Asymptotic distribution of principal components estimator of large spherical factor models. (Preliminary and Incomplete)

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Abstract

This paper studies the principal components estimator of the factor models with i.i.d. Gaussian idiosyncratic terms when the dimensionality of the data, n, and the number of observations, T, go to infinity proportionally. We focus on an empirically relevant situation when the cumulative effects of the normalized factors on the cross-sectional units do not overwhelmingly dominate the cumulative idiosyncratic influences. It is, therefore, assumed that the cumulative effects of the factors remain bounded as n rises. We show that, under such an assumption, the principal components estimators of the factors and factor loadings are inconsistent but asymptotically normal. We give explicit formulae for the amount of the inconsistency and for the asymptotic variance of the estimators. To illustrate potential implications of our results for econometric practice we estimate the amount of the forecast bias which would result from using the inconsistent principal components estimator of a factor in a simple diffusion index forecast model. As a Monte Carlo analysis shows, our asymptotic formulae work very well even in the samples as small as $n = 40$ and $T = 20$. They work better than the analogous formulae found by Bai (2003), who assumes strong domination of the factor effects over the idiosyncratic influences, and better than the analogous classical formulae, developed for the large T-small n situation. Our asymptotic formulae provide a link between the classical formulae and those of Bai in the sense that they converge to the classical formulae when n/T converges to zero, and to generalizations of Bai's formulae when the cumulative effects of the factors diverge to infinity.

1 Introduction

High-dimensional factor models have recently attracted an increasing amount of attention from researches in macroeconomics and finance. The factors extracted from hundreds of macroeconomic and financial variables observed for a period of several decades have been used for macroeconomic forecasting, monetary policy and business cycle analysis, arbitrage pricing theory tests, and portfolio performance evaluation (see, for example, Stock and Watson (2005), Bernanke, Boivin, and Eliasz (2004), Forni and Reichlin (1998), and Connor and Korajczyk (1988)). A popular technique for factor extraction is the principal components method which estimates the factors by the principal eigenvectors of a sample-covariance-type matrix. In this paper we study the asymptotic distribution of the principal components estimator when the dimensionality of the data, n, and the number of observations, T, go to infinity proportionally.

The consistency and asymptotic normality of the principal components estimator when both n and T go to infinity have been recently shown by Bai (2003). To prove his results, Bai makes a strong assumption equivalent to requiring that the ratio between the k-th largest and the $k+1$-th largest eigenvalues of the population covariance matrix of the data, where k is the number of factors, increase *proportionately* to n so that the cumulative effects of the normalized factors on the cross-sectional units strongly dominate the idiosyncratic influences asymptotically. In practice, the ratio of the eigenvalues of the finite sample analog of the population covariance matrix turns out to be rather small. For example, for the set of the 148 macroeconomic indicators used in Stock and Watson (2002), the ratio of the i-th to the $i+1$-th eigenvalues of the sample covariance matrix is smaller than 1.75 for any positive integer $i \leq 20$, where 20 is a generous *a priori* upper bound on the number of factors. Hence, for the macroeconomic data, the cumulative effect of the "least influential factor" on the cross-sectional units is comparable to the strongest idiosyncratic influence so that, if the ratio of the k-th to the $k+1$-th eigenvalues is increasing proportionally to n, the coefficient of proportionality must be very small and the usefulness of the "strong-factor asymptotics" is questionable.

In this paper, we, therefore, focus on the principal components estimation of models with factors having bounded cumulative effect on the cross-sectional units as the number of the units goes to infinity. We restrict our attention to the factor models with i.i.d. idiosyncratic terms. Such a simple framework makes normalized factors and factor loadings identifiable even though the idiosyncratic influences on the cross-sectional units remain non-negligible relative to the factor effects as n increases. In addition, the i.i.d. assumption implies that the vectors of the factor loadings coincide with the eigenvectors of the population covariance matrix so that the principal components method is a natural method for factor loadings estimation. As we show in the paper, even in this simple framework, the principal components estimator is inconsistent under our "weak-factor asymptotics". We give explicit formulae for the amount of this inconsistency and find the asymptotic distribution of the principal components estimator. A Monte Carlo analysis shows that our asymptotic formulae work very well even in the samples as small as $n = 40$ and $T = 20$. To illustrate potential implications of our results for econometric practice we estimate the amount of the forecast bias which would result from using the inconsistent principal components estimator of a factor in a simple diffusion index forecast model. An extension of our analysis to general approximate factor models is an important topic left for future research. The clean results of this paper can be viewed as building intuition for that future work.

Our main findings can be summarized in more detail as follows. Consider a factor model $X = LF' + \varepsilon$, where X is an $n \times T$ matrix of data, F and L are $T \times k$ and $n \times k$ matrices of factors and factor loadings, respectively, and ε is an $n \times T$ matrix of i.i.d. Gaussian idiosyncratic terms. The principal components estimator of F, $\hat{F}$, is defined as $\sqrt{T}$ times the matrix of the principal k eigenvectors of a sample-covariance-type matrix $X'X/T$, and the principal components estimator of L, $\hat{L}$, is defined as $X\hat{F}/T$.

We establish the following representation of the principal components estimator of the factors:

$$\hat{F} = F \cdot Q + F^{\perp}, \tag{1}$$

where Q is a random $k \times k$ matrix which tends in probability to a diagonal matrix with positive diagonal elements *strictly smaller than unity*, and $F^{\perp}$ is a random $n \times k$ matrix which has columns orthogonal to the columns of F, and is such that the joint distribution of the entries of $F^{\perp}$ conditional on F is invariant with respect to the multiplication of $F^{\perp}$ from the left by any orthogonal matrix having span (F) as an invariant subspace. Matrix Q centered by its probability limit and scaled by $\sqrt{T}$ has asymptotically jointly normal entries, and we find explicit formulae for the probability limit and for the covariance matrix of the asymptotic distribution of Q.

The above representation is illustrated in Figure 1. The principal components estimates $\hat{F}$ randomly "circle" around the true F so that the average projection of $\hat{F}$ on F, equal to $F \cdot \operatorname{plim} Q$, is a scaled-down version of F. When the cumulative effects of the factors on the cross-sectional units, measured by the diagonal elements of $L'L$, are large, $\operatorname{plim} Q$ is close to an identity matrix and $\hat{F}$ is close to F. When the cumulative effects are small, $\operatorname{plim} Q$ is close to zero and $\hat{F}$ is nearly orthogonal to F. In the extreme case, when the cumulative effect of one of the factors becomes below a certain threshold, representation (1) breaks down and the corresponding factor estimate starts to point to completely random direction. The width of the darker band on the sphere of radius $\sqrt{T}$ represents the size of the asymptotic variance of Q. The more narrow the band, the smaller the asymptotic variance of Q.

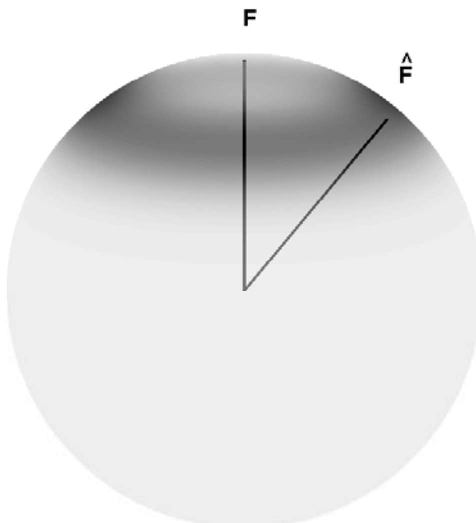

Figure 1: Distribution of $\hat{F}$. The darker areas on the sphere represent the regions of relatively higher probability for $\hat{F}$.

A formula completely analogous to (1) holds for the normalized principal components estimator of factor loadings $\hat{\mathcal{L}} \equiv \hat{L}\left(\hat{L}'\hat{L}\right)^{-1/2}$. Precisely, we show that $\hat{\mathcal{L}} = \mathcal{L} \cdot R + \mathcal{L}^{\perp}$, where $\mathcal{L}$ is a matrix of normalized factor loadings $L\left(L'L\right)^{-1/2}$ and a random matrix R has properties parallel to those of Q in (1).

Representations of type (1) can be used to obtain the asymptotic distributions of the principal components estimator of factors at particular time periods or factor loadings corresponding to specific cross-sectional units. We find such distributions in Theorems 3 and 4. The distributions are centered at the true values of the factors and factor loadings shrunk towards zero. As the cumulative effect of the factors on the cross-sectional units tend to infinity, the shrinkage disappears and our asymptotic formulae converge to what can be interpreted as generalizations of formulae found by Bai (2003). The Monte Carlo analysis shows that our asymptotic distribution provides a better approximation for the finite sample distribution than the asymptotic distribution found by Bai (2003) even for relatively "strong" factors.

In a special case when factors are i.i.d. Gaussian random variables, the principal components estimator of the normalized factor loadings is equal to the matrix of the principal eigenvectors of the sample covariance matrix of i.i.d. Gaussian data. The asymptotic distribution of such principal eigenvectors in case of fixed n and large T is well known (see

Anderson (1984), chapter 13). In this special case, our asymptotic distribution converges to the classical analog when the limit of the n/T ratio, c, converges to zero. The Monte Carlo analysis shows that our asymptotic distribution provides a better approximation for the finite sample distribution of the components of the eigenvectors in small samples.

In the paper we also find the asymptotic distribution of the principal eigenvalues of the sample covariance matrix XX'/T. It is easy to show that the i-th eigenvalue measures the square of the Euclidean length of the i-th column of $\hat{L}$. Hence, it can be interpreted as a principal components estimator of the cumulative effect of the i-th factor on the cross-sectional units. We find that the first k eigenvalues of the sample covariance matrix of the data converge in probability to values strictly larger than the first k eigenvalues of the population covariance matrix. When the "population eigenvalues" are large enough, the "sample eigenvalues" centered by their probability limits and multiplied by $\sqrt{T}$ are asymptotically jointly normal, and we find explicit formulae for the probability limits and the covariance matrix of the asymptotic distribution. If a "population eigenvalue" is below a certain threshold, the corresponding "sample eigenvalue" converges to a positive constant that does not depend on the population eigenvalue.

Our paper is closely related to several recent studies of eigenvalues and eigenvectors of the sample covariance matrix of high-dimensional data. For a 1-factor model with i.i.d. idiosyncratic terms, Johnstone and Lu (2004) showed that the sinus of the angle between principal eigenvector of the sample covariance matrix and that of the population covariance matrix remains separated from zero as n and T go to infinity proportionately. Paul (2004) precisely quantifies the amount of the inconsistency pointed out by Johnstone and Lu (2004) for the case of i.i.d. normal data such that all but k distinct eigenvalues of the population covariance matrix are the same. For the same model, Paul (2004) finds the asymptotic distribution of the k largest eigenvalues of the sample covariance matrix when the corresponding population eigenvalues are larger than a certain threshold. He points out that, when the population eigenvalues are below the threshold, the corresponding sample eigenvalues converge to a constant unrelated to the size of the population eigenvalues. The described "phase transition phenomenon" for the eigenvalues was also studied in papers by Péché (2003), Baik, Ben Arous and Péché (2004), and Baik and Silverstein (2004).

There are three main contributions of this paper that cannot be found in the above mentioned studies. First, we find the previously unknown asymptotic distribution of the eigenvectors of the sample covariance matrix. Second, we extend the asymptotic formulae for the eigenvalues found by Paul (2004) to the case of the non-i.i.d. data, where the non-i.i.d.-ness follows from the relatively arbitrary structure of the factors. We also rediscover the phase transition phenomenon for the eigenvalues, first pointed out by Péché (2003) and

Baik, Ben Arous and Péché (2004), for the case of the non-i.i.d.-factor model, and find similar "phase transition phenomenon" for the eigenvectors. Finally, the method of our proofs is different from the methods used in the previous literature, and is specifically tuned for the analysis of factor models.

The rest of the paper is organized as follows. In Section 2 we introduce the model, state our assumptions, and formulate our main results. Section 3 describes implications of our results for a simple diffusion index forecast model and provides a Monte Carlo analysis. The main steps of our proofs are given in Section 4. Section 5 concludes. All auxiliary results are proven in Appendix.

2 Model, assumptions, and main results

We consider a sequence of factor models indexed by n :

$$X^{(n)} = L^{(n)} F^{(n)\prime} + \varepsilon^{(n)} \tag{2}$$

where $X^{(n)}$ is an $n \times T^{(n)}$ matrix of data; $F^{(n)}$ is a $T^{(n)} \times k$ matrix of $T^{(n)}$ observations of k factors, where k does not depend on n; $L^{(n)}$ is an $n \times k$ deterministic matrix of factor loadings; and $\varepsilon^{(n)}$ is an $n \times T^{(n)}$ noise matrix with i.i.d. $N(0, \sigma^2)$ entries. We assume that (2) satisfies Assumptions 1 (or 1'), 2, and 3, formulated below.

In what follows, $A_{i\cdot}$ ($A_{\cdot i}$) denotes the i-th row (column) of matrix A and I_i denotes an i-dimensional identity matrix. Our first assumption comes in two varieties. Assumption 1 treats factors as random variables. It allows to identify factor loadings. Assumption 1' deals with deterministic factors. It allows to identify both factor loadings and factors. Both assumptions are standard (see Anderson (1984), pp. 552-553).

Assumption 1: *For each $n \geq 1$, factors $\left\{F_{t\cdot}^{(n)\prime}; t = 1, ..., T^{(n)}\right\}$ form a sample of length $T^{(n)}$ from a stationary zero-mean $k \times 1$ vector process, normalized so that $E\left(F_{t\cdot}^{(n)\prime} F_{t\cdot}^{(n)}\right) = I_k$. The loadings are normalized so that the first non-zero elements of the columns of $L^{(n)}$ are positive and $L^{(n)\prime} L^{(n)}$ is a $k \times k$ diagonal matrix with non-increasing positive elements along the diagonal.*

In a special case, when the rows of $F^{(n)}$ represent i.i.d. observations of normally distributed factors, model (2) becomes the so called spherical Gaussian case of the standard factor model (see Anderson (1984)).

Assumption 1': *For each $n \geq 1$, factors form a deterministic sequence of k-dimensional vectors. The factors are normalized so that $F^{(n)\prime} F^{(n)} / T^{(n)} = I_k$ and the loadings are nor-*

malized so that the first non-zero elements of the columns of $L^{(n)}$ *are positive and* $L^{(n)\prime}L^{(n)}$ *is a* $k \times k$ *diagonal matrix with non-increasing positive elements along the diagonal.*

The next assumption allows us to make orthogonal transformations of the data without changing the joint distribution of the noise components. A particularly important property of the Gaussian noise that we use in the paper is that the orthogonal matrix of eigenvectors of the sample covariance matrix of such noise has conditional Haar invariant distribution (see Anderson (1984), p.536).

Assumption 2: *For each* $n \geq 1$, *entries of* $\varepsilon^{(n)}$ *are i.i.d.* $N(0, \sigma^2)$ *random variables independent of the factors.*

Our last assumption describes the conditions needed to be satisfied for the asymptotic analysis below to be correct as n goes to infinity.

Assumption 3: *There exist a scalar* $c > 0$ *and a* $k \times k$ *diagonal matrix* $D \equiv \operatorname{diag}(d_1, ..., d_k)$, $d_1 > ... > d_k > 0$[1] *such that, as* $n \to \infty$,
i) $n/T^{(n)} - c = o\left(n^{-1/2}\right)$,
ii) $L^{(n)\prime}L^{(n)} - D = o\left(n^{-1/2}\right)$, *where the equality should be understood in the element by element sense,*
iii) $\sqrt{T^{(n)}}\left(\frac{1}{T^{(n)}}F^{(n)\prime}F^{(n)} - I_k\right) \xrightarrow{d} \Phi$, *where entries of* Φ *have a joint normal distribution (degenerate in the case of deterministic factors) with covariance function* $\operatorname{cov}(\Phi_{st}, \Phi_{s_1t_1}) \equiv \phi_{sts_1t_1}$.

Part i) of the assumption requires that n and $T^{(n)}$ be comparable even asymptotically. The requirement that the convergence is faster than $n^{-1/2}$ eliminates any possible effects of this convergence on our asymptotic results. In our opinion, the behavior of $n/T^{(n)}$ is likely to be application-specific and any consequential assumption on the rate of convergence of $n/T^{(n)}$ will be arbitrary. The assumption on the rate of convergence of $L^{(n)\prime}L^{(n)}$ is made for the same reason. The high-level assumption on the convergence of $\sqrt{T^{(n)}}\left(\frac{1}{T^{(n)}}F^{(n)\prime}F^{(n)} - I_k\right)$ is important because parameters $\phi_{sts_1t_1}$ enter our asymptotic formulae established below. A primitive assumption that implies the convergence is that the individual factors can be represented as infinite linear combinations, with absolutely summable coefficients, of i.i.d. random variables with a finite fourth moment (see Anderson (1971), Theorem 8.4.2). In a special case when $F_{t\cdot}^{(n)}$ are i.i.d. standard multivariate normal, the covariance function of the asymptotic distribution of $\sqrt{T^{(n)}}\left(\frac{1}{T^{(n)}}F^{(n)\prime}F^{(n)} - I_k\right)$ has a particularly simple form: $\phi_{iji_1j_1} = 2$ if $(i,j) = (i_1, j_1)$ and $i = j$, $\phi_{iji_1j_1} = 1$ if $(i,j) = (i_1, j_1)$ or $(i,j) = (j_1, i_1)$ and $i \neq j$, and $\phi_{iji_1j_1} = 0$ otherwise.

[1] We generalized Theorem 5 to the case of some or all of the diagonal elements of D being the same. To save the space, we do not report these results below.

In this paper, we study the principal components estimators $\hat{F}^{(n)}$ and $\hat{L}^{(n)}$ of factors and factor loadings, respectively. To define the estimators we introduce the following notation. Denote the largest k eigenvalues of matrices $\frac{1}{T}X^{(n)}X^{(n)\prime}$ and $\frac{1}{T}X^{(n)\prime}X^{(n)}$ as $\mu_1^{(n)} \geq ... \geq \mu_k^{(n)}$. Note that the matrices have the identical sets of largest $\min(n, T)$ eigenvalues, and we assume that $k < \min(n, T)$. Further, denote the corresponding eigenvectors for $\frac{1}{T}X^{(n)}X^{(n)\prime}$ and $\frac{1}{T}X^{(n)\prime}X^{(n)}$ as $u_1^{(n)}, ..., u_k^{(n)}$, and $v_1^{(n)}, ..., v_k^{(n)}$, respectively. Then the principal components estimator $\hat{F}^{(n)}$ is defined as a matrix with columns $v_1^{(n)}, ..., v_k^{(n)}$, and the principal components estimator $\hat{L}^{(n)}$ is defined as $\frac{1}{T}X\hat{F}$. It is easy to verify that the i-th column of $\hat{L}^{(n)}$ is equal to $\sqrt{\mu_i^{(n)}}u_i^{(n)}$. Therefore, the square of the Euclidean length of $\hat{L}_{\cdot i}^{(n)}$ which estimates the cumulative effect of the i-th factor on the cross-sectional units is equal to $\mu_i^{(n)}$, and the normalized principal components estimator of factor loadings $\hat{\mathcal{L}} \equiv \hat{L}\left(\hat{L}'\hat{L}\right)^{-1/2}$ is equal to a matrix with columns $u_1^{(n)}, ..., u_k^{(n)}$.

Of course, without further restrictions the eigenvectors $u_i^{(n)}$ and $v_i^{(n)}$ and, therefore, the principal components estimators $\hat{F}$ and $\hat{L}$ are defined only up to a change in the sign. To eliminate this indeterminacy, we require that the direction of the eigenvectors is chosen so that $u_i^{(n)\prime}L_{\cdot i}^{(n)} > 0$ and $v_i^{(n)\prime}F_{\cdot i}^{(n)} > 0$. Since neither $L^{(n)}$ nor $F^{(n)}$ are observed the requirement cannot be verified. This should be kept in mind in applications of the results stated below.

