tilde{e}) = o_p(n^{-1/3})$.

Now, suppose that Assumption 2iia holds and $m = o(T^{1/2-1/p} \log^{-1} T)^{6/13}$. In this case, $\hat{e}_{is} = \sum_{j=1}^{\infty} A_{ij} \hat{u}_{js}$, where \hat{u}_{js} is the dft of u_{jt} at frequency ω_s . For fixed j and $\omega_0 = 0$, Phillips (2007) shows that there exist iid complex normal variables ξ_{js} , $s = 1, \dots, m$, such that $\hat{u}_{js} - \xi_{js} = o_p\left(\frac{m}{T^{1/2-1/p}}\right)$ uniformly over $s \leq m$. Lemmas 1S, 2S and 3S below extend Phillips' proof to the case $\omega_0 \neq 0$ and show that there exist Gaussian processes u_{jt}^G with the same auto-covariance structure as u_{jt} and independent over $j \in \mathbb{N}$ such that the differences between the dft's $\hat{u}_{js} - \hat{u}_{js}^G \equiv r_{js}$ satisfy: $\sup_{j>0} E(\max_{s \leq m} |r_{js}|)^2 \leq Km^2 T^{2/p-1} \log^2 T$ for large enough T , where $K > 0$ depends only on $p, \mu_p, \sup_{j \geq 1} (\sum_{k=0}^{\infty} k |c_{jk}|)^p$ and $\sup_{j \geq 1} |C_j(e^{-i\omega_0})|$.

Note that the process $e_{it}^G = \sum_{j=1}^{\infty} A_{ij} u_{jt}^G$ satisfies Assumption 2ii. Indeed, let $c_{ij}^G(u) \equiv E e_{i,t+u}^G e_{jt}^G$. Then, since u_{jt}^G are independent over $j \in \mathbb{N}$ and have the same autocovariance structure as u_{jt} , we have: $c_{ij}^G(u) = \sum_{r=1}^{\infty} A_{ir} A_{jr} E u_{r,t+u} u_{rt}$. Therefore, $\sum_u (1 + |u|) |c_{ij}^G(u)| \leq \sum_{r=1}^{\infty} |A_{ir} A_{jr}| \sum_u (1 + |u|) |E u_{r,t+u} u_{rt}|$. On the other hand,

$$\begin{aligned} \sum_u (1 + |u|) |E u_{r,t+u} u_{rt}| &\leq \sum_u \sum_{k=|u|}^{\infty} (1 + |u|) |c_{rk}| |c_{rk-|u|}| = \sum_{k=0}^{\infty} \sum_{u:|u|\leq k} (1 + |u|) |c_{rk}| |c_{rk-|u|}| \\ &\leq \sum_{k=0}^{\infty} \sum_{u:|u|\leq k} (1 + k) |c_{rk}| |c_{rk-|u|}| \leq \left(\sum_{k=0}^{\infty} (1 + k) |c_{rk}| \right)^2. \end{aligned}$$

Hence, $\sup_{i,j} \sum_u (1 + |u|) |c_{ij}^G(u)| \leq \sup_i \sum_{r=1}^{\infty} A_{ir}^2 \sup_{r>0} (\sum_{k=0}^{\infty} (1 + k) |c_{rk}|)^2 < \infty$ by Assumption 2 iia.

Thus, the problem reduces to the Gaussian case analyzed above if we show that

$$\begin{aligned} \sigma_1^2(\hat{e} - \hat{e}^G) &= o_p(n^{-1/3}). \text{ But we have: } \sum_{i=1}^n \sum_{s=1}^m E |(\hat{e} - \hat{e}^G)_{is}|^2 = \\ &\sum_{i=1}^n \sum_{s=1}^m \sum_{j=1}^{\infty} A_{ij}^2 E |r_{js}|^2 \leq m \sum_{i=1}^n \sum_{j=1}^{\infty} A_{ij}^2 E (\max_{s \leq m} |r_{js}|)^2 \leq \\ &mn \left(\sup_{i>0} \sum_{j=1}^{\infty} A_{ij}^2 \right) Km^2 T^{2/p-1} \log^2 T = o(n^{-1/3}) \text{ if } n \sim m = o(T^{1/2-1/p} \log^{-1} T)^{6/13} \end{aligned}$$

as have been assumed. Therefore, Lemma 2 and Markov's inequality imply that $\sigma_1^2(\hat{e} - \hat{e}^G) = o_p(n^{-1/3})$. QED