We now formulate and discuss our main results postponing all proofs until Section 4. In what follows, we will omit the superscript (n) from our notations to make them easier to read. For any $q \leq k$, denote the matrix of the first q columns of $\hat{F}$ as $\hat{F}_{1:q}$, and let $F_q^{\perp}$ be a $T \times q$ matrix with columns orthogonal to the columns of F, and such that the joint distribution of its entries conditional on F is invariant with respect to multiplication from the left by any orthogonal matrix having $\operatorname{span}(F)$ as its invariant subspace. We establish the following

Theorem 1: *Let q be such that $d_i > \sqrt{c}\sigma^2$ for $i \leq q$ and $d_i \leq \sqrt{c}\sigma^2$ for $i > q$. Let Assumptions 1 (or 1'), 2, and 3 hold and let, in addition, $\phi_{ijst} = 0$ when $(i, j) \neq (s, t)$ and $(i, j) \neq (t, s)$. Then, we have:*
i)

$$\begin{aligned} \hat{F}_{1:q} &= F \cdot Q + F_q^{\perp}, \\ Q &= Q^{(1)} + \frac{1}{\sqrt{T}}Q^{(2)}, \end{aligned}$$

where $Q^{(1)}$ is diagonal with $Q_{ii}^{(1)} = \sqrt{\frac{d_i^2 - \sigma^4 c}{d_i(d_i + \sigma^2)}}$, and $\operatorname{vec} Q^{(2)}$ is asymptotically zero mean Gaussian vector with $\operatorname{Acov}\left(Q_{ij}^{(2)}, Q_{st}^{(2)}\right)$ given by the following formulae:

a) $\frac{\left(d_j^2+\sigma^2 d_i\right)}{(d_j-d_i)^2}+(\phi_{ijij}-1)\frac{d_j\left(d_j^2-c\sigma^4\right)}{(d_j+\sigma^2)(d_j-d_i)^2}$ *if* $(i,j)=(s,t)$ *and* $i\neq j$

b) $\frac{\sqrt{d_i d_j}\sqrt{(d_i+\sigma^2)(d_j+\sigma^2)\left(d_i^2-c\sigma^4\right)\left(d_j^2-c\sigma^4\right)}}{(d_j-d_i)^2(c\sigma^4-d_i d_j)}-\left(\phi_{ijij}-1\right)\frac{\sqrt{d_i d_j}\sqrt{\left(d_i^2-c\sigma^4\right)\left(d_j^2-c\sigma^4\right)}}{(d_j-d_i)^2\sqrt{(d_j+\sigma^2)(d_i+\sigma^2)}}$ *if* $(i,j)=(t,s)$ *and* $i\neq j$

c) $\frac{\left(c^2\sigma^8+d_i^4\right)\left(d_i+\sigma^2\right)}{2d_i\left(d_i^2-c\sigma^4\right)^2}+\frac{d_i\sigma^4(c-1)}{2\left(d_i^2-c\sigma^4\right)(d_i+\sigma^2)}+(\phi_{iiii}-2)\frac{\left(\left(d_i+\sigma^2\right)^2-\sigma^4(1-c)\right)^2 d_i}{4\left(d_i^2-c\sigma^4\right)(d_i+\sigma^2)^3}$ if $(i,j)=(t,s)$ *and* $i=j$

d) 0 *if* $(i,j)\neq(s,t)$ *and* $(i,j)\neq(t,s)$

ii) $\hat{F}_{q+1:k}=F\cdot\tilde{Q}+F_{k-q}^{\perp}$, *where* $\tilde{Q}\xrightarrow{p}0$, *as* $n\to\infty$.

A graphical interpretation of the above representation of $\hat{F}_{1:q}$ for the case of deterministic factors was given in Introduction. For the case of random factors, the interpretation is complicated by the fact that columns of F have random length, not necessarily equal to $\sqrt{T}$. Hence, "vector" F in Figure 1 does not "live" on the sphere and a potential graphical interpretation would not be so clean as for the case of deterministic factors. The theorem's requirement that $\phi_{ijst}=0$ when $(i,j)\neq(s,t)$ and $(i,j)\neq(t,s)$ holds, for example, if the different factors are mutually independent. It trivially holds if the factors are treated as non-random. The requirement was introduced solely to simplify formulae for $\mathrm{Acov}\left(Q_{ij}^{(2)},Q_{st}^{(2)}\right)$, which would otherwise become non-trivial even for the case $(i,j)\neq(s,t)$ and $(i,j)\neq(t,s)$.

Our next result is an analog of Theorem 1 for factor loadings. Denote the matrix of normalized factor loadings $L\left(L'L\right)^{-1/2}$ as matrix $\mathcal{L}$ and let $\mathcal{L}_q^{\perp}$ be an $n\times q$ random matrix with columns orthogonal to the columns of $\mathcal{L}$, and such that the joint distribution of its entries is invariant with respect to multiplication from the left by any orthogonal matrix having $\mathrm{span}\left(\mathcal{L}\right)$ as its invariant subspace. We have the following

Theorem 2: *Let* q *be such that* $d_i>\sqrt{c}\sigma^2$ *for* $i\leq q$ *and* $d_i\leq\sqrt{c}\sigma^2$ *for* $i>q$. *Let Assumptions 1 (or 1'), 2, and 3 hold and let, in addition,* $\phi_{ijst}=0$ *when* $(i,j)\neq(s,t)$ *and* $(i,j)\neq(t,s)$. *Then, we have:*
i)

$$\begin{aligned}\hat{\mathcal{L}}_{1:q} &= \mathcal{L}\cdot R+\mathcal{L}_q^{\perp},\\ R &= R^{(1)}+\frac{1}{\sqrt{T}}R^{(2)},\end{aligned}$$

where $R^{(1)}$ *is diagonal with* $R_{ii}^{(1)}=\sqrt{\frac{d_i^2-\sigma^4 c}{d_i(d_i+\sigma^2 c)}}$, *and* $\mathrm{vec}\,R^{(2)}$ *is asymptotically zero mean Gaussian vector with* $\mathrm{Acov}\left(R_{ij}^{(2)},R_{st}^{(2)}\right)$ *given by the following formulae:*

a) $\frac{d_j\left(d_j+\sigma^2\right)\left(d_i+\sigma^2\right)+d_i\left(\phi_{ijij}-1\right)\left(d_j^2-\sigma^4 c\right)}{(d_j+\sigma^2 c)(d_j-d_i)^2}$ *if* $(i,j)=(s,t)$ *and* $i\neq j$

b) $-\frac{\sqrt{d_i d_j}\sqrt{\left(d_i^2-\sigma^4 c\right)\left(d_j^2-\sigma^4 c\right)}}{(d_j-d_i)^2\sqrt{(d_i+\sigma^2 c)(d_j+\sigma^2 c)}}\left(\phi_{ijij}-1+\frac{\left(d_j+\sigma^2\right)\left(d_i+\sigma^2\right)}{(d_i d_j - c\sigma^4)}\right)$ *if* $(i,j)=(t,s)$ *and* $i\neq j$

c) $\frac{c\sigma^4 d_i\left(d_i+\sigma^2\right)^2}{2(d_i+c\sigma^2)\left(d_i^2-c\sigma^4\right)^2}\left(1+c\left(\frac{d_i+\sigma^2}{d_i+c\sigma^2}\right)^2\right)+(\phi_{iiii}-2)\frac{\left(\left(d_i+\sigma^2\right)^2-\sigma^4(1-c)\right)^2 c^2\sigma^4}{4d_i\left(d_i^2-\sigma^4 c\right)(d_i+c\sigma^2)^3}$ if $(i,j)=(t,s)$ *and* $i=j$

d) 0 *if* $(i,j)\neq(s,t)$ *and* $(i,j)\neq(t,s)$

ii) $\hat{\mathcal{L}}_{q+1:k}=\mathcal{L}\cdot\tilde{R}+\mathcal{L}_{k-q}^{\perp}$, where $\tilde{R}\xrightarrow{p}0$, *as* $n\to\infty$.

Theorems 1 and 2 can be used to obtain the asymptotic distributions of the principal components estimator of factors at particular time periods or factor loadings corresponding to specific cross-sectional units. We find such distributions in Theorems 3 and 4 below. Let δ_{ij} denote the Kronecker delta. Then we have:

Theorem 3: *Suppose the assumptions of Theorem 1 hold. Let* $\tau_1,...,\tau_r$ *be such that the probability limits of the* τ_1*-th,...,*τ_r*-th rows of matrix* $F/\sqrt{T}$ *as* n *and* T *go to infinity exist and equal to* $\bar{F}_{\tau_1\cdot},...,\bar{F}_{\tau_r\cdot}$*. Then,*
i) Random variables $\hat{F}_{\tau_g i}-Q_{ii}^{(1)}F_{\tau_1 i}$, $g=1,...,r$ and $i=1,...,q$ *are asymptotically jointly mean-zero Gaussian. The asymptotic covariance between* $\hat{F}_{\tau_s i}-Q_{ii}^{(1)}F_{\tau_s i}$ *and* $\hat{F}_{\tau_f p}-Q_{pp}^{(1)}F_{\tau_f p}$ *is equal to* $\sum_{s=1}^k \bar{F}_{\tau_g s}\bar{F}_{\tau_f s}\,\text{Avar}\left(Q_{si}^{(2)}\right)+\left(\delta_{gf}-\sum_{s=1}^k\bar{F}_{\tau_g s}\bar{F}_{\tau_f s}\right)\left(1-\left(Q_{ii}^{(1)}\right)^2\right)$ *when* $i=p$ *and to* $-\bar{F}_{\tau_g p}\bar{F}_{\tau_f i}\,\text{Acov}\left(Q_{pi}^{(2)},Q_{ip}^{(2)}\right)$ *when* $i\neq p$.
ii) for any $i>q$*, and any* $\tau\leq T$*,* $\hat{F}_{\tau i}/\sqrt{T}\xrightarrow{p}0$.

When factors are deterministic, allowing for non-zero limits $\bar{F}_{\tau_1\cdot},...,\bar{F}_{\tau_r\cdot}$ takes into account a possibility that special time periods exists for which the values of some or all factors are "unusually" large. Alternatively, non-zero limits $\bar{F}_{\tau_1\cdot},...,\bar{F}_{\tau_r\cdot}$ can be viewed as an ad hoc device to improve asymptotic approximation for relatively small T when the rows of $F/\sqrt{T}$ are not expected to be small. When the factors are random and satisfy Assumption 1, then, obviously, the probability limits $\bar{F}_{\tau_1\cdot},...,\bar{F}_{\tau_r\cdot}$ exist and equal to zero. In such a case, the above formula for the asymptotic covariance between $\hat{F}_{\tau_s i}-Q_{ii}^{(1)}F_{\tau_s i}$ and $\hat{F}_{\tau_f p}-Q_{pp}^{(1)}F_{\tau_f p}$ simplifies to $\delta_{gf}\frac{\sigma^2\left(d_i+\sigma^2 c\right)}{d_i(d_i+\sigma^2)}$ if $i=p$ and zero if $i\neq p$.

Theorem 3 can be compared to Theorem 1 of Bai (2003). He finds that, under his "strong-factor" requirement, $\sqrt{n}\left(\hat{F}_{t\cdot}-H'F_{t\cdot}\right)\xrightarrow{d}N(0,\Omega)$, where H and Ω are matrices that depends on the parameters describing factors, loadings, and noise. For our normalization of factors and factor loadings, it can be shown that H is equal to the identity matrix and Ω must be well approximated by $n\sigma^2 D^{-1}$ in large finite samples. Hence, Bai's asymptotic approximation of the finite sample distribution of $\hat{F}_{ti}-F_{ti}$ can be represented as $N\left(0,\frac{\sigma^2}{d_i}\right)$. The variance of the latter distribution is close to our asymptotic variance $\frac{\sigma^2\left(d_i+\sigma^2 c\right)}{d_i(d_i+\sigma^2)}$ when

d_i is very large, as it should be under the "strong-factor" assumption, or if c is close to 1. Note that the multiplier $Q_{ii}^{(1)}$, causing "inconsistency" of $\hat{F}_{ti}$ in our case, becomes very close to 1 as d_i increases. Hence, Bai's asymptotic formula is consistent with ours in the case of factors having very large cumulative effect on the cross-sectional units.

For factor loadings, we have the following

Theorem 4: *Suppose the assumptions of Theorem 2 hold. Let $j_1, ..., j_r$ be such that the limits of the j_1-th, ..., j_r-th rows of matrix $\mathcal{L}$, as n and T go to infinity,.exist and equal to $\bar{\mathcal{L}}_{j_1\cdot}, ..., \bar{\mathcal{L}}_{j_r\cdot}$. Then,*
i) Random variables $\sqrt{T}\left(\hat{\mathcal{L}}_{j_g i} - R_{ii}^{(1)}\mathcal{L}_{j_g i}\right)$, $g = 1, ..., r$ and $i = 1, ..., q$ are asymptotically jointly mean-zero Gaussian. The asymptotic covariance between $\sqrt{T}\left(\hat{\mathcal{L}}_{j_g i} - R_{ii}^{(1)}\mathcal{L}_{j_g i}\right)$ and $\sqrt{T}\left(\hat{\mathcal{L}}_{j_f p} - R_{pp}^{(1)}\mathcal{L}_{j_f p}\right)$ is equal to $\sum_{s=1}^{k}\bar{\mathcal{L}}_{j_g s}\bar{\mathcal{L}}_{j_f s}\operatorname{Avar}\left(R_{si}^{(2)}\right) + \left(\delta_{gf} - \sum_{s=1}^{k}\bar{\mathcal{L}}_{j_g s}\bar{\mathcal{L}}_{j_f s}\right)\left(1 - \left(R_{ii}^{(1)}\right)^2\right)$ when $i = p$ and to $-\bar{\mathcal{L}}_{j_g p}\bar{\mathcal{L}}_{j_f i}\operatorname{Acov}\left(R_{pi}^{(2)}, R_{ip}^{(2)}\right)$ when $i \neq p$.
ii) for any $i > q$, and any $j \leq n$, $\hat{\mathcal{L}}_{ji} \xrightarrow{p} 0$

For the special case when the factors are i.i.d. k-dimensional standard normal variables, the formula for the asymptotic covariance of the components of $\hat{\mathcal{L}}$ simplifies. We have:

Corollary 1: *Suppose that, in addition to the assumptions of theorem 4, the factors $F_{t\cdot}$ are i.i.d. standard multivariate random variables. Then, for any $i \leq q$*

$$\sqrt{T}\left(\left(\hat{\mathcal{L}}_{j_1 i} - R_{ii}^{(1)}\mathcal{L}_{j_1 i}\right), ..., \left(\hat{\mathcal{L}}_{j_r i} - R_{ii}^{(1)}\mathcal{L}_{j_r i}\right)\right) \xrightarrow{d} N(0, \Gamma),$$

where

$$\begin{aligned}\Gamma_{gf} &= \sum_{\substack{s=1\\ s\neq i}}^{k}\bar{\mathcal{L}}_{j_g s}\bar{\mathcal{L}}_{j_f s}\frac{d_i(d_i+\sigma^2)(d_s+\sigma^2)}{(d_i+c\sigma^2)(d_i-d_s)^2} + \left(\delta_{gf} - \sum_{s=1}^{k}\bar{\mathcal{L}}_{j_g s}\bar{\mathcal{L}}_{j_f s}\right)\frac{\sigma^2(d_i+\sigma^2)}{d_i(d_i+c\sigma^2)} \\ &\quad + \bar{\mathcal{L}}_{j_g i}\bar{\mathcal{L}}_{j_f i}\frac{c\sigma^4 d_i(d_i+\sigma^2)^2}{2(d_i+c\sigma^2)(d_i^2-c\sigma^4)^2}\left(1 + c\left(\frac{d_i+\sigma^2}{d_i+c\sigma^2}\right)^2\right)\end{aligned}$$

Note that when factors are i.i.d. Gaussian random variables, the principal components estimator of the normalized factor loadings is equal to the matrix of the principal eigenvectors of the sample covariance matrix of i.i.d. Gaussian data. The asymptotic distribution of such principal eigenvectors in case when only T goes to infinity is well known. According to theorem 13.5.1 of Anderson (1984),

$$\sqrt{T}\left(\hat{\mathcal{L}}_{\cdot i} - \mathcal{L}_{\cdot i}\right) \to N(0, \Pi), \tag{3}$$

where

$$\Pi_{gf} = \sum_{\substack{s=1 \\ s\neq i}}^{n} \mathcal{L}_{gs}\mathcal{L}_{fs} \frac{(d_i+\sigma^2)(d_s+\sigma^2)}{(d_i-d_s)^2} \tag{4}$$

and it is understood that $\mathcal{L}_{\cdot s}$ is defined as the eigenvector of the population covariance matrix corresponding to the s-th largest eigenvalues and $d_s = 0$ for $s > k$. Note that, $\sum_{s=k+1}^{n} \mathcal{L}_{gs}\mathcal{L}_{fs} = \delta_{gf} - \sum_{s=1}^{k} \mathcal{L}_{gs}\mathcal{L}_{fs}$ because the matrix of "population eigenvectors" is orthogonal. Therefore, we can rewrite (4) as

$$\Pi_{gf} = \sum_{\substack{s=1 \\ s\neq i}}^{k} \mathcal{L}_{gs}\mathcal{L}_{fs} \frac{(d_i+\sigma^2)(d_s+\sigma^2)}{(d_i-d_s)^2} + \left(\delta_{gf} - \sum_{s=1}^{k} \mathcal{L}_{gs}\mathcal{L}_{fs}\right) \frac{\sigma^2(d_i+\sigma^2)}{d_i^2}. \tag{5}$$

Since in the classical case n is fixed, the requirement that rows of $\mathcal{L}$ have limit as T goes to infinity is trivially satisfied. For the same reason, there is no need to focus attention on a subset of components $j_1, ..., j_r$ of the "population eigenvectors" so that formula (3) describes asymptotic behavior of all components of $\mathcal{L}_{\cdot i}$. More substantially, large dimensionality of the data introduces inconsistency (towards zero) to the components of $\hat{\mathcal{L}}_{\cdot i}$ viewed as estimates of the corresponding components of $\mathcal{L}_{\cdot i}$. Indeed, from Corollary 1, we see that the probability limit of $\hat{\mathcal{L}}_{j_s i}$ is equal to $\mathcal{L}_{j_s i}$ multiplied by $0 \leq R_{ii}^{(1)} < 1$. Comparing Π and Γ, we see that the high dimensionality of data introduces a new component to the asymptotic covariance matrix, which depends solely on the limits of the components of the i-th "population eigenvector". At the same time, it reduces the "classical component" of the asymptotic covariance by multiplying it by $\frac{d_i}{d_i+c\sigma^2}$. As c becomes very small, our formula for Γ_{gf} converges to the classical formula for Π_{gf}, as should be the case, intuitively.

The asymptotic result for high-dimensional data strikingly differs from the classical result when d_i is below the threshold $\sqrt{c}\sigma^2$. In such a case, $\hat{\mathcal{L}}_{\cdot i}$ has nothing to do with $\mathcal{L}_{\cdot i}$. It just points out the direction of maximal spurious "explanatory power" of the idiosyncratic terms. It is only when the cumulative effect of the i-th factor on the cross-sectional units passes the threshold that $\hat{\mathcal{L}}_{\cdot i}$ become related to $\mathcal{L}_{\cdot i}$. As d_i becomes larger and larger components of $\hat{\mathcal{L}}_{\cdot i}$ approximate those of $\mathcal{L}_{\cdot i}$ better and better, eventually matching them.

The rest of our results concerns with the asymptotic behavior of eigenvalues $\mu_1, ..., \mu_k$ which, as is explained above, can be interpreted as the principal components estimators of the cumulative effects of the 1st, 2nd,..., k-th factors, respectively, on the cross-sectional units. In fact, a potentially better estimator of the cumulative effect of the i-th factor would be $\mu_i - \hat{\sigma}^2$, where $\hat{\sigma}^2$ is any consistent estimator of σ^2. This can be understood by noting that the i-th eigenvalue of the population covariance matrix of data $EX_{\cdot t}X_{\cdot t}'$ is equal to $d_i + \sigma^2$, where d_i is the true cumulative effect. According to our next theorem, even such a corrected estimator would be inconsistent.

Theorem 5: *Let q be such that $d_i > \sqrt{c}\sigma^2$ for $i \leq q$ and $d_i \leq \sqrt{c}\sigma^2$ for $i > q$. For $i = 1, ..., q$ define constants $m_i = \frac{\left(d_i+\sigma^2\right)\left(d_i+\sigma^2 c\right)}{d_i}$. Under assumptions 1 or (1'), 2, and 3, we have:*

i) $\sqrt{T}\left(\mu_1 - m_1, ..., \mu_q - m_q\right)' \xrightarrow{d} N\left(0, \Sigma\right),$ *where*

$$\Sigma_{ij} = \left(\phi_{iijj} - 2\delta_{ij}\right)\frac{\left(d_i^2 - \sigma^4 c\right)\left(d_j^2 - \sigma^4 c\right)}{d_i d_j} + 2\delta_{ij}\frac{\left(d_i^2 + \sigma^2\right)^2\left(d_i^2 - \sigma^4 c\right)}{d_i^2}$$

ii) for any $i > q$, $\mu_i \xrightarrow{p} \left(1+\sqrt{c}\right)^2\sigma^2$

Theorem 5 extends Theorem 3 of Paul (2004) to the case of data having non-i.i.d factor structure. Note that according to the theorem $\mu_i - \sigma^2$ converges to $m_i - \sigma^2 = d_i + c\sigma^2\left(1 + \frac{\sigma^2}{d_i}\right) > d_i$. Hence, if we estimate the cumulative effect of the i-th factor by subtracting a true known σ^2 from μ_i, we are making systematic positive mistake which may be very large if c and σ^2 are large.

In the case of deterministic factors, the formula for the asymptotic covariance matrix significantly simplifies because $\phi_{iijj} \equiv 0$. The formula also simplifies for the case when the factors are i.i.d. standard multivariate normal random variables. In such a case, we have

Corollary 2: *If, in addition to assumptions of Theorem 5, factors $F_{t\cdot}$ are i.i.d. standard multivariate normal random variables, then $\sqrt{T}\left(\mu_1 - m_1, ..., \mu_q - m_q\right)' \xrightarrow{d} N\left(0, \Sigma\right)$, where Σ is diagonal matrix such that $\Sigma_{ii} = 2\left(d_i + \sigma^2\right)^2\left(1 - \frac{\sigma^4 c}{d_i^2}\right)$*

If we keep the framework of the above Corollary, but consider the classical case, when only T goes to infinity, then according to Theorem 13.5.1 of Anderson (1984), μ_i consistently estimates $d_i + \sigma^2$, and the asymptotic variance of μ_i is equal to $2\left(d_i + \sigma^2\right)^2$. This result can be recovered by setting $c = 0$ in Corollary 2. We see that the large dimensionality of the data introduces inconsistency but reduces the asymptotic variance of μ_i, viewed as an estimate of $d_i + \sigma^2$. Indeed, under our assumptions, the probability limit of μ_i is $d_i + \sigma^2$, multiplied by $\frac{d_i + \sigma^2 c}{d_i} > 1$, and the asymptotic variance is $2\left(d_i + \sigma^2\right)^2$ multiplied by $1 - \frac{\sigma^4 c}{d_i^2}$, which is positive if $i \leq q$, but less than 1.

A striking difference with the classical case occurs when the cumulative effect of the i-th factor on the cross-sectional units, measured by d_i, is below the threshold $\sqrt{c}\sigma^2$. In such a case, the i-th largest eigenvalue of $\frac{1}{T}XX'$ converges to a constant $\left(1+\sqrt{c}\right)^2\sigma^2$ which does not depend on d_i. Hence, if the cumulative effect of the i-th factors on the cross-sectional units is weak relative to the variance of idiosyncratic noise and/or if the number of the cross-sectional units in the sample is much larger than the number of the observations, the size of the i-th largest "sample eigenvalue" does not reflect the strength of the cumulative effect, but measures the maximal amount of variation in the data that can be spuriously "explained" by a linear combination of the idiosyncratic terms. The i-th largest "sample eigenvalue" starts to be related to the cumulative effect of the i-th factor only after the cumulative effect passes the threshold. This is the "phase transition phenomenon" mentioned in the introduction and studied by Péché (2003), Baik, Ben Arous and Péché (2004), Baik and Silverstein (2004), and Paul (2004).

3 Econometric implications and Monte Carlo analysis

To illustrate some consequences of our results for econometric practice, we consider the following simple diffusion index forecast model:

$$y_{t+h} = \beta_1 F_t + \beta_2 W_t + \eta_{t+h}, \tag{6}$$

where y is a variable to be forecasted h periods ahead, W is an observed explanatory variable and F is an unobserved index, which is equal to the factor in a single-factor model $X_{it} = L_i' F_t + \varepsilon_{it}$. We assume for simplicity that $W'W/T \xrightarrow{p} 1$ and $F'W/T \xrightarrow{p} \gamma$ as $T \to \infty$, where $W \equiv (W_1, ..., W_T)'$ and $F \equiv (F_1, ..., F_T)'$. Finally, we assume η_t are i.i.d. $N(0, \sigma_\eta^2)$, and independent on F_s, W_s, and ε_s for any s.

Since the factors are unobserved, equation (6) is usually estimated by a two-step procedure. At the first step, the factors are estimated by the principal components method. At the second step, an ordinary least squares regression of $Y \equiv (y_{1+h}, ..., y_T)'$ on $(W_1, ..., W_{T-h})$ and factor estimates $\left(\hat{F}_1, ..., \hat{F}_{T-h}\right)$ from the first step is run and OLS estimates of $\hat{\beta}_1$ and $\hat{\beta}_2$ are obtained. The forecast of y_{T+h} is then defined as $\hat{y}_{T+h|T} \equiv \hat{\beta}_1 \hat{F}_T + \hat{\beta}_2 W_T$. Bai and Ng (2005) have analyzed statistical properties of such a procedure under the "strong-factor" assumption. They found that $\hat{\beta}_1$ and $\hat{\beta}_2$ are consistent and asymptotically normal estimates of $\tilde{\beta}_1$ and β_2 ($\left|\tilde{\beta}_1\right| = |\beta_1|$ under our normalization of factors), and that the forecast error is approximately normal with variance equal to σ_η^2 plus a term converging to zero at the rate $\min\left(\sqrt{n}, \sqrt{T}\right)$ which is the same as $\sqrt{T}$ under our Assumption 3.

In the case when no "strong-factor" assumption is made the situation is very different. Precisely, we have the following

Proposition 1: *Let Assumptions 1 (or 1'), 2, and 3 hold, and suppose* $d_1 > \sqrt{c}\sigma^2$. *Define* $\varrho = \sqrt{\frac{d_1^2 - \sigma^4 c}{d_1(d_1+\sigma^2)}}$ *and let* $y_{T+h|T} = \beta_1 F_T + \beta_2 W_T$ *be the best (unobserved) forecast of* y_{T+h}. *As* $T \to \infty$, *we have:*
i) $\hat{\beta}_1 - \operatorname{sgn}\left(\hat{F}'F\right)\beta_1 \xrightarrow{p} -\frac{(1-\varrho)\left(1+\gamma^2\varrho\right)}{1-\gamma^2\varrho^2} \operatorname{sgn}\left(\hat{F}'F\right)\beta_1$

ii) $\hat{\beta}_2 - \beta_2 \xrightarrow{p} \frac{\gamma\left(1-\varrho^2\right)}{1-\gamma^2\varrho^2}\beta_1$
iii) $\hat{y}_{T+h|T} = y_{T+h|T} + \frac{1-\varrho^2}{1-\gamma^2\varrho^2}\beta_1\left(\gamma W_T - F_T\right) + o_p(1)$

Hence, $\hat{\beta}_1$, $\hat{\beta}_2$, and $\hat{y}_{T+h|T}$ are now inconsistent for $\text{sgn}\left(\hat{F}'F\right)\beta_1$, β_2, and $y_{T+h|T}$, respectively. The term $\text{sgn}\left(\hat{F}'F\right)$ in the proposition is needed because in practice, as was mentioned above, the principal components estimator of F_T is determined only up to a change in the sign. Although such an indeterminacy would affect the properties of $\hat{\beta}_1$, it does not affect estimation of β_2 and the quality of forecast $\hat{y}_{T+h|T}$.

Using the results from the previous section, we can further analyze properties of the diffusion index forecasts by computing the asymptotic variance of the corrected estimates. We, however, do not pursue such an exercise here leaving it to a separate research effort that would be more focused on the applications.

In the rest of this section we will perform a Monte Carlo analysis to check whether our asymptotic results approximate finite sample situations well. We perform four different experiments. In our first experiment, we simulate 1000 replications of data having 1-factor structure with $n = 40$, $T = 20$, F_{t1} an AR(1) process with AR coefficient 0.5 and variance 1, $\sigma^2 = 1$, $L_{i1} = \sqrt{d/n}$, and d on a grid 0.1:0.1:20. We repeat the experiment for $n = 200$, $T = 100$. Figure 2 shows the Monte Carlo and theoretical means and 5% and 95% quantiles of the regression coefficient in the regression of $\hat{F}$ on F as functions of d. Smooth solid lines correspond to the theoretical lines obtained using formulae of Theorem 1. According to that theorem, the regression coefficient should be equal to $Q^{(1)} + \frac{1}{\sqrt{T}}Q^{(2)}$. Note that the theoretical lines do not start from $d = 0.1$. It is because our formulae are valid for d larger than the threshold, which is equal to $\sqrt{2}$ for the experiment. Rough solid lines correspond to the Monte Carlo sample data. The left panel is for $n = 40$, $T = 20$. The right panel is for $n = 200$, $T = 100$.

The theoretical mean of the regression coefficient, $Q^{(1)}$, approximates the Monte Carlo mean reasonably well for $n = 40, T = 20$ and very well for $n = 200, T = 100$. The asymptotic quantiles tend to overestimate the amount of finite sample variation in the coefficient for relatively small cumulative effects of the factor. When the cumulative effect approaches the threshold $\sqrt{2}$, the amount of overestimation explodes.

In our next experiment, we simulate 1000 replications of data having 2-factor structure with $n = 40$, $T = 20$, F_{t1} and F_{t2} i.i.d. $N\left(0,1\right)$, $\sigma^2 = 1$, and the following factor loadings. We set $L_{\cdot 1}'L_{\cdot 1} = 10\sqrt{2}$ and $L_{\cdot 2}'L_{\cdot 2} = 2\sqrt{2}$, so that the cumulative effect of the first factor on the cross-sectional units is 10 times the threshold, and the cumulative effect of the second factor is only 2 times the threshold. The vectors of loadings are designed so that their first two components are "unusually" large and the other components are equal by absolute

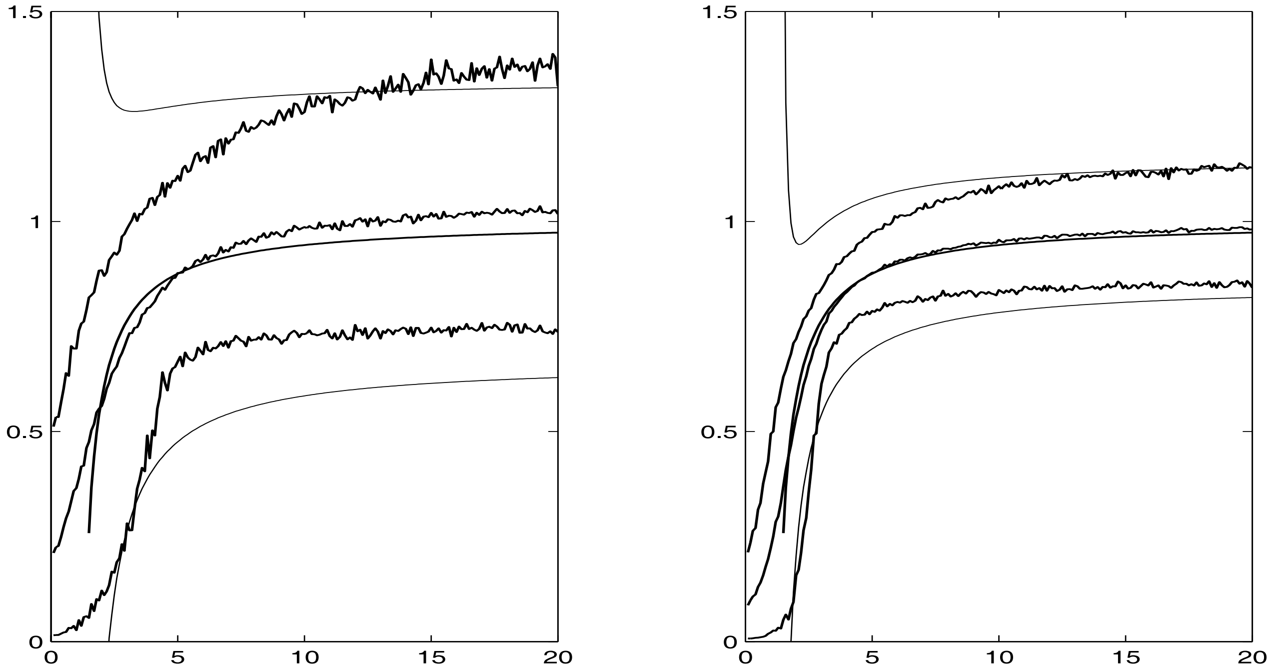

Figure 2: Monte Carlo and theoretical means and 5% and 95% quantiles of the regression coefficient in the regression of $\hat{F}$ on F as functions of d. Horizontal axis: d.

value. Precisely, $L_{11} = L_{21} = \left(10\sqrt{2}/3\right)^{1/2}$, $L_{i1} = \left(10\sqrt{2}/3\,(n-2)\right)^{1/2}$ for $i > 2$, and $L_{12} = -L_{22} = -\left(2\sqrt{2}/3\right)^{1/2}$, $L_{i1} = (-1)^i \left(2\sqrt{2}/3\,(n-2)\right)^{1/2}$ for $i > 2$.

Figure 3 shows the results of the second experiment. The upper three graphs correspond to the joint distributions of (from left to right) the (1st, 2nd), (2nd, 3rd), and (3rd, 4th) components of the normalized (to have unit length) vector of factor loadings corresponding to the first factor. The bottom three graphs correspond to the joint distributions of the same components of the normalized vector of factor loadings corresponding to the second factor. The dots on the graphs correspond to the Monte Carlo draws, the solid lines correspond to 95% confidence ellipses of our theoretical asymptotic distribution (see Corollary 1), the dashed lines correspond to the 95% confidence ellipses of the classical asymptotic distribution (see equation 5), and the dotted lines correspond to the 95% confidence ellipses of the asymptotic distribution under the "strong factor" requirement.

Starting from the upper left graph and going in the clockwise direction, the percentage of the Monte Carlo draws falling inside our ellipse, classical ellipse, and "strong factor ellipse" are, respectively, $(90, 63, 64)$, $(92, 91, 76)$, $(92, 94, 93)$, $(93, 98, 94)$, $(87, 64, 66)$, and $(84, 23, 47)$. Of course, ideally the percentage should be equal to 95. We see that our asymptotic distribution provides a much better approximation to the Monte Carlo distribution than the classical and the "strong factor" asymptotic distributions. The advantage of our distri-

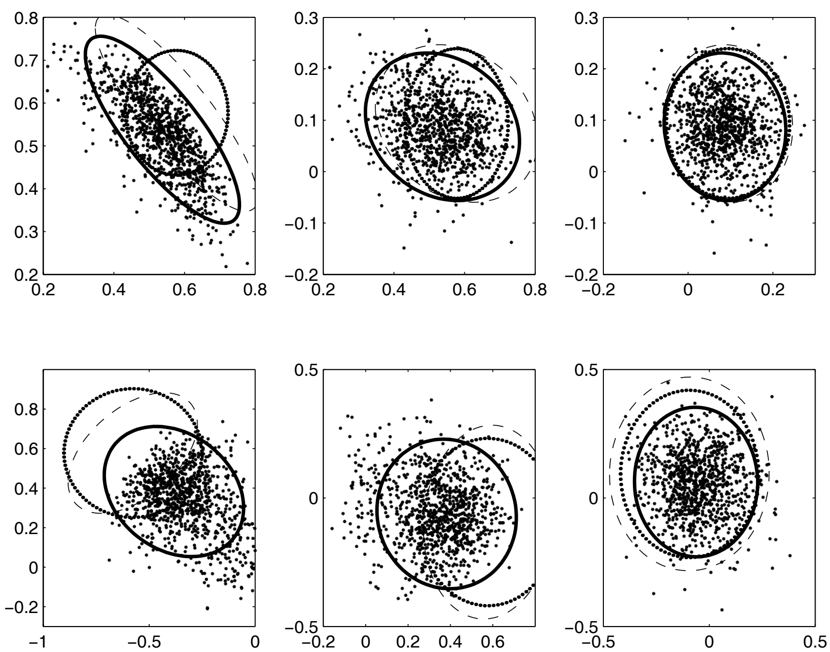

Figure 3: Monte Carlo draws and 95% asymptotic confidence ellipsoids for (from left to right) (1st, 2nd), (2nd, 3rd), (3rd, 4th) components of the normalized vectors of factor loadings. Upper panel: loadings correspond to the first factor. Lower level: loadings correspond to the second factor. Solid line: our asymptotics, dashed line: classical asymptotics. Dotted line: "strong factor" asymptotics.

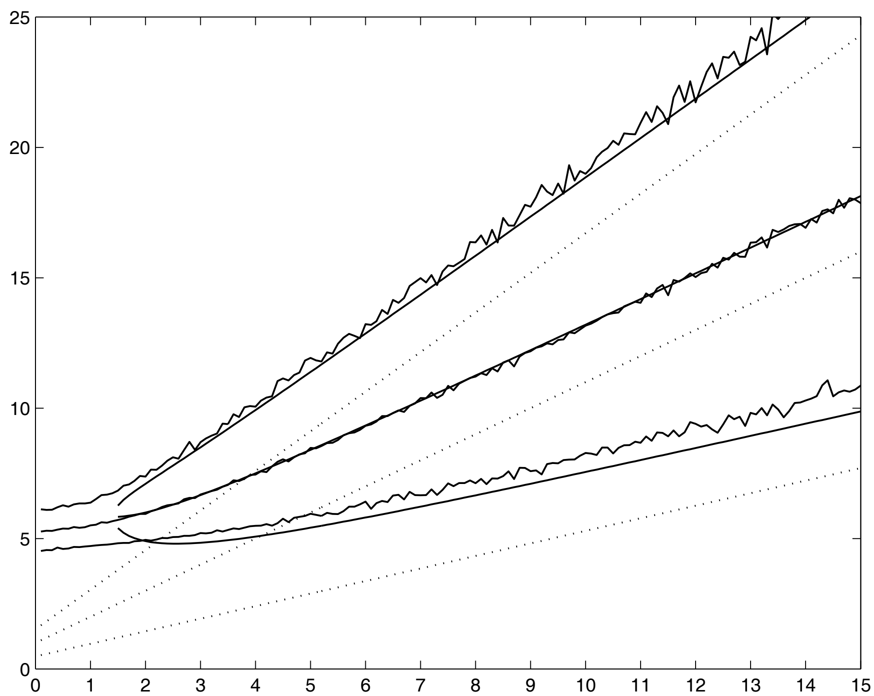

Figure 4: Monte Carlo and asymptotic means and 5% and 95% quantiles of the eigenvalue distribution. Smoothed solid lines- our asymptotics. Dotted lines- classical asymptotics. Horizontal axis: the cumulative effect d of the factor. $n = 40$, $T = 20$.

bution is particularly strong for relatively weak factors and unusually large factor loadings (loadings on the first and second cross-sectional units in our experiment).

In our third experiment, we simulate 1000 replications of data having 1-factor structure with $n = 40$, $T = 20$, F_{t1} i.i.d. $N(0,1)$, $\sigma^2 = 1$, $L_{i1} = \sqrt{d/n}$, and d on a grid 0.1:0.1:20. Figure 4 shows the Monte Carlo and theoretical means and 5% and 95% quantiles of the first eigenvalue of XX'/T as functions of d. Smooth solid lines correspond to the theoretical lines obtained using formulae in Corollary 2. Rough solid lines correspond to the Monte Carlo sample data. Dotted lines are classical theoretical lines (fixed n large T asymptotics). Remarkably, our asymptotic formula for the mean traces the actual finite sample mean very well for all d on the grid. The 5% and 95% asymptotic quantiles also work well. Clearly, our asymptotic distribution provides a much better approximation to the finite sample distribution than the classical distribution.

In our last experiment, we simulate 1000 replications of data from diffusion index forecast model (6) and a 1-factor model with $n = 40$, $T = 20$, $\beta_1 = 1$, $\beta_2 = 1$, $\sigma_\eta^2 = 1$, $\sigma^2 = 1$, $F_t = -1$ for $t \leq 10$, $F_t = 1$ for $t > 10$, $W_t = -2$ for $t \leq 5$, $W_t = 0$ for $t > 5$, and $L_{i1} = \sqrt{d/n}$ for d on a grid 0.1:0.1:20. For each replication, coefficients β_1 and β_2 were estimated using the first 19

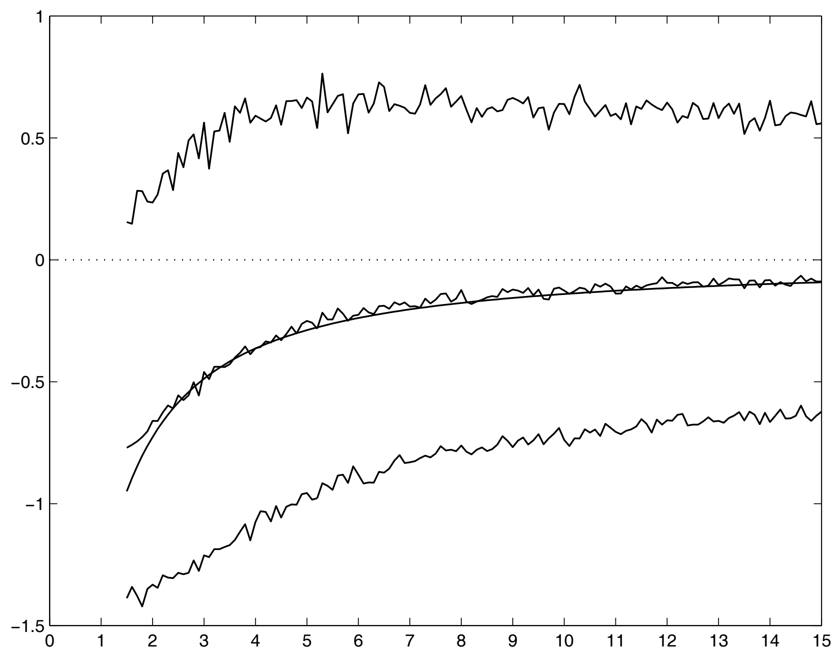

Figure 5: Monte Carlo and theoretical asymptotic means (and Monte Carlo 5% and 95% quantiles) of $\hat{y}_{T+h|T} - y_{T+h|T}$ as functions of the cumulative effect d. Solid line: our asymptotics. Dotted line: "strong factor" asymptotics.

(out of 20) entries of the simulated vector y, observed vector W, and the principal components estimator $\hat{F}$. The last entry of $\hat{F}$ was used to make a forecast of the last entry of y. Figure 5 shows the Monte Carlo and theoretical asymptotic means of the difference between the actual forecast $\hat{y}_{T+h|T} \equiv \hat{\beta}_1 \hat{F}_T + \hat{\beta}_2 W_T$ and the ideal forecast $y_{T+h|T} \equiv \beta_1 F_T + \beta_2 W_T$ as functions of the cumulative effect d of the factor on the cross-sectional units. We also report the Monte Carlo 5% and 95% quantiles to show how volatile the difference $\hat{y}_{T+h|T} - y_{T+h|T}$ is. The dotted line at zero is the theoretical asymptotic mean forecast error under the "strong factor assumption". Our asymptotic mean approximates the Monte Carlo mean very well. The inconsistency of the forecast remain noticeable even for relatively large cumulative effects of the factors.

4 The proofs

In this section, we, first, prove Theorem 5 and, then, prove Theorem 2, Theorem 4, and Proposition 1 in that order. The proofs of Theorems 1 and 3 are completely analogous to the proofs of Theorems 2 and 4 and we omit them to save the space.