Lemma S1 (Zaitsev, 2006). *Suppose that x_1, \dots, x_T are independent zero-mean random vectors in \mathbb{R}^d such that $L_p \equiv \sum_{t=1}^T E \|x_t\|^p < \infty$ with $p \geq 2$ and there exists a sequence $0 = m_0 < m_1 < \dots < m_\tau = T$ such that for $D_k \equiv \text{Var} [x_{m_{k-1}+1} + \dots + x_{m_k}]$, $k = 1, \dots, \tau$, we have: $I_d \leq \gamma^{-2} D_k \leq C \cdot I_d$ with $C \geq 1$ and $\gamma = 2eL_p^{1/p}$. Then there exists a probability space supporting both a sequence distributionally equivalent to x_1, \dots, x_T and a sequence of independent $N(0, \text{Var}(x_t))$ vectors y_t , $t = 1, \dots, T$, such that $\Pr \left(\max_{1 \leq t \leq T} \left\| \sum_{s=1}^t (x_s - y_s) \right\| > 5z \right) \leq 2L_p z^{-p} + \exp \left(-\frac{a_1 z}{\gamma d^{9/2} \log^* d} \right)$ for any $z > a_2 (d^8 \log^* d) \gamma \log^* \tau$, where $a_1, a_2 > 0$ depend only on C and where $\log^* a \equiv \max(1, \log a)$.*

This lemma is a slightly weakened version of Corollary 3 in Zaitsev (2006).

Lemma S2 *Under Assumption 2iia, there exists a probability space supporting a process distributionally equivalent to ε_{it} and a process $\varepsilon_{it}^G \sim iidN(0, 1)$ such that $E \left(\max_{1 \leq t \leq T} \frac{|R_{jt}|}{\sqrt{T}} \right)^2 \leq bT^{2/p-1} \log^2 T$ for large enough T , where $R_{jt} \equiv \sum_{l=1}^t e^{-i\omega_0 l} (\varepsilon_{jl} - \varepsilon_{jl}^G)$ and $b > 0$ depends only on p and μ_p .*

Proof: In Lemma S1, take $x_t = v_t \varepsilon_{jt}$, where $v_t \equiv (\cos \omega_0 t, -\sin \omega_0 t)'$, and assume that $\omega_0 \neq 0 \pmod{\pi}$. Then, for any l , the two singular values of $Var(x_{2l-1} + x_{2l})$ are: $\sigma_{1,2} \equiv 1 \pm |\cos \omega_0|$. Therefore, for $k = 1, \dots, \tau$, $\frac{\sigma_2(D_k)}{\gamma^2} \geq \frac{\sigma_2}{\gamma^2} \left[\frac{m_k - m_{k-1}}{2} \right]$ and $\frac{\sigma_1(D_k)}{\gamma^2} \leq \frac{\sigma_1}{\gamma^2} \left[\frac{m_k - m_{k-1} + 1}{2} \right]$ for any positive γ , and hence, for $\gamma = 2eL_p^{1/p}$ with $L_p = T\mu_p$. In particular, if we choose $m_k = k([2\gamma^2/\sigma_2] + 1)$ for $k \leq \tau - 1$ with $\tau = [T/m_1]$, we have: $\min_{k \leq \tau} \left[\frac{m_k - m_{k-1}}{2} \right] = \left[\frac{m_1}{2} \right] \geq \frac{\gamma^2}{\sigma_2}$ and $\max_{k \leq \tau} \left[\frac{m_k - m_{k-1} + 1}{2} \right] \leq m_1 \leq \frac{3\gamma^2}{\sigma_2}$, where the latter inequality holds because $\mu_p^{2/p} \equiv (E|\varepsilon_{jt}|^p)^{2/p} \geq E\varepsilon_{jt}^2 \equiv 1$, and thus, $\frac{\gamma^2}{\sigma_2} \geq 1$. Summarizing the above inequalities, we get: $I_2 \leq \gamma^{-2} D_k \leq 3 \frac{1+|\cos \omega_0|}{1-|\cos \omega_0|} I_2$ for $k = 1, \dots, \tau$. Hence, for each j , Zaitsev's inequality for the tail probability of $\max_{1 \leq t \leq T} \left\| \sum_{s=1}^t \{v_s \varepsilon_{js} - y_{js}\} \right\|$ is satisfied for independent $N(0, v_s v_s')$ vectors y_{js} , $s = 1, \dots, T$. By expanding the probability space, we can choose y_{js} 's independent across different j 's and embed the finite sequences y_{js} , $s = 1, \dots, T$ into the infinite ones: y_{js} , $s \in \mathbb{Z}$.

Now, define independent $N(0, 1)$ variables $\varepsilon_{js}^G \equiv y_{1,js} \cos \omega_0 s - y_{2,js} \sin \omega_0 s$, where $y_{1,js}$ and $y_{2,js}$ are the two components of vector $y_{js} \equiv (y_{1,js}, y_{2,js})'$. Note that

$$\operatorname{Re} \left(e^{-i\omega_0 s} \varepsilon_{js}^G \right) = y_{1,js} \cos^2 \omega_0 s - y_{2,js} \cos \omega_0 s \sin \omega_0 s, \quad (3)$$

$$\operatorname{Im} \left(e^{-i\omega_0 s} \varepsilon_{js}^G \right) = -y_{1,js} \sin \omega_0 s \cos \omega_0 s + y_{2,js} \sin^2 \omega_0 s. \quad (4)$$

Further, note that

$$y_{1,js} \sin(\omega_0 s) + y_{2,js} \cos(\omega_0 s) \equiv 0 \quad (5)$$

because $E((\sin(\omega_0 s), \cos(\omega_0 s)) y_{js})^2 = (\sin(\omega_0 s), \cos(\omega_0 s)) v_s v_s' (\sin(\omega_0 s), \cos(\omega_0 s))' \equiv 0$. Multiplying the left hand side of (5) by $\sin(\omega_0 s)$ and by $\cos(\omega_0 s)$ and adding the results to the right hand sides of (3) and (4), respectively, we find that the components of y_{js} equal the real and the imaginary parts of $e^{-i\omega_0 s} \varepsilon_{js}^G$. Therefore, we have: $\Pr\left(\max_{1 \leq t \leq T} |R_{jt}| > 5z\right) \leq 2L_p z^{-p} + \exp\left(-\frac{a_1 z}{\gamma 2^{9/2} \log 2}\right)$ for any $z > a_2 (2^8 \log 2) \gamma \log^* \tau$, where $R_{jt} \equiv \sum_{l=1}^t e^{-i\omega_0 l} (\varepsilon_{jl} - \varepsilon_{jl}^G)$ and $a_1, a_2 > 0$ depend only on $\omega_0 \neq 0 \pmod{\pi}$. For $\omega_0 = 0 \pmod{\pi}$, defining $x_t = e^{-i\omega_0 t} \varepsilon_{jt}$ and repeating a simplified scalar version of the above argument, we obtain the same tail probability estimate (with different a_1 and a_2 and $\tau = \lfloor T/m_1 \rfloor$ with $m_1 = \lfloor \gamma^2 \rfloor + 1$).

Now, we have: $E\left(\max_{1 \leq t \leq T} \frac{|R_{jt}|}{\sqrt{T}}\right)^2 = 2 \int_0^\infty x \Pr\left(\max_{1 \leq t \leq T} \frac{|R_{jt}|}{\sqrt{T}} > x\right) dx \leq \bar{x}^2 + 2 \int_{\bar{x}}^\infty x \left(2L_p \left(\frac{x\sqrt{T}}{5}\right)^{-p} + \exp\left(-\frac{a_1 x\sqrt{T}}{5\gamma 2^{9/2} \log 2}\right)\right) dx$ for $\bar{x} > 5T^{-1/2} a_2 (2^8 \log 2) \gamma \log^* \tau$. Recall that $\tau = \lfloor T/m_1 \rfloor = \lfloor T/(\lfloor 2\gamma^2/\sigma_2 \rfloor + 1) \rfloor$, $\gamma = 2eL_p^{1/p}$ with $L_p = T\mu_p$ and $\sigma_2 = 1 - |\cos \omega_0|$ (or, alternatively, $\tau = \lfloor T/m_1 \rfloor = \lfloor T/(\lfloor \gamma^2 \rfloor + 1) \rfloor$ for $\omega_0 = 0 \pmod{\pi}$). As $T \rightarrow \infty$, $\gamma \sim T^{1/p}$ and $\tau \sim T^{1-2/p}$ so that there exists a constant $b_2 > 0$ such that $\bar{x} \geq b_2 T^{1/p-1/2} \log T$ implies that $\bar{x} > 5T^{-1/2} a_2 (2^8 \log 2) \gamma \log^* \tau$ for large enough T . Furthermore, since the inequality $x \geq b_2 T^{1/p-1/2} \log T$ implies that $xT^{1/2-1/p} \rightarrow \infty$ as $T \rightarrow \infty$ and since $2L_p \left(\frac{x\sqrt{T}}{5}\right)^{-p} \sim (xT^{1/2-1/p})^{-p}$ and $\frac{a_1 x\sqrt{T}}{5\gamma 2^{9/2} \log 2} \sim xT^{1/2-1/p}$, we have: $2L_p \left(\frac{x\sqrt{T}}{5}\right)^{-p} > \exp\left(-\frac{a_1 x\sqrt{T}}{5\gamma 2^{9/2} \log 2}\right)$ for large enough T . To summarize, for large enough T and for $\bar{x} \geq b_2 T^{1/p-1/2} \log T$ with some positive constant b_2 , $E\left(\max_{1 \leq t \leq T} \frac{|R_{jt}|}{\sqrt{T}}\right)^2 \leq \bar{x}^2 + 2 \int_{\bar{x}}^\infty 4xL_p \left(\frac{x\sqrt{T}}{5}\right)^{-p} dx = \bar{x}^2 + b_1 T^{1-p/2} \bar{x}^{2-p}$, where $b_1 > 0$ depends only on μ_p and p . Setting $\bar{x} = b_2 T^{1/p-1/2} \log T$, we get: $E\left(\max_{1 \leq t \leq T} \frac{|R_{jt}|}{\sqrt{T}}\right)^2 \leq bT^{2/p-1} \log^2 T$ for large enough T , where $b > 0$ depends only on μ_p and p . QED