4.1 Proof of Theorem 5

The plan of the proof is as follows:

a) Make a transformation of data $X \rightsquigarrow \hat{X}$ *which does not change the eigenvalues of* $\frac{1}{T}XX'$ *so that* $\frac{1}{T}\hat{X}\hat{X}' = \Psi\Psi' + \Lambda$*, where* Λ *is a diagonal matrix and* Ψ *is an* $n \times k$ *matrix. Show that* x *is an eigenvalue of* $\frac{1}{T}\hat{X}\hat{X}'$ *if and only if* 1 *is an eigenvalue of* $\Psi'\left(xI_n - \Lambda\right)^{-1}\Psi$*.*

b) Establish convergence of $\Psi'\left(xI_n - \Lambda\right)^{-1}\Psi$ *considered as a random continuous matrix function of* x *to a non-random function* $M_0^{(1)}(x)$*, and prove that* $\sqrt{n}\left(\Psi'\left(xI_n - \Lambda\right)^{-1}\Psi - M_0^{(1)}(x)\right)$ *converges in distribution to a random matrix function with particular Gaussian finite-dimensional distributions.*

c) Show that the convergence in b) implies the convergence in distribution for the eigenvalues of $\Psi'\left(xI_n - \Lambda\right)^{-1}\Psi$*,* $\mu\left[\Psi'\left(xI_n - \Lambda\right)^{-1}\Psi\right]$*, and establish the asymptotic properties of the solutions to equation* $\mu\left[\Psi'\left(xI_n - \Lambda\right)^{-1}\Psi\right] = 1$ *and, hence, of the eigenvalues of* $\frac{1}{T}XX'$*.*

We now turn to the implementation of a). Let O_L and O_F be $n\times n$ and $T\times T$ orthogonal matrices such that the first k columns of O_L are equal to the columns of $L\left(L'L\right)^{-1/2}$ and the first k columns of O_F are equal to the columns of $F(F'F)^{-1/2}$. Define $\tilde{\varepsilon} = O_L'\varepsilon O_F$ and note that Assumption 2 implies that $\tilde{\varepsilon}$ has i.i.d. standard normal entries.

Denote a matrix that consists of columns $i, i+1, ..., j$ of a matrix A as $A_{i:j}$. Let

$$\frac{1}{T}\tilde{\varepsilon}_{k+1:T}\tilde{\varepsilon}_{k+1:T}' = O'\Lambda O \tag{7}$$

be the spectral decomposition of $\frac{1}{T}\tilde{\varepsilon}_{k+1:T}\tilde{\varepsilon}_{k+1:T}'$. If $c<1$, so that, for large enough n and T, $n < T-k$, then all eigenvalues of $\frac{1}{T}\tilde{\varepsilon}_{k+1:T}\tilde{\varepsilon}_{k+1:T}'$ are positive and different with probability 1. Hence, the eigenvectors collected in matrix O are uniquely defined up to multiplication by -1. We will assume that the sign of the first component of each of the eigenvectors is determined by a flip of a fair coin so that the distribution of O is defined uniquely. Note that, since $\tilde{\varepsilon}_{k+1:T}\tilde{\varepsilon}_{k+1:T}'$ is distributed according to Wishart $W\left(\sigma^2 I_n, T-k\right)$, O has the Haar invariant distribution (see Anderson (1984)). If $c \geq 1$, then $\frac{1}{T}\tilde{\varepsilon}_{k+1:T}\tilde{\varepsilon}_{k+1:T}'$ is singular for large enough n and T so that the last $n-T+k$ eigenvalues are 0 with probability 1. In such a case the last $n-T+k$ rows of O are not uniquely determined. We will assume that they are chosen at random[2] so that O has the Haar invariant distribution as in the case $c<1$.

[2] This choice can be made as follows. Consider the subspace S orthogonal to the first $T-k$ rows of O. Let $(v_{T-k+1}, ..., v_n)$ be any particular basis in S described in the original coordinates. Draw an orthogonal transformation O_1 from the Haar invariant distribution in S. O_1 is represented by an orthogonal matrix in the coordinates of basis $(v_{T-k+1}, ..., v_n)$. Define the $n-T+k$ last rows of O as $O_1\left(v_{T-k+1}, ..., v_n\right)'$.

Define

$$\hat{X} = OO_L'XO_F, \ \varphi = O_{1:k}, \ \eta = \frac{1}{\sigma}O\tilde{\varepsilon}_{1:k}, \text{ and } \Psi = \varphi\left(L'L\right)^{1/2}\left(\frac{F'F}{T}\right)^{1/2} + \frac{\sigma}{\sqrt{T}}\eta. \quad (8)$$

Then, $\frac{1}{T}\hat{X}\hat{X}' = \Psi\Psi' + \Lambda$. Note that the first k eigenvalues of $\frac{1}{T}\tilde{X}\tilde{X}'$ must coincide with the first k eigenvalues of $\frac{1}{T}XX'$. The first k eigenvectors of $\frac{1}{T}XX'$ denoted as $\mathcal{L}_{\cdot j}, j = 1, ..., k$ can be obtained from the first k eigenvectors of $\frac{1}{T}\hat{X}\hat{X}'$, which we will denote as $y_{\cdot j}, j = 1, ..., k$, using the formula $\mathcal{L}_{\cdot j} \equiv O_L O' y_{\cdot j}$.

There are three facts that make matrix $\frac{1}{T}\hat{X}\hat{X}'$ easy to analyze. First, since $\tilde{\varepsilon}_{k+1:T}\tilde{\varepsilon}_{k+1:T}'$ is distributed according to Wishart $W\left(\sigma^2 I_n, T-k\right)$, the diagonal matrix of its eigenvalues, which is proportional to Λ, and the matrix of its eigenvectors, O, are independent (see Anderson (1984), theorem 13.3.3)[3], and therefore Λ and φ are independent. Second, distribution of φ is invariant with respect to multiplication from the left by orthogonal matrices. Last, but not least, the empirical distribution of the elements along the diagonal of Λ defined as $\mathcal{F}^\Lambda \equiv \frac{\#\{\lambda_i \leq \lambda\}}{n}$, where λ_i are the eigenvalues of $\frac{1}{T}\tilde{\varepsilon}_{k+1:T}\tilde{\varepsilon}_{k+1:T}'$ sorted in decreasing order, almost surely converges to a non-random cumulative distribution function $\mathcal{F}_c$, which has density

$$\begin{aligned} f_c(\lambda) &= \begin{cases} \frac{1}{2\pi\lambda c\sigma^2}\sqrt{(b-\lambda)(\lambda-a)} & \textit{if } a \leq \lambda \leq b \\ 0 & \textit{otherwise} \end{cases} \\ a &= \left(1-\sqrt{c}\right)^2\sigma^2, \quad b = \left(1+\sqrt{c}\right)^2\sigma^2, \end{aligned} \quad (9)$$

and a point mass $1 - 1/c$ at $\lambda = 0$ if $c > 1$. This fact was established by Marčenko and Pastur (1967).

Let $\mu_i(A)$ denote the i-th largest eigenvalue of a symmetric matrix A, and let y_{ij} denote the i-th component of the eigenvector of $\frac{1}{T}\hat{X}\hat{X}'$, corresponding to eigenvalue $\mu_j\left(\frac{1}{T}XX'\right)$. Then, if μ_j is an eigenvalue of $\frac{1}{T}XX'$ such that $\mu_j \neq \lambda_i$ for any $i = 1, ..., n$, we have $\Psi_{i\cdot}\Psi' y_{\cdot j} + \lambda_i y_{ij} - \mu_j y_{ij} = 0$, and

$$y_{ij} = \frac{1}{\mu_j - \lambda_i}\Psi_{i\cdot}\Psi' y_{\cdot j}. \quad (10)$$

Multiplying this equality by $\Psi_{i\cdot}'$ and summing over all i, we get $\Psi' y_{\cdot j} = M_n^{(1)}\left(\mu_j\right)\Psi' y_{\cdot j}$, where $M_n^{(1)}(x) \equiv \sum_{i=1}^n \frac{\Psi_{i\cdot}'\Psi_{i\cdot}}{x-\lambda_i}$. Note that $M_n^{(1)}\left(\mu_j\right)$ has an eigenvalue equal to 1. In fact we can prove a stronger result:

[3] The theorem in Anderson (1984) covers only the case of $n \leq T - k$. In the case when $n > T - k$, the theorem is made applicable by our constructing O so that it has Haar invariant distribution.

Lemma 1: *Let $\mu \neq \lambda_i$, $i = 1, ..., n$. Then, μ is an eigenvalue of $\frac{1}{T}\hat{X}\hat{X}'$ (and therefore of $\frac{1}{T}XX'$) if and only if there exists $m \leq k$ such that $x = \mu$ satisfies equation*

$$\mu_m \left(M_n^{(1)}(x) \right) = 1. \tag{11}$$

A proof of this lemma as well as all other auxiliary propositions stated in this section can be found in Appendix.

Corollary 3: *Equation (11) has no more than n different solutions which are not equal to λ_i, $i = 1, ..., n$.*

We now turn to the implementation of part b) of the plan of our proof. Since the eigenvalues of $\frac{1}{T}\hat{X}\hat{X}'$ and $\frac{1}{T}XX'$ coincide and $\frac{1}{T}\hat{X}\hat{X}'$ differs from Λ by a positive semi-definite matrix $\Psi\Psi'$, the first k eigenvalues of $\frac{1}{T}XX'$ must be larger than or equal to $\lambda_1, ..., \lambda_k$. Bai, Silverstein and Yin (1988) show that λ_1 almost surely converges to $b \equiv (1+\sqrt{c})^2 \sigma^2$. Marčenko and Pastur (1967) result on the almost sure convergence of $\mathcal{F}^{\Lambda}$ to $\mathcal{F}_c$ then implies that all of $\lambda_1, ..., \lambda_k$ converge almost surely to b. Therefore, it is reasonable, first, to focus on the behavior of $M_n^{(1)}(x)$ for $x \in [\theta_1, \theta_2]$, where $\theta_2 \gg \theta_1 > b$.

Unfortunately, since λ_1 can occasionally belong to $[\theta_1, \theta_2]$, $M_n^{(1)}(x)$ is not a random element in the well-studied direct product of k^2 versions of the space of continuous functions on the interval $[\theta_1, \theta_2]$, equipped with the max sup norm. We denote such a direct product as $C[\theta_1, \theta_2]^{k^2}$. However, we can construct a matrix-valued function which belongs to $C[\theta_1, \theta_2]^{k^2}$ and is equal to $M_n^{(1)}(x)$ with high probability. Indeed, define $h(x, \lambda_i) = \max\left(x - \lambda_i, \frac{\theta_1 - b}{2}\right)$, and let $\hat{M}_n^{(1)}(x) \equiv \sum_{i=1}^{n} \frac{\Psi_{i\cdot}'\Psi_{i\cdot}}{h(x,\lambda_i)}$. Clearly, $\hat{M}_n^{(1)}(x) \in C[\theta_1, \theta_2]^{k^2}$. Besides

$$P\left(M_n^{(1)}(x) = \hat{M}_n^{(1)}(x), \forall x \in [\theta_1, \theta_2] \right) = P\left(\lambda_1 < \frac{\theta_1 + b}{2} \right) \to 1 \tag{12}$$

as $n \to \infty$. We will, therefore, infer the asymptotic properties of $M_n^{(1)}(x)$ from those of $\hat{M}_n^{(1)}(x)$.

To prove Theorems 2 and 4 we will also need to analyze the asymptotic properties of $M_n^{(2)}(x) \equiv \sum_{i=1}^{n} \frac{\Psi_{i\cdot}'\Psi_{i\cdot}}{(x-\lambda_i)^2}$ and $M_n^{(3)}(x) \equiv \sum_{i=1}^{n} \frac{\varphi_{i\cdot}'\Psi_{i\cdot}}{x-\lambda_i}$. Define

$$\hat{M}_n^{(1)}(x) = \sum_{i=1}^{n} \frac{\Psi_{i\cdot}'\Psi_{i\cdot}}{h(x,\lambda_i)}, \ \hat{M}_n^{(2)}(x) = \sum_{i=1}^{n} \frac{\Psi_{i\cdot}'\Psi_{i\cdot}}{h^2(x,\lambda_i)}, \ \hat{M}_n^{(3)}(x) = \sum_{i=1}^{n} \frac{\varphi_{i\cdot}'\Psi_{i\cdot}}{h(x,\lambda_i)} \tag{13}$$

and let

$$N_n^{(p)}(x) = \sqrt{n}\left(\hat{M}_n^{(p)}(x) - M_0^{(p)}(x) \right), p = 1, 2, 3, \tag{14}$$

where $M_0^{(1)}(x) = (D + \sigma^2 c I_k) \int \frac{d\mathcal{F}_c(\lambda)}{x-\lambda}$, $M_0^{(2)}(x) = (D + \sigma^2 c I_k) \int \frac{d\mathcal{F}_c(\lambda)}{(x-\lambda)^2}$, and $M_0^{(3)}(x) = D^{1/2} \int \frac{d\mathcal{F}_c(\lambda)}{x-\lambda}$. For $N_n^{(p)}(x)$ interpreted as random elements of $C^{k^2}[\theta_1, \theta_2]$, Appendix proves the following

Lemma 2: *Under Assumptions 1 (or 1'), 2, and 3, as n and T go to infinity,*

$$\left\{N_n^{(p)}(x), p = 1, 2, 3\right\} \xrightarrow{d} \left\{N^{(p)}(x), p = 1, 2, 3\right\}, \tag{15}$$

where, for any $\{x_1, ..., x_J\} \in [\theta_1, \theta_2]$, the joint distribution of entries of $\left\{N^{(p)}(x_j); p = 1, 2, 3, j = 1, ..., J\right\}$ is a $3Jk^2$-dimensional normal distribution with covariance between entry in row s and column t of $N^{(p)}(x_j)$ and entry in row s_1 and column t_1 of $N^{(r)}(x_{j_1})$ equal to $\Omega^{(p,r)}(\tau, \tau_1)$, where $\tau = (s, t, j)$ and $\tau_1 = (s_1, t_1, j_1)$, and $\Omega^{(p,r)}(\tau, \tau_1)$ is defined in Appendix.

We now turn to the implementation of the last part of our plan. By Lemma 1, eigenvalues of $\frac{1}{T}XX'$ which are not equal to $\lambda_1, ..., \lambda_n$ coincide with solutions to equations $\mu_j\left(M_n^{(1)}(x)\right) = 1$, $j = 1, ..., k$. On the other hand, Lemma 2 and (12) imply that, for any $\theta_2 > \theta_1 > b$, $M_n^{(1)}(x)$ is well-approximated by $M_0^{(1)}(x)$ for $x \in [\theta_1, \theta_2]$ and large enough n. Therefore, we, first, study solutions to equations $\mu_j\left(M_0^{(1)}(x)\right) = 1$, $j = 1, ..., k$, which are larger than b.

By definition of $M_0^{(1)}(x)$, we have: $\mu_j(M_0^{(1)}(x)) = (d_j + \sigma^2 c) \int \frac{d\mathcal{F}_c(\lambda)}{x-\lambda}$. Clearly, $\mu_j(M_0^{(1)}(x))$ is continuous, decreasing on $(b, +\infty)$, and tends to zero when $x \to +\infty$. Therefore, equation $\mu_j(M_0^{(1)}(x)) = 1$ has a unique solution $x_{0j} > b$ if and only if $\lim_{x \downarrow b} \mu_j(M_0^{(1)}(x)) > 1$. Straightforward calculation using definition of the Marcenko-Pastur law (9) shows that

$$\lim_{x \downarrow b} \int \frac{d\mathcal{F}_c(\lambda)}{x - \lambda} = \frac{1}{c\sigma^2} \frac{\sqrt{c}}{1 + \sqrt{c}}. \tag{16}$$

Therefore, $\lim_{x \downarrow b} \mu_j(M_0^{(1)}(x)) = \left(\frac{d_j}{c\sigma^2} + 1\right) \frac{\sqrt{c}}{1+\sqrt{c}}$ and $\lim_{x \downarrow b} \mu_m(M_0^{(1)}(x)) > 1$ if and only if $d_j > \sqrt{c}\sigma^2$.

Let q be such that $d_j > \sqrt{c}\sigma^2$ for $j \leq q$ and $d_j \leq \sqrt{c}\sigma^2$ otherwise. We will now show that

$$\mu_j\left(\frac{1}{T}XX'\right) \xrightarrow{p} x_{0j} \tag{17}$$

and $\mu_j(\frac{1}{T}XX') \xrightarrow{p} b$ for $q < j \leq k$. Note that the latter convergence is equivalent to statement ii) of the Theorem 5. Fix θ_1 and θ_2 so that $\theta_2 > \theta_1 > b$; $\{x_{0j} : j \leq q\} \in (\theta_1, \theta_2)$, and $(d_k + \sigma^2 c) \int \frac{d\mathcal{F}_c(\lambda)}{\theta_2 - \lambda} < \frac{1}{2}$. According to Weyl's inequality, for any Hermitian $k \times k$ matrices A, B, and C, we have:

$$\mu_{i+j-1}(C) \leq \mu_i(A) + \mu_j(B) \text{ whenever } i + j - 1 \leq k. \tag{18}$$

Using this inequality, we have:

$$\sup_{x\in[\theta_1,\theta_2]}\left|\mu_j(M_n^{(1)}(x))-\mu_j(M_0^{(1)}(x))\right| \leq \sup_{x\in[\theta_1,\theta_2]}\left|\mu_j(M_n^{(1)}(x))-\mu_j(\hat{M}_n^{(1)}(x))\right| + \frac{1}{\sqrt{n}}\max_{j=1,...,k}\sup_{x\in[\theta_1,\theta_2]}\left|\mu_j\left(N_n^{(1)}(x)\right)\right| \xrightarrow{p} 0, \tag{19}$$

where the convergence of the first term on right hand side of the above inequality follows from (12), and the convergence of the second term follows from Lemma 2.

Note that $\mu_j(M_n^{(1)}(x))$ are non-increasing functions of x so that if a solution to $\mu_j(M_n^{(1)}(x)) = 1$ exists, then it is unique or there are infinite number of solutions. However, the second possibility is ruled out by Corollary 3. Therefore, (19) implies that with probability tending to 1 as $n\to\infty$, each of the equations $\mu_j(M_{n1}(x)) = 1$, $j\leq q$ has a unique solution, which belongs to (θ_1,θ_2), and converges in probability to x_{0j}. In contrast, there are no solutions to equations $\mu_j(M_n^{(1)}(x)) = 1$, $q<m\leq k$, which are larger than θ_1. Combining these existence results with Lemma 1 and the fact that $\lambda_1<\theta_1$ with probability tending to 1 as $n\to\infty$, we obtain (17). Further, as we mentioned above, $\mu_j(\frac{1}{T}XX')$, $j=1,...,k$ cannot be smaller than $\lambda_1,...,\lambda_k$, all of which tend almost surely to b. On the other hand, as we have just shown, with probability tending to 1, the roots of $\mu_j(M_n^{(1)}(x)) = 1$, $q<j\leq k$ cannot be larger than θ_1, where θ_1 could have been chosen arbitrarily close to b. Therefore, statement ii) of Theorem 5 follows.

Let us now turn to the proof of statement i). First, note that for $j\leq q$, we have:

$$x_{0j} = m_j. \tag{20}$$

Indeed, as we have just shown, x_{0j} is the solution to $(d_j+\sigma^2 c)\int\frac{d\mathcal{F}_c(\lambda)}{x-\lambda} = 1$ and the probability limit of the j-th eigenvalue of $\frac{1}{T}XX'$. Therefore, it is also the probability limit of the j-th eigenvalue of $\frac{n}{T}\left(\frac{1}{n}X'X\right)$. Changing roles of factors and factor loadings, it is straightforward to show that y_{0j} defined as the solution to $\left(cd_j+\sigma^2\frac{1}{c}\right)\int\frac{d\mathcal{F}_{\frac{1}{c}}(\lambda)}{y-\lambda} = 1$ must be the probability limit of the j-th eigenvalue of $\frac{1}{n}X'X$. Hence, $x_{0j} = cy_{0j}$ and

$$\frac{d_j+\sigma^2}{c}\int\frac{\mathcal{F}_{\frac{1}{c}}(d\lambda)}{\frac{1}{c}x_{0j}-\lambda} = 1. \tag{21}$$

Now, it is straightforward to check that $f_{\frac{1}{c}}(\lambda) = c^2 f_c(c\lambda)$ and $\mathcal{F}_{\frac{1}{c}}$ does not have mass at zero if $c>1$ and has mass at zero equal to $1-c$ if $c<1$. Therefore, (21) can be rewritten as $c\,(d_j+\sigma^2)\left(\int\frac{\mathcal{F}_c(d\lambda)}{x_{0j}-\lambda}-\frac{1-\frac{1}{c}}{x_{0j}}\right) = 1$. Substituting $\int\frac{\mathcal{F}_c(d\lambda)}{x_{0j}-\lambda}$ by $(d_j+\sigma^2 c)^{-1}$ in the latter equation, we get $1 = c\left(d_j + \sigma^2\right)\left(\left(d_j + \sigma^2 c\right)^{-1} - \frac{1-\frac{1}{c}}{x_{0j}}\right)$, which implies (20). Note that (20) implies in particular that

$$\left(d_j + \sigma^2 c\right)\int \frac{d\mathcal{F}_c(\lambda)}{m_j - \lambda} = 1 \tag{22}$$

Next, we prove the following formula. For any $j \leq q$, we have:

$$\mu_j(\hat{M}_n^{(1)}(x)) = \mu_j\left(M_0^{(1)}(x)\right) + \frac{1}{\sqrt{n}} N_{n,jj}^{(1)}(x) + o_p\left(\frac{1}{\sqrt{n}}\right), \tag{23}$$

where $o_p\left(\frac{1}{\sqrt{n}}\right)$ is understood as a random element of $C\left[\theta_1, \theta_2\right]$, which, when multiplied by $\sqrt{n}$, tends in probability to zero as $n \to \infty$. To prove the formula we need an additional lemma.