Lemma 3S. *Let Assumption 2iia holds and let ε_{it}^G , $j \in \mathbb{N}, t \in \mathbb{Z}$, be the iid $N(0, 1)$ variables described in Lemma 2S. Define $u_{jt}^G \equiv C_j(L) \varepsilon_{jt}^G$ and consider the differences $r_{js} \equiv \hat{u}_{js} - \hat{u}_{js}^G$ between the dft's of u_{jt} and u_{jt}^G at frequencies ω_s with $s = 1, \dots, m$. Then, $\sup_{j>0} E(\max_{s \leq m} |r_{js}|)^2 \leq Km^2 T^{2/p-1} \log^2 T$ for large enough T , where $K > 0$*

depends only on p , μ_p , $\sup_{j \geq 1} (\sum_{k=0}^{\infty} k |c_{jk}|)^p$ and $\sup_{j \geq 1} |C_j(e^{-i\omega_0})|$.

Proof: Consider the following representation for $r_{js} \equiv \hat{u}_{js} - \hat{u}_{js}^G$:

$$\sqrt{T}r_{js} = e^{-i(\omega_s - \omega_0)T} \tilde{R}_{jT} - \sum_{t=1}^{T-1} \tilde{R}_{jt} e^{-i(\omega_s - \omega_0)t} (e^{-i(\omega_s - \omega_0)} - 1), \quad (6)$$

where $\tilde{R}_{jt} \equiv \sum_{l=1}^t e^{-i\omega_0 l} (u_{jl} - u_{jl}^G)$. Using a modified Beveridge-Nelson decomposition: $C_j(L) = C_j(e^{-i\omega_0}) + \tilde{C}_j(L)(L - e^{-i\omega_0})$, where $\tilde{C}_j(L) = \sum_{k=0}^{\infty} \tilde{c}_{jk} L^k$ with $\tilde{c}_{jk} = \sum_{s=k+1}^{\infty} e^{-i\omega_0(s-k-1)} c_{js}$, we get: $e^{-i\omega_0 l} (u_{jl} - u_{jl}^G) = C_j(e^{-i\omega_0}) e^{-i\omega_0 l} (\varepsilon_{jl} - \varepsilon_{jl}^G) + (\tilde{\varepsilon}_{j,l-1} - \tilde{\varepsilon}_{j,l-1}^G) - (\tilde{\varepsilon}_{jl} - \tilde{\varepsilon}_{jl}^G)$ with $\tilde{\varepsilon}_{jl} = e^{-i\omega_0(l+1)} \tilde{C}_j(L) \varepsilon_{jl}$ and $\tilde{\varepsilon}_{jl}^G = e^{-i\omega_0(l+1)} \tilde{C}_j(L) \varepsilon_{jl}^G$.

Therefore,

$$\tilde{R}_{jt} = C_j(e^{-i\omega_0}) R_{jt} + (\tilde{\varepsilon}_{j,0} - \tilde{\varepsilon}_{j,0}^G) - (\tilde{\varepsilon}_{jt} - \tilde{\varepsilon}_{jt}^G), \quad (7)$$

where $R_{jt} \equiv \sum_{l=1}^t e^{-i\omega_0 l} (\varepsilon_{jl} - \varepsilon_{jl}^G)$. Substituting (7) in (6) and using the fact that $|e^{-i(\omega_s - \omega_0)} - 1| \leq \frac{2\pi(m+1)}{T}$, we obtain:

$$\max_{1 \leq s \leq m} |r_{js}| \leq 2\pi(m+1) \left(|C_j(e^{-i\omega_0})| \max_{1 \leq t \leq T} \frac{|R_{jt}|}{\sqrt{T}} + 2 \max_{0 \leq t \leq T} \frac{|\tilde{\varepsilon}_{jt}|}{\sqrt{T}} + 2 \max_{0 \leq t \leq T} \frac{|\tilde{\varepsilon}_{jt}^G|}{\sqrt{T}} \right). \quad (8)$$

By Lemma 2S, $E \left(\max_{1 \leq t \leq T} \frac{|R_{jt}|}{\sqrt{T}} \right)^2 \leq bT^{2/p-1} \log^2 T$ for some $b > 0$, which depends only on p and μ_p , for large enough T . Furthermore, $\Pr \left(\max_{0 \leq t \leq T} \frac{|\tilde{\varepsilon}_{jt}|}{\sqrt{T}} > \delta \right) \leq \Pr \left(\sum_{t=0}^T \frac{|\tilde{\varepsilon}_{jt}|^p}{T^{p/2}} > \delta^p \right) \leq 2T^{1-p/2} E |\tilde{\varepsilon}_{jt}|^p / \delta^p$. But by Minkowski's inequality, $E |\tilde{\varepsilon}_{jt}|^p = E (|\sum_{k=0}^{\infty} \tilde{c}_{jk} \varepsilon_{jt-k}|^p) < \left(\sum_{k=0}^{\infty} |\tilde{c}_{jk}| (E |\varepsilon_{jt-k}|^p)^{1/p} \right)^p \leq (\sum_{k=0}^{\infty} k |c_{jk}|)^p \mu_p$. Hence, $\Pr \left(\max_{0 \leq t \leq T} \frac{|\tilde{\varepsilon}_{jt}|}{\sqrt{T}} > \delta \right) \leq 2T^{1-p/2} (\sum_{k=0}^{\infty} k |c_{jk}|)^p \mu_p / \delta^p$ and therefore, $E \left(\max_{0 \leq t \leq T} \frac{|\tilde{\varepsilon}_{jt}|}{\sqrt{T}} \right)^2 = 2 \int_0^{\infty} x \Pr \left(\max_{0 \leq t \leq T} \frac{|\tilde{\varepsilon}_{jt}|}{\sqrt{T}} > x \right) dx \leq T^{2/p-1} + 4 \int_{T^{1/p-1/2}}^{\infty} x^{1-p} T^{1-p/2} (\sum_{k=0}^{\infty} k |c_{jk}|)^p \mu_p dx \leq aT^{2/p-1}$ for some $a > 0$ which depends only on p , μ_p and $\sup_{j \geq 1} (\sum_{k=0}^{\infty} k |c_{jk}|)^p$. Using similar argument, we can show that $E \left(\max_{0 \leq t \leq T} \frac{|\tilde{\varepsilon}_{jt}^G|}{\sqrt{T}} \right)^2 \leq cT^{2/p-1}$ for some $c > 0$ which depends only on p and $\sup_{j \geq 1} (\sum_{k=0}^{\infty} k |c_{jk}|)^p$. Using the above estimates

for the second moments together with (8), we get: $\sup_{j \geq 1} E(\max_{1 \leq s \leq m} |r_{js}|)^2 \leq Km^2 T^{2/p-1} \log^2 T$ for large enough T and for $K > 0$ which depends only on p , μ_p , $\sup_{j \geq 1} (\sum_{k=0}^{\infty} k |c_{jk}|)^p$ and $\sup_{j \geq 1} |C_j(e^{-i\omega_0})|$. QED

A detail of the proof of Theorem 1

In the proof of Theorem 1, we mention that we can establish the fact that $\bar{c}_{m,n} - c_{m,n} = O(1/n)$ by finding bounds on function $\bar{f}(c) \equiv \int \left(\frac{\lambda c}{1-\lambda c}\right)^2 d\bar{H}_n(\lambda)$ in terms of function $f(c) \equiv \int \left(\frac{\lambda c}{1-\lambda c}\right)^2 dH_n(\lambda)$. Here we derive such bounds and use them to prove that $\bar{c}_{m,n} - c_{m,n} = O(1/n)$.

Since $\left(\frac{\lambda c}{1-\lambda c}\right)^2$ is an increasing function of λ for $\lambda c < 1$, inequalities $l_{k+i,n} \leq \bar{l}_{in} \leq l_{in}$ for $n-k \leq i \leq 1$ imply that

$$f(c) - \frac{k}{n} \left(\frac{l_{1n}c}{1-l_{1n}c}\right)^2 \leq \frac{n-k}{n} \bar{f}(c) \leq f(c) \quad (9)$$

for $c \in [0, l_{1n}^{-1})$. By definition, $\bar{c}_{m,n}$ and $c_{m,n}$ are the solutions to equations $\bar{f}(c) = \frac{m-k}{n-k}$ and $f(c) = \frac{m}{n}$, respectively. Furthermore, $\bar{f}(c)$ and $f(c)$ are increasing functions of c on $c \in [0, \bar{l}_{1n}^{-1})$ and on $c \in [0, l_{1n}^{-1})$, respectively. Hence, inequalities (9) would imply $\bar{c}_{m,n} - c_{m,n} = O(1/n)$ if we show that for any $n > N$, $f(c_{m,n} + \frac{M}{n}) - \frac{k}{n} \left(\frac{l_{1n}(c_{m,n} + M/n)}{1-l_{1n}(c_{m,n} + M/n)}\right)^2 \geq \frac{m-k}{n}$ and $f(c_{m,n} - \frac{M}{n}) \leq \frac{m-k}{n}$, where $N > 0$ and $M > 0$ are constants yet to be chosen.