Let $A^{(1)}$ be a symmetric $k \times k$ matrix and A be a $k \times k$ diagonal matrix with distinct positive diagonal elements. That is $A = \operatorname{diag}\left(a_1, a_2, ..., a_p\right),\ a_1 > a_2 > ... > a_p > 0$. And let $r_0 = \frac{1}{2}\min_{j=1,\dots,k}\left|a_j - a_{j+1}\right|$, where we define a_{k+1} as zero. Note that A and $A^{(1)}$ can be interpreted as matrix representations of linear operators acting in the space R^k. For any linear bounded operator on R^k, B, we define its norm as $\|B\| = \left(\max\operatorname{eval}\left(B^*B\right)\right)^{1/2}$, which is the operator norm induced by the standard Euclidean norm of vectors in R^k. Let $S_j = \operatorname{diag}\left(\frac{1}{a_1 - a_j}, \dots \frac{1}{a_{j-1} - a_j}, 0, \frac{1}{a_{j+1} - a_j}, \dots, \frac{1}{a_k - a_j}\right)$. We have the following

Lemma 3: *Let $A(\varkappa) = A + \varkappa A^{(1)}$. For real $\varkappa$ such that $|\varkappa| < \frac{r_0}{\left\|A^{(1)}\right\|}$, the following two statements hold:*

i) Exactly one eigenvalue of $A(\varkappa)$ belongs to the segment $\left(a_j - r_0, a_j + r_0\right)$. Denoting this eigenvalue as $a_j(\varkappa)$, we have: $\left|\frac{1}{\varkappa}\left(a_j(\varkappa) - a_j\right) - A_{jj}^{(1)}\right| \leq \frac{|\varkappa|\left\|A^{(1)}\right\|}{r_0 - |\varkappa|\left\|A^{(1)}\right\|}$.

ii) Let $P_j(\varkappa)$ be the orthogonal projection on the invariant subspace of $A(\varkappa)$ corresponding to eigenvalue $a_j(\varkappa)$. Then $e_j(\varkappa) \equiv \frac{P_j(\varkappa)e_j}{\|P_j(\varkappa)e_j\|}$ is an eigenvector of $A(\varkappa)$ corresponding to eigenvalue $a_j(\varkappa)$, and $\left\|\frac{1}{\varkappa}\left(e_j(\varkappa) - e_j\right) + SA^{(1)}e_j\right\| \leq 2\frac{|\varkappa|\left\|A^{(1)}\right\|^2}{\left(r_0 - |\varkappa|\left\|A^{(1)}\right\|\right)^2}$.

Formula (23) follows from the above Lemma. Indeed, let $r_0 = \frac{1}{2}\min_{j=1,\dots,p}\left|d_j - d_{j+1}\right|\int\frac{d\mathcal{F}_c(\lambda)}{\theta_1 - \lambda}$, where d_{p+1} is defined as 0. Then, by Lemma 3, if $\sup_{x\in[\theta_1,\theta_2]}\left\|N_n^{(1)}(x)\right\| < \sqrt{n}r_0$, we have $\sup_{x\in[\theta_1,\theta_2]}\left|\sqrt{n}\left(\mu_j(\hat{M}_n^{(1)}(x)) - \mu_j\left(M_0^{(1)}(x)\right)\right) - N_{n,jj}^{(1)}(x)\right| \leq \frac{1}{\sqrt{n}}\frac{\sup_{x\in[\theta_1,\theta_2]}\left\|N_n^{(1)}(x)\right\|}{r_0 - \frac{1}{\sqrt{n}}\sup_{x\in[\theta_1,\theta_2]}\left\|N_n^{(1)}(x)\right\|}$ for any integer $j \le k$. Therefore, for any $\varepsilon > 0$

$$\begin{aligned} & P\left(\sup_{x\in[\theta_1,\theta_2]}\left|\sqrt{n}\left(\mu_j(\hat{M}_n^{(1)}(x))-\mu_j\left(M_0^{(1)}(x)\right)\right)-N_{n,jj}^{(1)}(x)\right|>\varepsilon\right) \\ \le\ & P\left(\sup_{x\in[\theta_1,\theta_2]}\left\|N_n^{(1)}(x)\right\|\ge\sqrt{n}r_0\right)+P\left(\frac{1}{\sqrt{n}}\frac{\sup_{x\in[\theta_1,\theta_2]}\left\|N_n^{(1)}(x)\right\|}{r_0-\frac{1}{\sqrt{n}}\sup_{x\in[\theta_1,\theta_2]}\left\|N_n^{(1)}(x)\right\|}>\varepsilon\right). \end{aligned}$$

But both terms in the sum on right hand side of the above inequality tend to zero as $n \to \infty$ as follows from Lemma 2, and hence, (23) holds.

Define function $\nu_j(y)$ for $y > 0$ so that it is equal to b if $y > \lim_{x\downarrow b}\mu_j(M_0^{(1)}(x))$ and to x satisfying equation $y = \mu_j(M_0^{(1)}(x))$ otherwise. Since $\frac{d}{dx}\mu_j(M_0^{(1)}(x)) = -\left(d_j + \sigma^2 c\right)\int\frac{d\mathcal{F}_c(\lambda)}{(x-\lambda)^2}$, it is easy to see that $\lim_{x\downarrow b}\frac{d}{dx}\mu_j(M_0(x)) = +\infty$, and, hence, $\nu_j(y)$ is differentiable for $y > 0$. Applying ν_j to both sides of (23) and using the first order Taylor expansion of the right hand side, we have for $x \in [\theta_1, \theta_2]$:

$$\nu_j\left(\mu_j(\hat{M}_n^{(1)}(x))\right) = x + \nu_j'\left(\tau_n(x)\right)\frac{1}{\sqrt{n}}N_{n,jj}^{(1)}(x) + o_p\left(\frac{1}{\sqrt{n}}\right), \tag{24}$$

where $\tau_n(x)$ is a random element of $C[\theta_1, \theta_2]$ such that $\left|\tau_n(x) - \mu_j(M_0^{(1)}(x))\right| \le \frac{1}{\sqrt{n}}N_{n,jj}^{(1)}(x) + o_p\left(\frac{1}{\sqrt{n}}\right)$. Note that the latter inequality implies that $\tau_n(x) \xrightarrow{p} \mu_j(M_0^{(1)}(x))$ as $n \to \infty$. (24) can be conveniently rewritten as

$$\sqrt{n}\left(x - \nu_j\left(\mu_j(\hat{M}_n^{(1)}(x))\right)\right) = -\nu_j'\left(\tau_n(x)\right)N_{n,jj}^{(1)}(x) + o_p(1). \tag{25}$$

Let Ω_n be a set of elementary events for which the first q eigenvalues of $\frac{1}{T}XX'$ belong to $[\theta_1, \theta_2]$, the other eigenvalues are less than θ_1, and $\lambda_1 < \frac{\theta_1+b}{2}$. Note that event Ω_n implies that $\hat{M}_n^{(1)}(x) = M_0^{(1)}(x)$ for any $x \in [\theta_1, \theta_2]$. (17), statement ii) of Theorem 5, and the almost sure convergence of λ_1 to b imply that $P(\Omega_n) \to 1$ as $n \to \infty$. For $j = 1, ..., q$, define random variables x_{nj} as follows:

$$x_{nj} = 1_{\Omega_n}\mu_j\left(\frac{1}{T}XX'\right) + 1_{\bar{\Omega}_n}x_{0j}, \tag{26}$$

where 1_{Ω_n} and $1_{\bar{\Omega}_n}$ denote indicators of event Ω_n and event complementary to Ω_n respectively. Note that $x_{nj} \in [\theta_1, \theta_2]$, and, by (17), $x_{nj} \xrightarrow{p} x_{0j}$. By Lemma 1, Ω_n implies that the j-th eigenvalue of $\frac{1}{T}XX'$ is the root of $\mu_j\left(\hat{M}_n^{(1)}(x)\right) = 1$. Therefore, (26) implies that $\nu_j\left(\mu_j(\hat{M}_n^{(1)}(x_{nj}))\right) = 1_{\Omega_n}\nu_j(1) + 1_{\bar{\Omega}_n}\nu_j\left(\mu_j(\hat{M}_n^{(1)}(x_{0j}))\right) = 1_{\Omega_n}x_{0j} + 1_{\bar{\Omega}_n}\nu_j\left(\mu_j(\hat{M}_n^{(1)}(x_{0j}))\right)$.

Substituting x by x_{nj} in (25), and using the above representation of $\nu_j\left(\mu_j(\hat{M}_n^{(1)}(x_{nj}))\right)$, we get:

$$\sqrt{n}\left(\mu_j\left(\frac{1}{T}XX'\right)-x_{0j}\right) = -1_{\bar{\Omega}_n}\sqrt{n}\left(2x_{0j}-\mu_j\left(\frac{1}{T}XX'\right)-\nu_j\left(\mu_j(\hat{M}_n^{(1)}(x_{0j}))\right)\right) -\nu_j'\left(\tau_n(x_{nj})\right)N_{n,jj}^{(1)}(x_{nm})+o_p\left(1\right). \quad (27)$$

The first term on right hand side of the above equality tends to zero in probability because $P\left(\Omega_n\right)\to 1$. Further, since, as we showed above, $x_{nj}\xrightarrow{p}x_{0j}\equiv m_j$, we have: $\nu_j'\left(\tau_n(x_{nj})\right)\xrightarrow{p}\nu_j'\left(1\right)$. Finally, $N_{n,jj}^{(1)}(x_{nm})-N_{n,jj}^{(1)}(m_j)\xrightarrow{p}0$, which follows from Lemma 2 and the following additional

Lemma 4: *Let $f_n(x)$ and $f_0(x)$ be random elements of $C\left[\theta_1,\theta_2\right]$ such that $f_n(x)\xrightarrow{d}f_0(x)$ as $n\to\infty$. And let x_n be random variables with values form $[\theta_1,\theta_2]$ and such that $x_n\xrightarrow{p}x_0$, where $x_0\in[\theta_1,\theta_2]$ Then $f_n(x_n)-f_n(x_0)\xrightarrow{p}0$.*

Therefore, equality (27) can be rewritten as

$$\sqrt{n}\left(\mu_j\left(\frac{1}{T}XX'\right)-m_j\right)=-\nu_j'\left(1\right)N_{n,jj}^{(1)}(m_j)+o_p\left(1\right). \quad (28)$$

Finally, we have: $\nu_j'\left(1\right)=\left(\mu_j'\left(M_0^{(1)}(m_j)\right)\right)^{-1}=\left(-\left(d_j+\sigma^2c\right)\int\frac{d\mathcal{F}_c(\lambda)}{(m_j-\lambda)^2}\right)^{-1}$. The latter expression can be simplified as follows. Consider m_j as a function of d_j : $m_j=\left(d_j+\sigma^2\right)\left(d_j+\sigma^2c\right)/d_j$. Differentiating both sides of (22) with respect to d_j, we get: $\left(d_j+\sigma^2c\right)^{-1}-\left(d_j+\sigma^2c\right)\int\frac{d\mathcal{F}_c(\lambda)}{(m_j-\lambda)^2}\left(1-\frac{\sigma^4c}{d_j^2}\right)=0$. Solving this equation for the integral, we get:

$$\int\frac{d\mathcal{F}_c(\lambda)}{\left(m_j-\lambda\right)^2}=\frac{d_j^2}{\left(d_j^2-\sigma^4c\right)\left(d_j+\sigma^2c\right)^2}, \quad (29)$$

and therefore

$$-\nu_j'\left(1\right)=\frac{\left(d_j^2-\sigma^4c\right)\left(d_j+\sigma^2c\right)}{d_j^2}. \quad (30)$$

Now, (28), (30), and Lemma 2 imply statement i) of Theorem 5.□

4.2 Proof of Theorem 2

First, note that representation $\hat{\mathcal{L}}_{1:q}=\mathcal{L}\cdot R+\mathcal{L}_q^{\perp}$, where $\mathcal{L}_q^{\perp}$ is a matrix with q columns orthogonal to span$\left(\mathcal{L}\right)$ is a trivial coordinate decomposition statement. The value of Theorem 2 is, therefore, contained in describing properties of $\mathcal{L}_q^{\perp}$ and R. Recall that the columns of $\hat{\mathcal{L}}_{1:q}$ are equal to the q principal eigenvectors of $\frac{1}{T}XX'$. By Assumption 2, the joint distribution of elements of X is invariant with respect to multiplication of X from the left by any orthogonal matrix leaving columns of L unchanged. This immediately implies that the joint distribution of entries of $\mathcal{L}_q^{\perp}$ is invariant with respect to the multiplication of $\mathcal{L}_q^{\perp}$ from the left by any orthogonal matrix that has $\operatorname{span}(\mathcal{L}) = \operatorname{span}(L)$ as its invariant subspace. In the rest of the proof we, therefore, focus on the properties of R.

Recall that $\hat{\mathcal{L}}_{\cdot i}$ is equal to the i-th unit-length eigenvector of $\frac{1}{T}XX'$, u_i. It must be equal to the the i-th eigenvector of $\frac{1}{T}\hat{X}\hat{X}'$, $y_{\cdot i}$, multiplied from the left by $O_L O'$, that is:

$$\hat{\mathcal{L}}_{\cdot i} = u_i = O_L O' y_{\cdot i}. \tag{31}$$

By definition, the columns of O_L constitute an orthonormal basis in n-dimensional space, and $O_{L,\cdot r} = \mathcal{L}_{\cdot r}$ for $r \leq k$. Therefore, $R_{\cdot i}$ can be interpreted as the vector of the first k coordinates of $\hat{\mathcal{L}}_{\cdot i}$ in the basis O_L.

Let Ω_n, and x_{ni} be as in the proof of Theorem 5. Using (31), (8), (10), and (13), we obtain:

$$R_{\cdot i} = 1_{\Omega_n} \hat{M}_n^{(3)}(x_{ni})\, \Psi' y_{\cdot i} + 1_{\bar{\Omega}_n} R_{\cdot i}.$$

We, first, find the probability limit of $R_{\cdot i}$ as $n \to \infty$. Lemma 2, Lemma 4, the fact that $x_{nj} \xrightarrow{p} m_j$, and (22) imply that

$$\hat{M}_n^{(3)}(x_{ni}) \xrightarrow{p} M_0^{(3)}(m_j) \equiv D^{1/2}\left(d_i + \sigma^2 c\right)^{-1}. \tag{32}$$

Further, denote $\Psi' y_{\cdot i} / \|\Psi' y_{\cdot i}\|$ as v_{ni}, and let $\tilde{v}_{ni}$ be a unit length eigenvector of $\hat{M}_n^{(1)}(x_{ni})$ normalized so that its i-th component is positive. By Lemma 1 and definition of x_{ni},

$$v_{ni} = 1_{\Omega_n} \tilde{v}_{ni} + 1_{\bar{\Omega}_n} v_{ni}.$$

But, by Lemma 3, $\tilde{v}_{ni} \xrightarrow{p} e_i$. Therefore,

$$v_{ni} \xrightarrow{p} e_i. \tag{33}$$

Further, from (10) we have, for any elementary event from Ω_n, $1 = \sum_{j=1}^{n} y_{ji}^2 = y_{\cdot i}' \Psi \hat{M}_n^{(2)}(x_{ni})\, \Psi' y_{\cdot i}$, and hence

$$\|\Psi' y_{\cdot i}\| = 1_{\Omega_n}\left(\tilde{v}_{ni}' \hat{M}_n^{(2)}(x_{ni})\, \tilde{v}_{ni}\right)^{-1/2} + 1_{\bar{\Omega}_n} \|\Psi' y_{\cdot i}\|. \tag{34}$$

But by Lemma 2, and Lemma 4, $\hat{M}_n^{(2)}(x_{ni}) \xrightarrow{p} (D + \sigma^2 c I_k) \int \frac{d\mathcal{F}_c(\lambda)}{(m_i - \lambda)^2}$. Using this fact, (34), and (33) we get:

$$\|\Psi' y_{\cdot i}\| \xrightarrow{p} \left(\left(d_i + \sigma^2 c\right) \int \frac{d\mathcal{F}_c(\lambda)}{(m_i - \lambda)^2}\right)^{-\frac{1}{2}}. \tag{35}$$

Combining (32), (33), (35), and (29), we get:

$$R_{\cdot i} \xrightarrow{p} \left(\frac{d_i^2 - \sigma^4 c}{d_i \left(d_i + \sigma^2 c\right)} \right)^{1/2} e_i$$

which establishes the form of $R^{(1)}$.

Now, we will study the asymptotic behavior of R around its probability limit $R^{(1)}$. Noting that $1_{\bar{\Omega}_n}$ multiplied by any random variable is $o_p(1)$, we have:

$$\begin{aligned} \sqrt{n}\left(R_{\cdot i} - R_{\cdot i}^{(1)}\right) &= \sqrt{n}\left(\hat{M}_n^{(3)}\left(x_{ni}\right) - D^{1/2}\int \frac{d\mathcal{F}_c(\lambda)}{m_i - \lambda}\right)\tilde{v}_{ni}\left\|\Psi' y_{\cdot i}\right\| \\ &+ D^{1/2}\int \frac{d\mathcal{F}_c(\lambda)}{m_i - \lambda}\sqrt{n}\left(\tilde{v}_{ni} - e_i\right)\left\|\Psi' y_{\cdot i}\right\| \\ &+ D^{1/2}\int \frac{d\mathcal{F}_c(\lambda)}{m_i - \lambda} e_i \sqrt{n}\left(\left\|\Psi' y_{\cdot i}\right\| - p\lim\left\|\Psi' y_{\cdot i}\right\|\right) + o_p(1) \end{aligned} \tag{36}$$

Note that the first term in the sum on right hand side of the above equality is equal to $N_n^{(3)}(x_{ni})\tilde{v}_{ni}\left\|\Psi' y_{\cdot i}\right\| + D^{1/2}\sqrt{n}\left(\int \frac{d\mathcal{F}_c(\lambda)}{x_{ni}-\lambda} - \int \frac{d\mathcal{F}_c(\lambda)}{m_i-\lambda}\right)\tilde{v}_{ni}\left\|\Psi' y_{\cdot i}\right\|$. Using Taylor expansion of function $\int \frac{d\mathcal{F}_c(\lambda)}{x-\lambda}$ around $x = m_i$, the latter can be transformed into $N_n^{(3)}(x_{ni})\tilde{v}_{ni}\left\|\Psi' y_{\cdot i}\right\| - D^{1/2}\int \frac{d\mathcal{F}_c(\lambda)}{(m_i-\lambda)^2}\sqrt{n}\left(x_{ni} - m_i\right)\tilde{v}_{ni}\left\|\Psi' y_{\cdot i}\right\| + o\left(\sqrt{n}\left(x_{ni} - m_i\right)\right)$. Note that, by Theorem 5, $o\left(\sqrt{n}\left(x_{ni} - m_i\right)\right)$ is $o_p(1)$, and therefore, summarizing the above transformations, we have:

$$\begin{aligned} \sqrt{n}\left(R_{\cdot i} - R_{\cdot i}^{(1)}\right) &= N_n^{(3)}(x_{ni})\tilde{v}_{ni}\left\|\Psi' y_{\cdot i}\right\| - D^{1/2}\int \frac{d\mathcal{F}_c(\lambda)}{\left(m_i - \lambda\right)^2}\sqrt{n}\left(x_{ni} - m_i\right)\tilde{v}_{ni}\left\|\Psi' y_{\cdot i}\right\| \\ &+ D^{1/2}\int \frac{d\mathcal{F}_c(\lambda)}{m_i - \lambda}\sqrt{n}\left(\tilde{v}_{ni} - e_i\right)\left\|\Psi' y_{\cdot i}\right\| \\ &+ D^{1/2}\int \frac{d\mathcal{F}_c(\lambda)}{m_i - \lambda} e_i \sqrt{n}\left(\left\|\Psi' y_{\cdot i}\right\| - p\lim\left\|\Psi' y_{\cdot i}\right\|\right) + o_p(1) \end{aligned} \tag{37}$$

Let us focus on the last term in the above expression. Obviously, $\left\|\Psi' y_{\cdot i}\right\| = \left(\left\|\Psi' y_{\cdot i}\right\|^{-2}\right)^{-1/2}$. Using Taylor expansion of function $x^{-1/2}$ around probability limit of $\left\|\Psi' y_{\cdot i}\right\|^{-2}$, we get:

$$\begin{aligned} \left(\left\|\Psi' y_{\cdot i}\right\|^{-2}\right)^{-1/2} &= \left(p\lim\left\|\Psi' y_{\cdot i}\right\|^{-2}\right)^{-1/2} \\ &- \frac{1}{2}\left(p\lim\left\|\Psi' y_{\cdot i}\right\|^{-2}\right)^{-3/2}\left(\left\|\Psi' y_{\cdot i}\right\|^{-2} - p\lim\left\|\Psi' y_{\cdot i}\right\|^{-2}\right) \\ &+ o\left(\left\|\Psi' y_{\cdot i}\right\|^{-2} - p\lim\left\|\Psi' y_{\cdot i}\right\|^{-2}\right). \end{aligned}$$