Since $f(c_{m,n}) = \frac{m}{n}$, we have: $f(c_{m,n} \pm \frac{M}{n}) \geq \frac{m}{n} \pm \frac{M}{n} \min_{|c-c_{m,n}| \leq M/n} f'(c)$ for any $n > N_1(M)$, where $N_1(M)$ is so large that $(c_{m,n} + \frac{M}{n})l_{1n} < 1$ and $c_{m,n} - \frac{M}{n} > 0$ for any $n > N_1(M)$. That such an $N_1(M)$ exists, follows from the assumption that $\limsup l_{1n}c_{m,n} < 1$ and from inequality $\liminf c_{m,n} > 0$. The latter inequality holds because $\frac{m}{n} = \int \left(\frac{\lambda c_{m,n}}{1-\lambda c_{m,n}}\right)^2 dH_n(\lambda) \leq \left(\frac{l_{1n}c_{m,n}}{1-l_{1n}c_{m,n}}\right)^2$ so that $\liminf c_{m,n} \geq$

$\liminf \left(\sqrt{\frac{m}{n}} \right) \frac{1 - \limsup l_{1n} c_{m,n}}{\limsup l_{1n}}$, where $\liminf \left(\sqrt{\frac{m}{n}} \right) > 0$ by assumption that $\frac{m}{n}$ remain in a compact subset of $(0, \infty)$, and $\limsup l_{1n} c_{m,n} < 1$ and $\limsup l_{1n} < \infty$ by Assumption 3.

Further, note that $f''(c) \equiv \int \frac{2\lambda^2 + 4\lambda^3 c}{(1-\lambda c)^4} dH_n(\lambda) > 0$ for $c \in [0, l_{1n}^{-1})$. Therefore, $f(c)$ is convex on $[0, l_{1n}^{-1})$, and we have: $\min_{|c - c_{m,n}| \leq M/n} f'(c) = f'(c_{m,n} - \frac{M}{n}) \equiv \int \frac{2\lambda^2 c}{(1-\lambda c)^3} dH_n(\lambda) \Big|_{c=c_{m,n} - M/n} \geq \frac{2l_{nn}^2(c_{m,n} - M/n)}{(1-l_{nn}(c_{m,n} - M/n))^3} > 2l_{nn}^2(c_{m,n} - M/n)$. But by Assumption 3, $\liminf l_{nn} > 0$. Therefore, there exist $N_2(M)$ and $\gamma > 0$ such that $\min_{|c - c_{m,n}| \leq M/n} f'(c) \geq \gamma$ for any $n > N_2(M)$. And hence, $f(c_{m,n} \pm \frac{M}{n}) \geq \frac{m}{n} \pm \frac{M}{n} \gamma$ for any $n > \max(N_1(M), N_2(M))$.

Finally, let $N_3(M)$ and $C \geq k$ be such that for any $n > N_3(M)$, $\left(\frac{l_{1n}(c_{m,n} + M/n)}{1 - l_{1n}(c_{m,n} + M/n)} \right)^2 \leq \frac{C}{k}$. Choose $M > \frac{C}{\gamma}$ and $N = \max(N_1(M), N_2(M), N_3(M))$. Then, for any $n > N$, $f(c_{m,n} + \frac{M}{n}) - \frac{k}{n} \left(\frac{l_{1n}(c_{m,n} + M/n)}{1 - l_{1n}(c_{m,n} + M/n)} \right)^2 > \frac{m}{n} + \frac{M}{n} \gamma - \frac{C}{n} \geq \frac{m-k}{n}$ and $f(c_{m,n} - \frac{M}{n}) < \frac{m}{n} - \frac{M}{n} \gamma < \frac{m}{n} - \frac{C}{n} \leq \frac{m-k}{n}$ as desired, and thus, $\bar{c}_{m,n} - c_{m,n} = O(1/n)$. QED

Proof of Theorem 3

Let $\lambda_j(A)$ denote the i -th largest eigenvalue of a Hermitian matrix A . Let $\tilde{F}_t = F_t + \sqrt{-1}F_{t+T/2}$. Below, we will assume that T is an even number. If it is not, we will redefine T as $T - 1$. We have the following:

Lemma 4S: *Suppose Assumption 1m holds. Then, there exists a constant $B > 0$ such that $\Pr \left(\lambda_k \left(\frac{2}{T} \sum_{t=1}^{T/2} \tilde{F}_t \tilde{F}_t' \right) < B \right) \rightarrow 0$ as $T \rightarrow \infty$.*

Proof: Let us denote $\frac{1}{t_2 - t_1} \sum_{t=t_1+1}^{t_2} F_t F_{t+u}'$ as $\hat{\Gamma}^{(t_1, t_2)}(u)$ and the i, j -th entry of matrix $\hat{\Gamma}^{(t_1, t_2)}(u)$ as $\hat{\Gamma}_{ij}^{(t_1, t_2)}(u)$. Note that, by definition of \tilde{F}_j , $\frac{2}{T} \sum_{j=1}^{T/2} \tilde{F}_j \tilde{F}_j' = 2\hat{\Gamma}^{(0, T)}(0) + \sqrt{-1} \left(\hat{\Gamma}^{(T/2, T)}(-T/2) - \hat{\Gamma}^{(0, T/2)}(T/2) \right)$. Using Weyl's inequalities for eigenvalues of a

sum of Hermitian matrices (see Horn and Johnson (1985) Theorem 4.3.7), we obtain:

$$\left| \lambda_k \left(\frac{2}{T} \sum_{t=1}^{\frac{T}{2}} \tilde{F}_t \tilde{F}_t' \right) - \lambda_k \left(2\hat{\Gamma}^{(0,T)}(0) \right) \right| \leq \left\| \hat{\Gamma}^{(\frac{T}{2},T)} \left(-\frac{T}{2} \right) - \hat{\Gamma}^{(0,\frac{T}{2})} \left(\frac{T}{2} \right) \right\| \quad (10)$$

According to formula 3.3 on 209 of Hannan (1970), the variance of $\hat{\Gamma}_{ij}^{(t_1,t_2)}(s)$ equals

$$\frac{1}{t_2-t_1} \sum_{u=-t_2+t_1+1}^{t_2-t_1-1} \left(1 - \frac{|u|}{t_2-t_1} \right) \times \{ \Gamma_{ii}(u) \Gamma_{jj}(u) + \Gamma_{ij}(u+s) \Gamma_{ji}(u-s) + cum(F_{i0}, F_{j,s}, F_{i,u}, F_{j,u+s}) \}.$$

Since by Assumption 1m, for any i and j , $\Gamma_{ij}(v) \rightarrow 0$ as $v \rightarrow \infty$ and $cum(F_{i0}, F_{j,s}, F_{i,u}, F_{j,u+s}) \rightarrow$

0 as $\max(|s|, |u|, |s+u|) \rightarrow \infty$, the variances of $2\hat{\Gamma}_{ij}^{(0,T)}(0)$, of $\hat{\Gamma}_{ij}^{(T/2,T)}(-T/2)$ and

of $\hat{\Gamma}_{ij}^{(0,T/2)}(T/2)$ converge to zero as $T \rightarrow \infty$. Therefore, $2\hat{\Gamma}_{ij}^{(0,T)}(0)$ converges in prob-

ability to its mean $2\Gamma_{ij}(0)$ and, since by Assumption 1m $\Gamma_{ij}(-T/2) - \Gamma_{ij}(T/2) \rightarrow 0$,

$\hat{\Gamma}_{ij}^{(T/2,T)}(-T/2) - \hat{\Gamma}_{ij}^{(0,T/2)}(T/2)$ converges in probability to zero. Since the eigenvalues

are continuous functions of the entries of the matrix, $\lambda_k \left(2\hat{\Gamma}^{(0,T)}(0) \right)$ converges in

probability to $2\lambda_k(\Gamma(0)) > 0$. Further, $\left\| \hat{\Gamma}_{ij}^{(T/2,T)}(-T/2) - \hat{\Gamma}_{ij}^{(0,T/2)}(T/2) \right\|$ converges

in probability to zero. The latter two convergence results and inequality (10) imply

that the statement of the lemma holds with $B = \lambda_k(\Gamma(0))$. QED

Lemma 5S: *Let Assumptions 1m-4m hold, and n and T go to infinity so that n/T remains in a compact subset of $(0, \infty)$. Then, for any positive integer r , the joint distribution of $\sigma_{T/2,n}^{-1}(\tilde{\gamma}_{k+1} - \mu_{T/2,n}), \dots, \sigma_{T/2,n}^{-1}(\tilde{\gamma}_{k+r} - \mu_{T/2,n})$ weakly converges to the r -dimensional TW_2 distribution.*

Proof: The proof of this lemma is almost identical to the proof of Theorem 1 in the Appendix. We introduce the following notation to minimize the discrepancies.

Let $m = T/2$, $\tilde{X} = \sqrt{2\pi} [\tilde{X}_1, \dots, \tilde{X}_m]$, $\tilde{F} = \sqrt{2\pi} [\tilde{F}_1, \dots, \tilde{F}_m]$ and $\tilde{e} = \sqrt{2\pi} [\tilde{e}_1, \dots, \tilde{e}_m]$.