Substituting $p\lim\left\|\Psi' y_{\cdot i}\right\|^{-2}$ and $\left\|\Psi' y_{\cdot i}\right\|^{-2}$ by their expressions from (35) and (34), we obtain a formula for $\left(\|\Psi' y_{\cdot i}\|^{-2}\right)^{-1/2}$, which (together with (22) and (29)) implies:

$$\begin{aligned}
& D^{1/2}\int \frac{d\mathcal{F}_c(\lambda)}{m_i-\lambda} e_i\sqrt{n}\left(\|\Psi' y_{\cdot i}\| - p\lim \|\Psi' y_{\cdot i}\|\right) \\
= \; & \varrho_i e_i \sqrt{n}\left(\tilde{v}_{ni}'\hat{M}_n^{(2)}(x_{ni})\,\tilde{v}_{ni} - \left(d_i+\sigma^2 c\right)\int \frac{d\mathcal{F}_c(\lambda)}{(m_i-\lambda)^2}\right) \\
& + o\left(\sqrt{n}\left(\|\Psi' y_{\cdot i}\|^{-2} - p\lim \|\Psi' y_{\cdot i}\|^{-2}\right)\right) + o_p(1) \\
= \; & \varrho_i e_i \left(\tilde{v}_{ni}+e_i\right)'\hat{M}_n^{(2)}(x_{ni})\sqrt{n}\left(\tilde{v}_{ni}-e_i\right) + \varrho_i e_i e_i' N_n^{(2)}(x_{ni}) e_i \\
& - 2\varrho_i e_i\left(d_i+\sigma^2 c\right)\int \frac{d\mathcal{F}_c(\lambda)}{(m_i-\lambda)^3}\sqrt{n}\left(x_{ni}-m_i\right) \\
& + o\left(\sqrt{n}\left(\|\Psi' y_{\cdot i}\|^{-2} - p\lim \|\Psi' y_{\cdot i}\|^{-2}\right)\right) + o_p(1),
\end{aligned} \tag{38}$$

where

$$\varrho_i = -\frac{1}{2}\left(\frac{\left(d_i^2-\sigma^4 c\right)^3\left(d_i+\sigma^2 c\right)}{d_i^5}\right)^{1/2}.$$

Combining (37) and (38), we get:

$$\sqrt{n}\left(R_{\cdot i} - R_{\cdot i}^{(1)}\right) = A_i^{(1)} + A_i^{(2)} + A_i^{(3)} + A_i^{(4)} + A_i^{(5)},$$

where

$$\begin{aligned}
A_i^{(1)} &= N_n^{(3)}(x_{ni})\tilde{v}_{ni}\|\Psi' y_{\cdot i}\|, \\
A_i^{(2)} &= \varrho_i e_i e_i' N_n^{(2)}(x_{ni}) e_i, \\
A_i^{(3)} &= -\left(D^{1/2}\int \frac{d\mathcal{F}_c(\lambda)}{(m_i-\lambda)^2}\tilde{v}_{ni}\|\Psi' y_{\cdot i}\| + 2\varrho_i e_i\left(d_i+\sigma^2 c\right)\int \frac{d\mathcal{F}_c(\lambda)}{(m_i-\lambda)^3}\right)\sqrt{n}\left(x_{ni}-m_i\right), \\
A_i^{(4)} &= \left(D^{1/2}\int \frac{d\mathcal{F}_c(\lambda)}{m_i-\lambda}\|\Psi' y_{\cdot i}\| + \varrho_i e_i\left(\tilde{v}_{ni}+e_i\right)'\hat{M}_n^{(2)}(x_{ni})\right)\sqrt{n}\left(\tilde{v}_{ni}-e_i\right), \\
A_i^{(5)} &= o\left(\sqrt{n}\left(\|\Psi' y_{\cdot i}\|^{-2} - p\lim \|\Psi' y_{\cdot i}\|^{-2}\right)\right) + o_p(1).
\end{aligned}$$

Using the same argument as that for the proof of (29), we obtain

$$\int \frac{d\mathcal{F}_c(\lambda)}{(m_i-\lambda)^3} = \frac{\left(d_i^3+c^2\sigma^6\right)d_i^3}{\left(d_i+c\sigma^2\right)^3\left(d_i^2-c\sigma^4\right)^3} \tag{39}$$

$$\int \frac{d\mathcal{F}_c(\lambda)}{(m_i-\lambda)^4} = \frac{\left(d_i^6+c^4\sigma^{12}+c\sigma^4 d_i^4+4c^2\sigma^6 d_i^3+c^3\sigma^8 d_i^2\right)d_i^4}{\left(d_i+c\sigma^2\right)^4\left(d_i^2-c\sigma^4\right)^5} \tag{40}$$

By (33), (35), the fact that $x_{ni} \xrightarrow{p} m_i$, Lemma 4, and formula (39) the distribution limit of $\left\{\sum_{j=1}^{5} A_i^{(j)}, i=1,...,q\right\}$ must be the same as that of $\left\{\sum_{j=1}^{5} \tilde{A}_i^{(j)}, i=1,...,q\right\}$, where

$$
\begin{aligned}
\tilde{A}_i^{(1)} &= N_n^{(3)}(m_i)e_i\left(\frac{\left(d_i^2-\sigma^4 c\right)\left(d_i+\sigma^2 c\right)}{d_i^2}\right)^{1/2}, \\
\tilde{A}_i^{(2)} &= \varrho_i e_i e_i' N_n^{(2)}(m_i)e_i, \\
\tilde{A}_i^{(3)} &= -e_i\frac{\sigma^4 c}{2\varrho_i d_i^2}\sqrt{n}\left(x_{ni}-m_i\right), \\
\tilde{A}_i^{(4)} &= \left(\frac{d_i^2-\sigma^4 c}{\left(d_i+\sigma^2 c\right)d_i}\right)^{1/2}\left(d_i^{-1/2}D^{1/2}-e_i e_i'\right)\sqrt{n}\left(\tilde{v}_{ni}-e_i\right), \\
\tilde{A}_i^{(5)} &= o\left(\sqrt{n}\left(\left\|\Psi' y_{\cdot i}\right\|^{-2}-p\lim\left\|\Psi' y_{\cdot i}\right\|^{-2}\right)\right).
\end{aligned}
$$

Further, (26), (28), and (30) imply that $\tilde{A}_i^{(3)}$ can be redefined as

$$
\tilde{A}_i^{(3)} = -e_i\frac{\sigma^4 c}{2\varrho_i d_j^2}\frac{\left(d_i^2-c\sigma^4\right)\left(d_i+c\sigma^2\right)}{d_i^2}e_i' N_n^{(1)}\left(m_i\right)e_i
$$

without changing the distribution limit of $\left\{\sum_{j=1}^{5} \tilde{A}_i^{(j)}, i=1,...,q\right\}$.

We now focus on $\tilde{A}_i^{(4)}$ and will relate $\sqrt{n}\left(\tilde{v}_{ni}-e_i\right)$ to elements of matrix $N_n^{(1)}\left(m_i\right)$. From statement ii) of Lemma 3, we have

$$
\left\|\sqrt{n}\left(\tilde{v}_{ni}-e_i\right)+\tilde{S}\left(x_{ni}\right)N_n^{(1)}\left(x_{ni}\right)e_i\right\| \leq \frac{2}{\sqrt{n}}\frac{\left\|N_n^{(1)}\left(x_{ni}\right)\right\|^2}{\left(r_0-\frac{1}{\sqrt{n}}\left\|N_n^{(1)}\left(x_{ni}\right)\right\|\right)^2}
$$

if $\left\|N_n^{(1)}\left(x_{ni}\right)\right\| < \sqrt{n}r_0$, where r_0 was defined in the proof of Theorem 5 and

$$
\tilde{S}\left(x\right) = \left(\int\frac{d\mathcal{F}_c(\lambda)}{x-\lambda}\right)^{-1}\operatorname{diag}\left(\frac{1}{d_1-d_i},...,\underbrace{0}_{i\text{-th position}},...,\frac{1}{d_k-d_i}\right)
$$

Repeating arguments used in the proof of Theorem 5 following formula (23), we have

$$
\sqrt{n}\left(\tilde{v}_{ni}-e_i\right) = -\tilde{S}\left(x_{ni}\right)N_n^{(1)}\left(x_{ni}\right)e_i+o_p(1)
$$

Therefore, since $x_{ni}\xrightarrow{p} m_i$, $\tilde{A}_i^{(4)}$ can be redefined as

$$
\tilde{A}_i^{(4)} = -\left(\frac{d_i^2-\sigma^4 c}{\left(d_i+\sigma^2 c\right)d_i}\right)^{1/2}\left(d_i^{-1/2}D^{1/2}-e_i e_i'\right)\tilde{S}\left(m_i\right)N_n^{(1)}\left(m_i\right)e_i
$$

without changing the distribution limit of $\left\{\sum_{j=1}^{5} \tilde{A}_i^{(j)}, i = 1, ..., q\right\}$.

Finally, it is easy to see that $\tilde{A}_i^{(5)} = o\left(\sqrt{n}\left(\|\Psi' y_{\cdot i}\|^{-2} - p\lim \|\Psi' y_{\cdot i}\|^{-2}\right)\right) = o_p(1)$. To summarize, the asymptotic distribution of $\sqrt{n}\,(\alpha - \rho e_i)$ is the same as that of $\sum_{i=1}^{4} \tilde{A}^{(i)}$, where

$$\tilde{A}^{(1)} = \varkappa_i N_n^{(3)}(m_i) e_i, \quad (41)$$

$$\tilde{A}^{(2)} = -\frac{(d_i^2 - c\sigma^4)}{2d_i^{3/2}} \varkappa_i e_i e_i' N_n^2(m_i) e_i, \quad (42)$$

$$\tilde{A}^{(3)} = \frac{\sigma^4 c}{(d_i^2 - c\sigma^4)\, d_i^{1/2}} \varkappa_i e_i e_i' N_n^{(1)}(m_i)\, e_i, \quad (43)$$

$$\tilde{A}^{(4)} = -\varkappa_i \hat{S}(m_i)\, N_n^{(1)}(m_i)\, e_i, \quad (44)$$

$$\hat{S}(m_i) = \operatorname{diag}\left(\frac{d_1^{1/2}}{d_1 - d_i}, ..., \underbrace{0}_{i\text{-th position}}, ..., \frac{d_k^{1/2}}{d_k - d_i}\right)$$

and

$$\varkappa_i = \left(\frac{(d_i^2 - \sigma^4 c)(d_i + \sigma^2 c)}{d_i^2}\right)^{1/2}$$

Using Lemma 2, we conclude that the joint asymptotic distribution of elements of $\sqrt{n}\left(R - R^{(1)}\right)$ is normal. The elements of the covariance matrix of the asymptotic distribution of $\sqrt{n}\left(R - R^{(1)}\right)$ can be found[4] using (41),(42),(43),(44), and the expressions for the covariance of $N_n^{(1)}(m_i), N_n^{(2)}(m_i)$, and $N_n^{(3)}(m_i)$, summarized in the definition of $\Omega^{(\cdot,\cdot)}$ given in Appendix.

Let us now complete the proof by considering the case when $d_i \leq \sqrt{c}\sigma^2$. Since $\frac{1}{T}\hat{X}\hat{X}' = \Psi\Psi' + \Lambda$ and $\Psi\Psi'$ is positive definite with probability 1, $\mu_i\left(\frac{1}{T}XX'\right)$ is almost surely larger than $\lambda_i > \lambda_{k+1}$. Let $\bar{y}_{\cdot i}, \bar{\Psi}$, and $\bar{\Lambda}$ be $(n-k)\times 1$ vector, $(n-k)\times k$ matrix and $(n-k)\times(n-k)$ matrix respectively obtained from $y_{\cdot i}, \Psi$, and Λ by throwing away first k components, first k rows, and first k rows and columns respectively. Similar to (10), we have: $\bar{y}_{\cdot i} = \left(\mu_i\left(\frac{1}{T}XX'\right) I_{n-k} - \bar{\Lambda}\right)^{-1} \bar{\Psi}\left(\Psi' y_{\cdot i}\right)$, and therefore

$$\left(\Psi' y_{\cdot i}\right)' \bar{\Psi}' \left(\mu_i\left(\frac{1}{T}XX'\right) I_{n-k} - \bar{\Lambda}\right)^{-2} \bar{\Psi}\left(\Psi' y_{\cdot i}\right) = \bar{y}_{\cdot i}' \bar{y}_{\cdot i} \leq 1. \quad (45)$$

But $\min\operatorname{eval} \bar{\Psi}'\left(\mu_i\left(\frac{1}{T}XX'\right) I_{n-k} - \bar{\Lambda}\right)^{-2} \bar{\Psi} \to \infty$. Indeed, consider a number $\gamma > b \equiv \sigma^2\left(1 + c^{1/2}\right)^2$. For large enough n, with high probability $\mu_i\left(\frac{1}{T}XX'\right) < \gamma$ because, by Theorem 5, $\mu_i\left(\frac{1}{T}XX'\right) \xrightarrow{p} b$. Therefore, for large enough n with high probability

[4] To obtain these formulas we used symbolic manipulation software of the Scientific Workplace, version 5.

$\min \text{eval}\, \bar{\Psi}' \left(\mu_i \left(\frac{1}{T}XX'\right) I_{n-k} - \bar{\Lambda}\right)^{-2} \bar{\Psi} \geq \min \text{eval}\, \bar{\Psi}' \left(\gamma I_{n-k} - \bar{\Lambda}\right)^{-2} \bar{\Psi}$. But by Lemma 2 the right hand side of the last inequality converges to $\min \text{eval}\left(D + \sigma^2 c I_k\right) \int \frac{\mathcal{F}_c(d\lambda)}{(\gamma-\lambda)^2}$ which can be made arbitrarily large by choosing γ close enough to b. Equation (45) implies therefore that $\Psi' y_{\cdot i} \xrightarrow{p} 0$.

Now, let $\varkappa$ be a number $0 < \varkappa < 1$ and let $\tilde{y}_{\cdot i}, \tilde{\Psi}$, and $\tilde{\Lambda}$ be $(n - [\varkappa n]) \times 1$ vector, $(n - [\varkappa n]) \times k$ matrix and $(n - [\varkappa n]) \times (n - [\varkappa n])$ matrix respectively obtained from $y_{\cdot i}, \Psi$, and Λ by throwing away first $[\varkappa n]$ components, first $[\varkappa n]$ rows, and first $[\varkappa n]$ rows and columns respectively. Again, we have $\tilde{y}_{\cdot i} = \left(\mu_i \left(\frac{1}{T}XX'\right) I_{n-[\varkappa n]} - \tilde{\Lambda}\right)^{-1} \tilde{\Psi}\left(\Psi' y_{\cdot i}\right)$. Note that $\max \text{eval}\left(\mu_i \left(\frac{1}{T}XX'\right) I_{n-[\varkappa n]} - \tilde{\Lambda}\right)^{-1} = \left(\mu_i \left(\frac{1}{T}XX'\right) - \lambda_{[\varkappa n]+1}\right)^{-1}$, which by Marčenko and Pastur (1967) result and Theorem 5 converges to $\left(b - \mathcal{F}_c^{-1}(1-\varkappa)\right)^{-1} < \infty$. Therefore, Euclidean length of each column of $\left(\mu_i \left(\frac{1}{T}XX'\right) I_{n-[\varkappa n]} - \tilde{\Lambda}\right)^{-1} \tilde{\Psi}$ is bounded in probability and, since $\Psi' y_{\cdot i} \xrightarrow{p} 0$, $\|\tilde{y}_{\cdot i}\| \xrightarrow{p} 0$. Loosely speaking, for any $\varkappa$, with high probability for large enough n, almost all "mass" in vector y_i is concentrated in the first $\varkappa 100\%$ of its components.

Finally, the j-th coordinate of $\hat{\mathcal{L}}_{\cdot i}$ in the basis O_L are equal to $(O_{\cdot j})' y_i$. We have $\left|(O_{\cdot j})' y_{\cdot i}\right| = \left|\left(O_{1:[\varkappa n],j}\right)' y_{i,1:[\varkappa n]} + \left(O_{[\varkappa n]+1:n,j}\right)' \tilde{y}_{\cdot i}\right| \leq \left\|O_{1:[\varkappa n],j}\right\| + \|\tilde{y}_{\cdot i}\|$. The last term in the right hand side of the above inequality converges in probability to zero. As to the first term, since O is Haar distributed, $\left\|O_{1:[\varkappa n],j}\right\|^2$ has the same distribution as $\frac{1}{\|\varsigma\|^2} \sum_{j=1}^{[\varkappa n]} \varsigma_j^2$, where ς is an $n \times 1$ standard normal vector. Clearly, $\frac{1}{\|\varsigma\|^2} \sum_{j=1}^{[\varkappa n]} \varsigma_j^2 \xrightarrow{p} \varkappa$. Therefore, $\Pr\left(\left|(O_{\cdot j})' y_{\cdot i}\right| > 2\varkappa\right) \to 0$ as $n \to \infty$ for any $0 < \varkappa < 1$. In other words, all coordinates of $\hat{\mathcal{L}}_{\cdot i}$ in basis O_L converge in probability to zero.□

4.3 Proof of Theorem 4

First, note that since the distribution of the data X does not depend on the multiplication of X from the left by any orthogonal matrix having $\text{span}(L)$ as its invariant subspace, the joint distribution of the coordinates of the columns of $\hat{\mathcal{L}}$ in the basis formed by the columns of O_L does not depend on how the $k+1$-th, $k+2$-th, ..., n-th columns of O_L are chosen.

Denote an $n \times 1$ unit-length vector with all entries but the j-th equal to zero as e_j. Let the $k+1$-th column of O_L be chosen as $M(L)e_{j_1}/\left\|M(L)e_{j_1}\right\|$, where $M(L)$ denotes the operator of taking the residual from the orthogonal projection on $\text{span}(L)$, the $k+2$-th column be chosen as $M([L, e_{j_1}])e_{j_2}/\left\|M([L, e_{j_1}])e_{j_2}\right\|$, ..., and the $k+r$-th column be chosen as $M(\left[L, e_{j_1}, ..., e_{j_{r-1}}\right])e_{j_r}/\left\|M(\left[L, e_{j_1}, ..., e_{j_{r-1}}\right])e_{j_r}\right\|$. For example, if $r = 2$ and $j_1 = 1$ and

$j_2 = 2$, then matrix O_L has the following structure

$$O_L = \left[\begin{array}{c|c} LD^{-\frac{1}{2}} & \begin{array}{ccccc} x & 0 & 0 & \cdots & 0 \\ y & z & 0 & \cdots & 0 \\ \hline & & * & & \end{array} \end{array} \right], \tag{46}$$

where $x = \|M(L)e_1\|$, $y = e_2' M(L)e_1 / \|M(L)e_1\|$, and $z = \|M([L, e_1])e_2\|$. Note that:

$$x^2 = e_{j_1}' M(L) e_{j_1} = 1 - e_{j_1}' L \left(L'L\right)^{-1} L' e_{j_1} = 1 - \sum_{i=1}^{k} \mathcal{L}_{j_1 i}^2 \tag{47}$$

$$y = \frac{1}{x} e_{j_2}' M(L) e_{j_1} = -\frac{1}{x} \sum_{i=1}^{k} \mathcal{L}_{j_1 i} \mathcal{L}_{j_2 i}. \tag{48}$$

Let us denote the $n-k$ coordinates of the columns of $\hat{\mathcal{L}}_{1:q}$ in the basis formed by the columns of O_L as $R^{\perp}$. That is, $R_{ij}^{\perp}$ is the scalar product of $\hat{\mathcal{L}}_{\cdot j}$ and the $k+i$-th column of O_L. Then, $\hat{\mathcal{L}}_{j_s i} = \mathcal{L}_{j_s \cdot} \cdot R_{\cdot i} + \sum_{t=1}^{r} O_{L, j_s t} \cdot R_{ti}^{\perp}$. Hence, we can obtain the asymptotic joint distribution of $\left\{ \hat{\mathcal{L}}_{j_s i}; s = 1, ..., r; i = 1, ..., q \right\}$ from the asymptotic joint distribution of the entries of R and the first r columns of $R^{\perp}$.

It is easy to see that matrix $\tilde{R}^{\perp} \equiv R^{\perp} \left(I_q - R'R\right)^{-1/2}$, where R is as defined in Theorem 2, has orthonormal columns. Moreover, as a consequence of the invariance of the distribution of X with respect to the orthogonal transformations leaving L unchanged, the joint distribution of the entries of $\tilde{R}^{\perp}$ conditional on R is invariant with respect to multiplication of $\tilde{R}^{\perp}$ from the left by any orthogonal matrix. This implies that the joint distribution of the entries of $\tilde{R}^{\perp}\alpha$ conditional on R, where α is any $q \times 1$ unit-length vector, is the same as the joint distribution of the entries of $\xi / \|\xi\|$, where ξ is an $(n-k) \times 1$ vector with i.i.d. Gaussian entries.

As a consequence of the above result, the entries of $\tilde{R}^{\perp}\alpha$ are independent from the entries of $R\alpha$, and their unconditional joint distribution is the same as that of the entries of $\xi / \|\xi\|$. This fact, together with Theorem 2 and Cramer-Wold theorem (see White (1999), p.114), implies that the entries of $\sqrt{n}\left(R - R^{(1)}\right)$ and of the first r raws of $\sqrt{n}R^{\perp}$, where r is any fixed positive number, are asymptotically independent and have asymptotic joint zero-mean Gaussian distribution. The covariance matrix of the asymptotic distribution of the first r raws of $\sqrt{n}R^{\perp}$ is diagonal and $\text{Avar}\left(\sqrt{n}R_{ji}^{\perp}\right) = 1 - \left(R_{ii}^{(1)}\right)^2$.

The asymptotic joint Gaussianity of the entries of $\sqrt{n}\left(R - R^{(1)}\right)$ and $\sqrt{n}R^{\perp}$ implies that $\left\{ \sqrt{n}\left(\hat{\mathcal{L}}_{j_g i} - R_{ii}^{(1)} \mathcal{L}_{j_g i}\right); g = 1, ..., r; i = 1, ..., q \right\}$ are asymptotically jointly mean-zero Gaussian. We will now find the variances and covariances of the asymptotic distribution.