Then, by definition, $\tilde{\gamma}_i = \lambda_i \left(\frac{\tilde{X} \tilde{X}'}{2\pi m} \right)$ for all $i = 1, \dots, n$. The remaining proof of Lemma

5S repeats the proof of Theorem 1 starting from the second paragraph of that proof

with the following changes: matrices $S_n^e(\omega_0)$ and $\bar{S}_n^e(\omega_0)$ must be replaced by Σ_n^e

and $\bar{\Sigma}_n^e$, where $\Sigma_n^e \equiv Ee_t(n)e_t'(n)$; the word ‘‘Assumption 3’’ must be replaced by

“Assumption 3m”; the words “by Assumptions 1 and 4” must be replaced by “by Lemma 4S and Assumption 4m”. QED

The convergence of \tilde{R} to $\max_{0 < i \leq k_1 - k_0} \frac{\lambda_i - \lambda_{i+1}}{\lambda_{i+1} - \lambda_{i+2}}$ when $k = k_0$ follows from Lemma 5S. When $k_0 < k \leq k_1$, $\tilde{R} \geq \frac{\tilde{\gamma}_k - \tilde{\gamma}_{k+1}}{\tilde{\gamma}_{k+1} - \tilde{\gamma}_{k+2}}$. Therefore, we only need to show that $\frac{\tilde{\gamma}_k - \tilde{\gamma}_{k+1}}{\tilde{\gamma}_{k+1} - \tilde{\gamma}_{k+2}} \xrightarrow{p} \infty$. Using notation of Lemma 5S, we have: $\tilde{\gamma}_i = \lambda_i \left(\frac{\tilde{X}\tilde{X}'}{2\pi m} \right)$ for all $i = 1, \dots, n$. Using Weyl’s inequalities for singular values (see Lemma 3), we obtain: $\left| \lambda_i^{1/2} \left(\frac{\tilde{X}\tilde{X}'}{2\pi m} \right) - \lambda_i^{1/2} \left(\frac{\hat{\Lambda}_0 \hat{F} \hat{F}' \hat{\Lambda}'_0}{2\pi m} \right) \right| \leq \lambda_1^{1/2} \left(\frac{\tilde{e}\tilde{e}'}{2\pi m} \right)$ for $i = 1, \dots, n$, where $\lambda_1 \left(\frac{\tilde{e}\tilde{e}'}{2\pi m} \right) = O_p(1)$ by Lemma 1. Take $i = k$. By Assumption 4m and Lemma 4S, $\lambda_k \left(\frac{\hat{\Lambda}_0 \hat{F} \hat{F}' \hat{\Lambda}'_0}{2\pi m} \right) \xrightarrow{p} \infty$. Therefore, $\lambda_k \left(\frac{\tilde{X}\tilde{X}'}{2\pi m} \right) \xrightarrow{p} \infty$, and hence, $\tilde{\gamma}_k \xrightarrow{p} \infty$. Now, take $i > k$. Then $\lambda_i^{1/2} \left(\frac{\hat{\Lambda}_0 \hat{F} \hat{F}' \hat{\Lambda}'_0}{2\pi m} \right) = 0$. Therefore, $\lambda_i \left(\frac{\tilde{X}\tilde{X}'}{2\pi m} \right) = O_p(1)$, and hence, $\tilde{\gamma}_i = O_p(1)$. Summing up, $\tilde{\gamma}_k - \tilde{\gamma}_{k+1} \xrightarrow{p} \infty$, while $\tilde{\gamma}_{k+1} - \tilde{\gamma}_{k+2} = O_p(1)$. Hence, $\frac{\tilde{\gamma}_k - \tilde{\gamma}_{k+1}}{\tilde{\gamma}_{k+1} - \tilde{\gamma}_{k+2}} \xrightarrow{p} \infty$. QED

References

- BRILLINGER, D. R. (1981) Time series. Data analysis and theory. Holden Day, Inc., San Francisco.
- HANNAN, E.J. (1970) Multiple Time Series. John Wiley and Sons, Inc., New York, London, Sydney, Toronto.
- HORN, R.A. AND C. R. JOHNSON (1985) Matrix Analysis, Cambridge University Press, Cambridge, New York.
- PHILLIPS, P.C.B. (2007) “Unit root log periodogram regression”, *Journal of Econometrics* 138, pp. 104-124.
- ZAITSEV, A. YU. (2006) “Estimates for the strong approximation in the multi-dimensional invariance principle”, *Zapiski Nauchnyh Seminarov POMI* 339, 37-53. <http://www.pdmi.ras.ru/zns/>