Consider the random variables $\sqrt{n}\left(\hat{\mathcal{L}}_{j_g i}-R_{ii}^{(1)}\mathcal{L}_{j_g i}\right)$ and $\sqrt{n}\left(\hat{\mathcal{L}}_{j_f p}-R_{pp}^{(1)}\mathcal{L}_{j_f p}\right)$. Without loss of generality assume that $g=1$, $f=2$. If $g\neq 1$ and/or $f\neq 2$, construct O_L so that its $k+1$-th column is $M(L)e_{j_g}/\left\|M(L)e_{j_g}\right\|$ and its $k+2$-th column is $M(\left[L,e_{j_g}\right])e_{j_f}/\left\|M(\left[L,e_{j_g}\right])e_{j_f}\right\|$. From (46), we have: $\sqrt{n}\left(\hat{\mathcal{L}}_{j_g i}-R_{ii}^{(1)}\mathcal{L}_{j_g i}\right)=\sum_{1\leq s\leq k}\mathcal{L}_{j_g s}\sqrt{n}\left(R_{si}-R_{si}^{(1)}\right)+x\sqrt{n}R_{1i}^{\perp}$, and $\sqrt{n}\left(\hat{\mathcal{L}}_{j_f p}-R_{pp}^{(1)}\mathcal{L}_{j_f p}\right)=\sum_{1\leq s\leq k}\mathcal{L}_{j_f s}\sqrt{n}\left(R_{sp}-R_{sp}^{(1)}\right)+y\sqrt{n}R_{1p}^{\perp}+z\sqrt{n}R_{2p}^{\perp}$. These two formulae together with (47), (48), and the formulae for the asymptotic covariance of entries of $\sqrt{n}\left(R-R^{(1)}\right)$ and of the first two rows of $\sqrt{n}R^{\perp}$ established above and in Theorem 2 imply the formula for the asymptotic covariance matrix claimed by Theorem 4.

Part ii of the theorem follows from part ii of Theorem 2 and the fact that the entries of the first row of $\sqrt{n}\tilde{R}^{\perp}$, where $\tilde{R}^{\perp}$ is defined similarly to $R^{\perp}$, converge in distribution. This fact can be established similarly to the analogous fact for $\sqrt{n}R^{\perp}$.□

4.4 Proof of Proposition 1

Define $\eta=\left(\eta_{1+h},...,\eta_{T+h}\right)'$, $Z\equiv[F,W]$, $\hat{Z}\equiv\left[\hat{F},W\right]$, $\beta\equiv(\beta_1,\beta_2)'$, and $\hat{\beta}\equiv\left(\hat{\beta}_1,\hat{\beta}_2\right)'$. Then $\hat{\beta}=\left(\frac{\hat{Z}'\hat{Z}}{T}\right)^{-1}\frac{\hat{Z}'Z}{T}\beta+\left(\frac{\hat{Z}'\hat{Z}}{T}\right)^{-1}\frac{\hat{Z}'\eta}{T}$. Using Theorem 1, we obtain: $\left(\hat{Z}'\hat{Z}\right)_{21}=\text{sign}\left(\hat{F}'F\right)\cdot\left(W'FQ+W'F^{\perp}\right)$. Consider the component $W'F^{\perp}$ of the latter sum. Define $W^{\perp}$ as $W-F\left(F'F\right)^{-1}F'W$. As follows from the invariance of the data distribution with respect to the multiplication of X' from the left by any orthogonal matrix having $\text{span}\left(F\right)$ as its invariant subspace and from the assumed independence of W and the matrix of the idiosyncratic terms in the underlying factor model, the joint conditional on F and W distribution of the coordinates of $F^{\perp}$ in the subspace orthogonal to $\text{span}\left(F\right)$ does not depend on the choice of basis in this subspace and is equal to the joint distribution of the entries of vector $\xi/\left\|\xi\right\|$, where $\xi\sim N\left(0,I_{T-1}\right)$ (see proof of Theorem 4 for a proof of a similar statement for $\mathcal{L}^{\perp}$). In particular, we can choose the first vector of the basis to be proportional to $W^{\perp}$. Then, since $W'F^{\perp}=W^{\perp\prime}F^{\perp}$, we must have $\frac{1}{\sqrt{T}}W'F^{\perp}$ converges in distribution to a Gaussian random variable, and, therefore, $\frac{1}{T}F^{\perp\prime}W\xrightarrow{p}0$. Using Theorem 1, it is now easy to see that now that $\frac{1}{T}\left(\hat{Z}'\hat{Z}\right)_{12}=\frac{1}{T}\left(\hat{Z}'\hat{Z}\right)_{21}\xrightarrow{p}\text{sign}\left(\hat{F}'F\right)\varrho\gamma$. Similarly, we can show that $\frac{1}{T}\left(\hat{Z}'\eta\right)_1\xrightarrow{p}0$. Combining the latter two convergence results with the above representation for $\hat{\beta}$ and easily verifiable facts that $\frac{1}{T}\left(\hat{Z}'\hat{Z}\right)_{11}=1$, $\frac{1}{T}\left(\hat{Z}'\hat{Z}\right)_{22}=\frac{1}{T}\left(\hat{Z}'Z\right)_{22}\xrightarrow{p}1$, $\frac{1}{T}\left(\hat{Z}'Z\right)_{21}\xrightarrow{p}\gamma$, $\frac{1}{T}\left(\hat{Z}'Z\right)_{12}\xrightarrow{p}\text{sign}\left(\hat{F}'F\right)\varrho\gamma$, and $\frac{1}{T}\left(\hat{Z}'\eta\right)_2\xrightarrow{p}0$, we obtain parts i and ii of Proposition 1. Part iii of the proposition follows from statements i and ii, Theorem 3 and the following identity: $\hat{y}_{T+h|T}-y_{T+h|T}=\left(\hat{\beta}_1-\text{sign}\left(\hat{F}'F\right)\beta_1\right)\hat{F}_T+\beta_1\left(\text{sign}\left(\hat{F}'F\right)\hat{F}_T-F_T\right)+\left(\hat{\beta}_2-\beta_2\right)W_T$.□

5 Conclusion

In this paper we have shown that the principal components estimators of factors and factor loadings are inconsistent but asymptotically normal as n and T go to infinity proportionally when the cumulative effects of the normalized factors on the cross-sectional units are assumed to be bounded. We have found explicit formulae for the amount of the inconsistency and for the asymptotic covariance matrix of the estimators, and explained the potential consequences of the inconsistency for the forecasts based on the diffusion index forecast models. Our Monte Carlo analysis suggests that the asymptotic formulae found in the paper work well even for such small samples as $n = 40$, $T = 20$.

Our assumption that the cumulative effects of the factors are bounded contrasts the usual assumption of the unbounded effects made in the approximate factor models. This conflict should not preclude using our results in the empirical applications of such models. Our formulae simply provide an alternative asymptotic approximation to the finite sample distributions of interest to the applications. As we have shown in the paper, our asymptotic approximations converge to those proposed by Bai (2003) when the assumed bounds on the cumulative effects of the factors increase. Hence, in the applications where factors have very large cumulative effects in the sample, our asymptotic approximation should work similarly to Bai's. On the other hand, when factors do not have large cumulative effect in the sample of interest, our results will provide a better approximation than results based on the assumption of strong asymptotic domination of factors over the idiosyncratic influences.

In principle, our analysis can be modified so that the assumption of the bounded factor effects is not made. The modified analysis would then study a second order asymptotic approximation of the principal components estimator. Although such an approach is appealing because it reliefs us from the necessity of making assumptions inconsistent with the traditional approximate factor models, it creates some additional technical problems which may be difficult to solve. As we explained above, the asymptotic equivalence of our framework and the more traditional framework is not important for empirical applications. Therefore, we have not pursued the second-order-asymptotics idea.

Our assumption of the i.i.d. Gaussian idiosyncratic terms strongly reduces the applicability of our results to macroeconomic and financial problems. We view this paper as a first step in a more general research program that would study the alternative asymptotics of the principal components estimator in the models with correlated non-Gaussian idiosyncratic terms. An extension of our results would probably be based on the generalization of the key Marčenko-Pastur (1967) result which this paper relies on to the case of the sample-covariance-type matrices of the correlated data studied by Silverstein (1995) and Vasilchuk

(2001). Such an extension should solve a number of new challenging technical problems. It is left for future work.

6 Appendix

Definition of covariance function Ω from Lemma 1:

For $\tau=(s,t,j)$, $\tau_1=(s_1,t_1,j_1)$, and integers p_1 and p_2 such that $1\le p_1\le p_2\le 2$, we define Ω as follows.

$$\begin{aligned}
\Omega^{(p_1,p_2)}(\tau,\tau_1) &= \frac{c}{4}(d_s+d_t)(d_{s_1}+d_{t_1})\phi_{sts_1t_1}\int\frac{d\mathcal{F}_c(\lambda)}{(x_j-\lambda)^{p_1}}\int\frac{d\mathcal{F}_c(\lambda)}{(x_{j_1}-\lambda)^{p_2}}\\
\Omega^{(p_1,3)}(\tau,\tau_1) &= \frac{c}{4}(d_s+d_t)\sqrt{d_{s_1}}\phi_{sts_1t_1}\int\frac{d\mathcal{F}_c(\lambda)}{(x_j-\lambda)^{p_1}}\int\frac{d\mathcal{F}_c(\lambda)}{x_{j_1}-\lambda}\\
\Omega^{(3,3)}(\tau,\tau_1) &= \frac{c}{4}\sqrt{d_sd_{s_1}}\phi_{sts_1t_1}\int\frac{d\mathcal{F}_c(\lambda)}{x_j-\lambda}\int\frac{d\mathcal{F}_c(\lambda)}{x_{j_1}-\lambda}
\end{aligned}$$

if $(s_1,t_1)\neq(s,t)$ and $(s_1,t_1)\neq(t,s)$;

$$\begin{aligned}
\Omega^{(p_1,p_2)}(\tau,\tau_1) &= \left[\frac{c}{4}(d_s+d_t)^2\phi_{stst}-(1+\delta_{st})d_sd_t\right]\int\frac{d\mathcal{F}_c(\lambda)}{(x_j-\lambda)^{p_1}}\int\frac{d\mathcal{F}_c(\lambda)}{(x_{j_1}-\lambda)^{p_2}}\\
&\quad+\left[(1+\delta_{st})\left(\sigma^4c^2+d_sd_t\right)+\sigma^2c\left(d_s+d_t+2\delta_{st}\sqrt{d_sd_t}\right)\right]\\
&\quad\cdot\int\frac{d\mathcal{F}_c(\lambda)}{(x_j-\lambda)^{p_1}(x_{j_1}-\lambda)^{p_2}}\\
\Omega^{(p_1,3)}(\tau,\tau_1) &= \left[\frac{c}{4}(d_s+d_t)\sqrt{d_s}\phi_{stst}-(1+\delta_{st})\sqrt{d_s}d_t\right]\int\frac{d\mathcal{F}_c(\lambda)}{(x_j-\lambda)^{p_1}}\int\frac{d\mathcal{F}_c(\lambda)}{x_{j_1}-\lambda}\\
&\quad+\left[(1+\delta_{st})\sqrt{d_s}d_t+\sigma^2c\left(\sqrt{d_s}+\delta_{st}\sqrt{d_t}\right)\right]\int\frac{d\mathcal{F}_c(\lambda)}{(x_j-\lambda)^{p_1}(x_{j_1}-\lambda)}\\
\Omega^{(3,3)}(\tau,\tau_1) &= \left(\frac{c}{4}d_s\phi_{stst}-(1+\delta_{st})d_t\right)\int\frac{d\mathcal{F}_c(\lambda)}{x_j-\lambda}\int\frac{d\mathcal{F}_c(\lambda)}{x_{j_1}-\lambda}\\
&\quad+\left((1+\delta_{st})d_t+\sigma^2c\right)\int\frac{d\mathcal{F}_c(\lambda)}{(x_j-\lambda)(x_{j_1}-\lambda)}
\end{aligned}$$

if $(s_1,t_1)=(s,t)$; and

$$\begin{aligned}
\Omega^{(p_1,p_2)}(\tau,\tau_1) &= \Omega^{(p_1,p_2)}((t,s,j),(s_1,t_1,j_1))\\
\Omega^{(p_1,3)}(\tau,\tau_1) &= \Omega^{(p_1,3)}((t,s,j),(s_1,t_1,j_1))\\
\Omega^{(3,3)}(\tau,\tau_1) &= \left(\frac{c}{4}\phi_{stst}-(1+\delta_{st})\right)\sqrt{d_sd_t}\int\frac{d\mathcal{F}_c(\lambda)}{x_j-\lambda}\int\frac{d\mathcal{F}_c(\lambda)}{x_{j_1}-\lambda}\\
&\quad+\left((1+\delta_{st})\sqrt{d_sd_t}+\delta_{st}\sigma^2c\right)\int\frac{d\mathcal{F}_c(\lambda)}{(x_j-\lambda)(x_{j_1}-\lambda)}
\end{aligned}$$

if $(s_1,t_1)=(t,s)$.

Proof of Lemma 1:

Suppose $x_0 \neq \lambda_i$, $i = 1, ..., n$ and x_0 satisfies (11). Let v be an eigenvector of $M_n^{(1)}(x_0)$ corresponding to the unit eigenvalue. Define $z_i = \frac{1}{x_0 - \lambda_i} \Psi_i . v$ and let $z = (z_1, ..., z_n)'$. We have: $\Psi' z = M_n^{(1)}(x_0) v = v$, and hence, $z = \frac{1}{x_0 - \lambda_i} \Psi_i . \Psi' z$, which proves that z is an eigenvector of $\frac{1}{T} \hat{X} \hat{X}'$ corresponding to eigenvalue x_0. Since the eigenvalues of $\frac{1}{T} \hat{X} \hat{X}'$ and $\frac{1}{T} X X'$ coincide, x_0 must be an eigenvalue of $\frac{1}{T} X X'$ which proves the "if" statement of the Lemma. The "only if" statement of the Lemma has been established in Section 4.□

Proof of lemma 2:

We, first, formulate and prove the key technical lemma of this paper. Let $g_j(\lambda)$, $j = 1, ..., J$, be analytic functions of real variable λ on an open interval $(\bar{a}, \bar{b})$ containing the support of the Marčenko-Pastur distribution, that is the set $\{0, [a, b]\}$ if $c > 1$, and the segment $[a, b]$ if $c \geq 1$, where $a = (1 - \sqrt{c})^2 \sigma^2$ and $b = (1 + \sqrt{c})^2 \sigma^2$. Further, let $\varsigma^{(n)}$ be an array of $n \times m$ matrices with i.i.d. standard normal entries independent of $\lambda_1, ..., \lambda_n$. In what follows we will omit the superscript n in $\varsigma^{(n)}$ to simplify notations. Finally, denote the set of triples $\{(j, s, t) : 1 \leq j \leq J, 1 \leq s \leq t \leq m\}$ as Θ_1. Then, we have the following

Lemma 5: *Let Assumptions 2 and 3 hold. Then, the joint distribution of random variables* $\left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^n g_j(\lambda_i) (\varsigma_{is} \varsigma_{it} - \delta_{st}) ; (j, s, t) \in \Theta_1 \right\}$ *weakly converge to a multivariate normal distribution as* $n \to \infty$. *Covariance between components* (j, s, t) *and* (j_1, s_1, t_1) *of the limiting distribution is equal to* 0 *when* $(s, t) \neq (s_1, t_1)$, *and to* $(1 + \delta_{st}) \int g_j(\lambda) g_{j_1}(\lambda) d\mathcal{F}_c(\lambda)$ *when* $(s, t) = (s_1, t_1)$.

Proof: To prove this lemma we will need two well known results, which we formulate below as two additional lemmas.

Lemma 6: *(McLeish (1974)) Let* $\{X_{n,i}, \mathcal{F}_{n,i}; i = 1, 2, ..., n\}$ *be a martingale difference array on the probability triple* $(\Omega, \mathcal{F}, P)$. *If the following conditions are satisfied:*
a) Lindeberg's condition: for all $\varepsilon > 0, \sum_i \int_{|X_{n,i}| > \varepsilon} X_{n,i}^2 dP \to 0, n \to \infty$
b) $\sum_{i=1}^n X_{n,i}^2 \xrightarrow{p} 1,$
then $\sum_{i=1}^n X_{n,i} \xrightarrow{w} N(0, 1).$

Proof: This is a consequence of Theorem (2.3) of McLeish (1974). Two conditions of the theorem, i) $\max_{i \leq n} |X_{n,i}|$ is uniformly bounded in L_2 norm, and ii) $\max_{i \leq n} |X_{n,i}| \xrightarrow{p} 0$, are replaced here by the Lindeberg condition. As explained in McLeish (1974), since for any ε, $\max_{i \leq n} X_{n,i}^2 \leq \varepsilon^2 + \sum_i X_{n,i}^2 I(|X_{n,i}| > \varepsilon)$ and since $P\{\max_{i \leq n} |X_{n,i}| > \varepsilon\} = P\left\{\sum_i X_{n,i}^2 I(|X_{n,i}| > \varepsilon) > \varepsilon^2\right\}$, both conditions i) and ii) follow from the Lindeberg condition.□

Lemma7: *(Hall and Heyde) Let* $\{X_{n,i}, \mathcal{F}_{ni}; 1 \leq i \leq n\}$ *be a martingale difference array and define*

$$V_{n,j}^2 = \sum_{i=1}^j E\left(X_{n,i}^2 | \mathcal{F}_{n,i-1}\right) \text{ and } U_{n,j}^2 = \sum_{i=1}^j X_{n,i}^2 \text{ for } 1 \leq j \leq n.$$

Suppose that the conditional variances $V_{n,n}^2$ *are tight, that is* $\sup_n P\left(V_{n,n}^2 > \varepsilon\right) \to 0$ *as* $\varepsilon \to \infty$, *and that the* *conditional Lindeberg condition holds, that is for all* $\varepsilon > 0$, $\sum_i E\left[X_{n,i}^2 I\left(|X_{n,i}| > \varepsilon\right)|\mathcal{F}_{n,i-1}\right] \xrightarrow{p} 0$. *Then*

$$\max_j \left|U_{n,j}^2 - V_{n,j}^2\right| \xrightarrow{p} 0.$$

Proof: This is a shortened version of Theorem 2.23 in Hall and Heyde (1980).□

Returning to the proof of Lemma 5, let real numbers a_1 and b_1 be such that $[a_1, b_1]$ is included in $\left(\bar{a}, \bar{b}\right)$, but itself includes the support of the Marčenko-Pastur law. Define functions $h_j(\lambda), j = 1, ..., J$, so that $h_j(\lambda) = g_j(\lambda)$ for $\lambda \in [a_1, b_1]$, and $h_j(\lambda) = 0$ otherwise. Note that $|h_j(\lambda)| < B$ for any $j = 1, ..., J$ and any λ, where B is a constant larger than $\max_{j=1,...,J} \sup_{\lambda \in [a_1,b_1]} |g_j(\lambda)|$. Note also that since, as shown in Bai, Silverstein and Yin (1988), λ_1 almost surely converges to b, $P\{\exists j \leq J, i \leq n \text{ such that } h_j(\lambda_i) \neq g_j(\lambda_i)\} \to 0$ as $n \to \infty$.

Consider random variables $X_{n,i} = \frac{1}{\sqrt{n}} \sum_{(j,s,t)\in\Theta_1} \gamma_{jst} h_j(\lambda_i) \left(\varsigma_{is}\varsigma_{it} - \delta_{st}\right)$, where γ_{jst} are some constants. Let $\mathcal{F}_{n,i}$ be sigma-algebra generated by $\lambda_1, ..., \lambda_n$ and $\varsigma_{js}; 1 \leq j \leq i, 1 \leq s \leq m$. Clearly, $\{X_{n,i}, \mathcal{F}_{n,i}; i = 1, 2, ..., n\}$ form a martingale difference array. We now will check that the Lindeberg condition of McLeish's proposition is satisfied for $X_{n,i}$.

Let R be the number of different triples $(j, s, t) \in \Theta_1$. Consider an arbitrary order in Θ_1. In the Hölder's inequality:

$$\sum_{r=1}^{R} a_r b_r \leq \left(\sum_{r=1}^{R} (a_r)^p\right)^{1/p} \left(\sum_{r=1}^{R} (b_r)^q\right)^{1/q}, \tag{49}$$

which holds for $a_r > 0, b_r > 0, p > 1, q > 1$, and $(1/p) + (1/q) = 1$, take $a_r = \left|\frac{1}{\sqrt{n}}\gamma_{jst} h_j(\lambda_i)\left(\varsigma_{is}\varsigma_{it} - \delta_{st}\right)\right|$, where (j, s, t) is the r-th triple in Θ_1, $b_r = 1$, and $p = 2 + \delta$ for some $\delta > 0$. Then, (49) implies:

$$|X_{n,i}|^{2+\delta} \leq R^{1+\delta} \sum_{(j,s,t)\in\Theta_1} \left|\gamma_{jst} h_j(\lambda_i) \frac{\varsigma_{is}\varsigma_{it} - \delta_{st}}{\sqrt{n}}\right|^{2+\delta} \leq R^{1+\delta} B^{2+\delta} \sum_{(j,s,t)\in\Theta_1} \left|\gamma_{jst} \frac{\varsigma_{is}\varsigma_{it} - \delta_{st}}{\sqrt{n}}\right|^{2+\delta}.$$

Recalling that ς_{is} are i.i.d. standard normal random variables, we have: $\sum_i E|X_{n,i}|^{2+\delta}$ tends to zero as $n \to \infty$, which means that the Lyapunov condition holds for $X_{n,i}$. As is well known, Lyapunov's condition implies Lindeberg's condition. Hence, condition a) of McLeish's proposition is satisfied for $X_{n,i}$.

Now, let us consider $\sum_{i=1}^{n} X_{n,i}^2$. Since convergence in mean implies convergence in probability, the conditional Lindeberg condition is satisfied for $X_{n,i}$ because the unconditional Lindeberg condition is satisfied as checked above. Further, in notations of Hall and Heyde's proposition, we have:

$$\begin{aligned} V_{n,n}^2 &= \frac{1}{n}\sum_{i=1}^{n} E(\sum_{\substack{(j,s,t)\in\Theta_1,\\ (j_1,s_1,t_1)\in\Theta_1}} \gamma_{jst}\gamma_{j_1s_1t_1} h_j(\lambda_i)h_{j_1}(\lambda_i)\left(\varsigma_{is}\varsigma_{it}-\delta_{st}\right)\left(\varsigma_{is_1}\varsigma_{it_1}-\delta_{s_1t_1}\right)|\mathcal{F}_{n,i-1}) \\ &= \frac{1}{n}\sum_{i=1}^{n} \sum_{\substack{(j,s,t)\in\Theta_1,\\ (j_1,s_1,t_1)\in\Theta_1}} \gamma_{jst}\gamma_{j_1s_1t_1} h_j(\lambda_i)h_{j_1}(\lambda_i)E(\left(\varsigma_{is}\varsigma_{it}-\delta_{st}\right)\left(\varsigma_{is_1}\varsigma_{it_1}-\delta_{s_1t_1}\right)|\mathcal{F}_{n,i-1}) \\ &= \frac{1}{n}\sum_{i=1}^{n} \sum_{\substack{(j,s,t)\in\Theta_1,\\ (j_1,s_1,t_1)\in\Theta_1}} \gamma_{jst}\gamma_{j_1s_1t_1} h_j(\lambda_i)h_{j_1}(\lambda_i)E(\left(\varsigma_{is}\varsigma_{it}-\delta_{st}\right)\left(\varsigma_{is_1}\varsigma_{it_1}-\delta_{s_1t_1}\right)) \end{aligned}$$

It is straightforward to check that $E(\left(\varsigma_{is}\varsigma_{it}-\delta_{st}\right)\left(\varsigma_{is_1}\varsigma_{it_1}-\delta_{s_1t_1}\right))$ is equal to 2 if $s=s_1=t=t_1$, to 1 if $s=s_1\neq t=t_1$, and to 0 otherwise. Hence, $V_{n,n}^2=\sum_{\substack{1\leq j\leq J\\ 1\leq j_1\leq J}}\left[\left(\sum_{1\leq s\leq t\leq m}\gamma_{jst}\gamma_{j_1st}\left(1+\delta_{st}\right)\right)\frac{1}{n}\sum_{i=1}^{n}h_j(\lambda_i)h_{j_1}(\lambda_i)\right]$.

Consider now $\tilde{V}_{n,n}^2=\sum_{\substack{1\leq j\leq J\\ 1\leq j_1\leq J}}\left[\left(\sum_{1\leq s\leq t\leq m}\gamma_{jst}\gamma_{j_1st}\left(1+\delta_{st}\right)\right)\frac{1}{n}\sum_{i=1}^{n}g_j(\lambda_i)g_{j_1}(\lambda_i)\right]$. Since $P\left(\tilde{V}_{n,n}^2\neq V_{n,n}^2\right)\to 0$ as $n\to\infty$, $\tilde{V}_{n,n}^2$ and $V_{n,n}^2$ must converge in probability to the same limit, or must both diverge. But, by Theorem 1.1 of Bai and Silverstein (2004), $\frac{1}{n}\sum_{i=1}^{n}g_j(\lambda_i)g_{j_1}(\lambda_i)-\int g_j(\lambda)g_{j_1}(\lambda)d\mathcal{F}_{\frac{n}{T}}(\lambda)$ converges in probability to zero. Therefore, since $\mathcal{F}_{\frac{n}{T}}(\lambda)$ weakly converge to $\mathcal{F}_c(\lambda)$ as $n\to\infty$, we have

$$\tilde{V}_{n,n}^2\xrightarrow{p}\Sigma\equiv\sum_{\substack{1\leq j\leq J\\ 1\leq j_1\leq J}}\left[\left(\sum_{1\leq s\leq t\leq m}\gamma_{jst}\gamma_{j_1st}\left(1+\delta_{st}\right)\right)\int g_j(\lambda)g_{j_1}(\lambda)d\mathcal{F}_c(\lambda)\right]. \tag{50}$$

Hence, $V_{n,n}^2$ also converges in probability to Σ. In particular, $V_{n,n}^2$ is tight and Hall and Heyde's proposition applies. From Hall and Heyde's proposition, we know that $\sum_{i=1}^{n}X_{n,i}^2$ must converge to the same limit as $V_{n,n}^2$. Therefore, using McLeish's result, we get:

$$\sum_{i=1}^{n}X_{n,i}\xrightarrow{w}N(0,\Sigma). \tag{51}$$

Let us now define $Y_{n,i}=\sum_{(j,s,t)\in\Theta_1}\gamma_{jst}g_j(\lambda_i)\frac{\varsigma_{is}\varsigma_{it}-\delta_{st}}{\sqrt{n}}$. Since $P\left(\sum_{i=1}^{n}Y_{n,i}\neq\sum_{i=1}^{n}X_{n,i}\right)\to 0$ as $n\to\infty$, (51) implies that

$$\sum_{i=1}^{n}Y_{n,i}\xrightarrow{w}N(0,\Sigma). \tag{52}$$

Finally, Lemma 5 follows from Cramer-Wold result (see White (1999), p.114), (52), and definition of Σ (50).□

Now we turn to the proof of Lemma 2. Recall that the construction of $N_n^{(p)}(x)$ (see (13) and (14)) uses an $n\times k$ random matrix

$$\Psi=\varphi\left(L'L\right)^{1/2}\left(\frac{F'F}{T}\right)^{1/2}+\sigma\sqrt{\frac{n}{T}}\frac{1}{\sqrt{n}}\eta \tag{53}$$

Note that distribution of $N_n^{(p)}(x)$ will not change if we substitute φ in (53) by $\xi(\xi'\xi)^{-1/2}$, where ξ is an

$n \times k$ matrix with i.i.d. standard normal entries independent from η, F, and $\lambda_1, ..., \lambda_n$. Indeed, the columns of $\xi(\xi'\xi)^{-1/2}$ are orthogonal and of unit length. Further, the joint distribution of elements of $\xi(\xi'\xi)^{-1/2}$ is invariant with respect to multiplication from the left by any orthogonal matrix. Hence, this distribution coincides with the joint distribution of the elements of the first k columns of random orthogonal matrix having Haar invariant distribution. But the latter is the joint distribution of elements of φ. In the rest of the proof, we, therefore, will substitute φ by $\xi(\xi'\xi)^{-1/2}$ and redefine $\hat{M}_n^{(p)}(x)$ as

$$\hat{M}_n^{(1)}(x) = \sum_{i=1}^{n} \frac{\tilde{\Psi}'_{i\cdot}\tilde{\Psi}_{i\cdot}}{h(x,\lambda_i)},\ \hat{M}_n^{(2)}(x) = \sum_{i=1}^{n} \frac{\tilde{\Psi}'_{i\cdot}\tilde{\Psi}_{i\cdot}}{h^2(x,\lambda_i)},\ \hat{M}_n^{(3)}(x) = \sum_{i=1}^{n} \frac{(\xi'\xi)^{-1/2}\xi'_{i\cdot}\tilde{\Psi}_{i\cdot}}{h(x,\lambda_i)},$$

where $\tilde{\Psi} = \xi(\xi'\xi)^{-1/2}(L'L)^{1/2}\left(\frac{F'F}{T}\right)^{1/2} + \sigma\sqrt{\frac{n}{T}}\frac{1}{\sqrt{n}}\eta$.

We can prove (15) by, first, checking the convergence of the finite dimensional distributions

$$\left\{N_{n,st}^{(p)}(x_j); p = 1,2,3,(s,t,j) \in \Theta\right\} \xrightarrow{d} \left\{N_{st}^{(p)}(x_j); p = 1,2,3,(s,t,j) \in \Theta\right\} \tag{54}$$

where Θ denotes the set of all integer triples (s,t,j) satisfying $1 \le s,t \le k$ and $1 \le j \le J$, and, second, by demonstrating the tightness of all entries of $\left\{N_n^{(p)}(x); p = 1,2,3\right\}$. In the proof below, we show that

$$\left\{N_{n,st}^{(1)}(x_j); (s,t,j) \in \Theta\right\} \xrightarrow{d} \left\{N_{st}^{(1)}(x_j); (s,t,j) \in \Theta\right\} \tag{55}$$

and prove the tightness for $N_n^{(1)}(x)$. The proof of (54) and of the tightness of $\left\{N_n^{(p)}(x); p = 1,2,3\right\}$ is fully analogous to the proof of (55) and of the tightness of $N_n^{(1)}(x)$ and we omit it here. A more complete exposition would not add any insight to what we about to say below, but would require considerable extra space.

It is straightforward to check that $N_n^{(1)}(x) = \sum_{v=1}^{10} S^{(v)}(x)$, where

$S^{(1)}(x) = \left(\frac{F'F}{T}\right)^{1/2}(L'L)^{1/2}\left(\frac{\xi'\xi}{n}\right)^{-1/2}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\xi'_{i\cdot}\xi_{i\cdot}-I_k}{h(x,\lambda_i)}\right)\left(\frac{\xi'\xi}{n}\right)^{-1/2}(L'L)^{1/2}\left(\frac{F'F}{T}\right)^{1/2}$,

$S^{(2)}(x) = \left(\frac{F'F}{T}\right)^{1/2}(L'L)^{1/2}\left(\frac{\xi'\xi}{n}\right)^{-1}(L'L)^{1/2}\sqrt{\frac{n}{T}}\sqrt{T}\left(\left(\frac{F'F}{T}\right)^{1/2} - I_k\right)\sum_{i=1}^{n}\frac{1}{nh(x,\lambda_i)}$,

$S^{(3)}(x) = \sqrt{\frac{n}{T}}\sqrt{T}\left(\left(\frac{F'F}{T}\right)^{1/2} - I_k\right)(L'L)^{1/2}\left(\frac{\xi'\xi}{n}\right)^{-1}(L'L)^{1/2}\sum_{i=1}^{n}\frac{1}{nh(x,\lambda_i)}$,

$S^{(4)}(x) = (L'L)^{1/2}\sqrt{n}\left(I_k - \left(\frac{\xi'\xi}{n}\right)\right)\left(\frac{\xi'\xi}{n}\right)^{-1}(L'L)^{1/2}\sum_{i=1}^{n}\frac{1}{nh(x,\lambda_i)}$,

$S^{(5)}(x) = \sqrt{n}\,(L'L - D)\sum_{i=1}^{n}\frac{1}{nh(x,\lambda_i)}$,

$S^{(6)}(x) = \sigma\sqrt{\frac{n}{T}}\left(\frac{F'F}{T}\right)^{1/2}(L'L)^{1/2}\left(\frac{\xi'\xi}{n}\right)^{-1/2}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\xi'_{i\cdot}\eta_{i\cdot}}{h(x,\lambda_i)}\right)$,

$S^{(7)}(x) = \sigma\sqrt{\frac{n}{T}}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\eta'_{i\cdot}\xi_{i\cdot}}{h(x,\lambda_i)}\right)\left(\frac{\xi'\xi}{n}\right)^{-1/2}(L'L)^{1/2}\left(\frac{F'F}{T}\right)^{1/2}$,

$S^{(8)}(x) = \sigma^2\left(\frac{n}{T}\right)\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\eta'_{i\cdot}\eta_{i\cdot}-I_k}{h(x,\lambda_i)}$,

$S^{(9)}(x) = \sigma^2\sqrt{n}\left(\frac{n}{T} - c\right)I_k\sum_{i=1}^{n}\frac{1}{nh(x,\lambda_i)}$,

$S^{(10)}(x) = -\left(D + \sigma^2 cI_k\right)\sqrt{n}\left(\int\frac{d\mathcal{F}_c(\lambda)}{x-\lambda} - \sum_{i=1}^{n}\frac{1}{nh(x,\lambda_i)}\right)$

By Assumption 3 and the fact that, by definition, ξ is an $n \times k$ matrix with i.i.d. standard normal entries, we have:

$$\frac{F'F}{T} \xrightarrow{p} I_k,\ \frac{\xi'\xi}{n} \xrightarrow{p} I_k,\ L'L - D = o\left(\sqrt{n}\right), \text{ and } \frac{n}{T} - c = o\left(\sqrt{n}\right). \tag{56}$$

Further, by Theorem 1 of Bai and Silverstein (2004), $\sqrt{n}\left(\int \frac{d\mathcal{F}_{n/T}(\lambda)}{x-\lambda} - \sum_{i=1}^{n} \frac{1}{n}\frac{1}{x-\lambda_i}\right) \xrightarrow{p} 0$ for any $x \in [\theta_1, \theta_2]$. Our assumption that $n/T - c = o(1/\sqrt{n})$ and the definition of Marčenko-Patur law imply that $\sqrt{n}\left(\int \frac{d\mathcal{F}_{n/T}(\lambda)}{x-\lambda} - \int \frac{d\mathcal{F}_{c}(\lambda)}{x-\lambda}\right) \xrightarrow{p} 0$, and hence

$$\sqrt{n}\left(\int \frac{d\mathcal{F}_c(\lambda)}{x-\lambda} - \sum_{i=1}^{n} \frac{1}{n}\frac{1}{x-\lambda_i}\right) \xrightarrow{p} 0 \tag{57}$$

for any $x \in [\theta_1, \theta_2]$. Since, as shown in Bai, Silverstein and Yin (1988), λ_1 almost surely converges to $b \equiv (1+\sqrt{c})^2 \sigma^2$,

$$P\left\{\sum_{i=1}^{n} \frac{1}{\sqrt{n}}\left(\frac{1}{x-\lambda_i} - \frac{1}{h(x,\lambda_i)}\right) \neq 0\right\} \to 0 \tag{58}$$

as $n \to \infty$ and, therefore, (57) implies that

$$\sqrt{n}\left(\int \frac{d\mathcal{F}_c(\lambda)}{x-\lambda} - \sum_{i=1}^{n} \frac{1}{nh(x,\lambda_i)}\right) \xrightarrow{p} 0. \tag{59}$$

(56) and (59) imply that $\left\{\sum_{v=1}^{10} S_{st}^{(v)}(x_j); (s,t,j) \in \Theta\right\}$ and $\left\{\sum_{v=1}^{10} \tilde{S}_{st}^{(v)}(x_j); (s,t,j) \in \Theta\right\}$ weakly converge to the same limit or do not converge together, where

$\tilde{S}^{(1)}(x) = D^{1/2}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} \frac{\xi_{i\cdot}'\xi_{i\cdot} - I_k}{h(x,\lambda_i)}\right) D^{1/2}$,
$\tilde{S}^{(2)}(x) = D\sqrt{c}\sqrt{T}\left(\left(\frac{F'F}{T}\right)^{1/2} - I_k\right)\int \frac{d\mathcal{F}_c(\lambda)}{x-\lambda}$,
$\tilde{S}^{(3)}(x) = \sqrt{c}\sqrt{T}\left(\left(\frac{F'F}{T}\right)^{1/2} - I_k\right) D\int \frac{d\mathcal{F}_c(\lambda)}{x-\lambda}$,
$\tilde{S}^{(4)}(x) = D^{1/2}\sqrt{n}\left(I_k - \left(\frac{\xi'\xi}{n}\right)\right) D^{1/2}\int \frac{d\mathcal{F}_c(\lambda)}{x-\lambda}$,
$\tilde{S}^{(5)}(x) = 0$,
$\tilde{S}^{(7)}(x) = \sigma\sqrt{c}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} \frac{\eta_{i\cdot}'\xi_{i\cdot}}{h(x,\lambda_i)}\right) D^{1/2}$,
$\tilde{S}^{(8)}(x) = \sigma^2 c\frac{1}{\sqrt{n}}\sum_{i=1}^{n} \frac{\eta_{i\cdot}'\eta_{i\cdot} - I_k}{h(x,\lambda_i)}$,
$\tilde{S}^{(9)}(x) = \tilde{S}^{(10)}(x) = 0$.

Let us, first, consider the limit of $\left\{\tilde{S}_{st}^{(2)}(x_j) + \tilde{S}_{st}^{(3)}(x_j), (s,t,j) \in \Theta\right\}$. Since $\left(\frac{F'F}{T}\right)^{1/2} = \left(I + \left(\frac{F'F}{T} - I\right)\right)^{1/2} = I + \frac{1}{2}\left(\frac{F'F}{T} - I\right) + o_p\left(\frac{1}{\sqrt{T}}\right)$, using Assumption 3, we get $\sqrt{T}\left(\left(\frac{F'F}{T}\right)^{1/2} - I_k\right) \xrightarrow{w} \frac{1}{2}\Phi$. The latter convergence and definition of $\tilde{S}^{(2)}(x)$, $\tilde{S}^{(3)}(x)$, and Φ imply that

$$\left\{\tilde{S}^{(2)}(x_j) + \tilde{S}^{(3)}(x_j), 1 \leq j \leq J\right\} \xrightarrow{w} \left\{\frac{\sqrt{c}}{2}(D\Phi + \Phi D)\int \frac{d\mathcal{F}_c(\lambda)}{x_j - \lambda}, 1 \leq j \leq J\right\}$$

and, hence, $\left\{\tilde{S}_{st}^{(2)}(x_j) + \tilde{S}_{st}^{(3)}(x_j), (s,t,j) \in \Theta\right\}$ weakly converge to $\left\{\Sigma_{stj}^{(1)}, (s,t,j) \in \Theta\right\}$ having joint zero-mean Gaussian distribution such that

$$\operatorname{cov}\left(\Sigma_{stj}^{(1)},\Sigma_{s_1t_1j_1}^{(1)}\right)=\frac{c}{4}\left(d_s+d_t\right)\left(d_{s_1}+d_{t_1}\right)\phi_{sts_1t_1}\int\frac{d\mathcal{F}_c(\lambda)}{x_j-\lambda}\int\frac{d\mathcal{F}_c(\lambda)}{x_{j_1}-\lambda}. \tag{60}$$

Now, let us consider the limit of $\left\{\sum_{v\neq 2,3}\tilde{S}_{st}^{(v)}(x_j);(s,t,j)\in\Theta\right\}$. By definition, we have:

$$\begin{aligned}\sum_{v\neq 2,3}\tilde{S}_{st}^{(v)}(x_j) &= \sqrt{d_sd_t}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\xi_{is}\xi_{it}-\delta_{st}}{h(x,\lambda_i)} \\ &-\sqrt{d_sd_t}\int\frac{d\mathcal{F}_c(\lambda)}{x_j-\lambda}\sum_{i=1}^{n}\frac{\xi_{is}\xi_{it}-\delta_{st}}{\sqrt{n}} \\ &+\sigma\sqrt{cd_s}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\xi_{is}\eta_{it}}{h(x,\lambda_i)}+\sigma\sqrt{cd_t}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\xi_{it}\eta_{is}}{h(x,\lambda_i)} \\ &+\sigma^2c\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\eta_{is}\eta_{it}-\delta_{st}}{h(x,\lambda_i)}\end{aligned} \tag{61}$$

Since $[\xi,\eta]$ is an $n\times 2k$ matrix with i.i.d. standard normal entries, Lemma 5 and (61) imply that $\left\{\sum_{v\neq 2,3}\tilde{S}_{st}^{(v)}(x_j);(s,t,j)\in\Theta\right\}$ weakly converge to $\left\{\Sigma_{stj}^{(2)},(s,t,j)\in\Theta\right\}$ having joint normal distribution such that $\operatorname{cov}\left(\Sigma_{stj}^{(2)},\Sigma_{s_1t_1j_1}^{(2)}\right)=0$ if $(s,t)\neq(s_1,t_1)$ and $\operatorname{cov}\left(\Sigma_{stj}^{(2)},\Sigma_{s_1t_1j_1}^{(2)}\right)$ is equal to

$$\begin{aligned}&\left[\left(1+\delta_{st}\right)\left(\sigma^4c^2+d_sd_t\right)+\sigma^2c\left(d_s+d_t+2\delta_{st}\sqrt{d_sd_t}\right)\right]\int\frac{d\mathcal{F}_c(\lambda)}{\left(x_j-\lambda\right)\left(x_{j_1}-\lambda\right)} \\ &-\left(1+\delta_{st}\right)d_sd_t\int\frac{d\mathcal{F}_c(\lambda)}{x_j-\lambda}\int\frac{d\mathcal{F}_c(\lambda)}{x_{j_1}-\lambda}\end{aligned} \tag{62}$$

otherwise.

Finally, since $\left\{\tilde{S}_{st}^{(2)}(x_j)+\tilde{S}_{st}^{(3)}(x_j),(s,t,j)\in\Theta\right\}$ are, by definition, independent from $\left\{\sum_{v\neq 2,3}\tilde{S}_{st}^{(v)}(x_j);(s,t,j)\in\Theta\right\}$, $\left\{\Sigma_{stj}^{(1)},(s,t,j)\in\Theta\right\}$ must be independent from $\left\{\Sigma_{stj}^{(2)},(s,t,j)\in\Theta\right\}$ and $\left\{\sum_{v=1}^{10}\tilde{S}_{st}^{(v)}(x_j);(s,t,j)\in\Theta\right\}\xrightarrow{w}\left\{\Sigma_{stj}^{(1)}+\Sigma_{stj}^{(2)};(s,t,j)\in\Theta\right\}$, having joint zero-mean Gaussian distribution such that $\operatorname{cov}\left(\Sigma_{stj}^{(1)}+\Sigma_{stj}^{(2)},\Sigma_{s_1t_1j_1}^{(1)}+\Sigma_{s_1t_1j_1}^{(2)}\right)=\operatorname{cov}\left(\Sigma_{stj}^{(1)},\Sigma_{s_1t_1j_1}^{(1)}\right)+\operatorname{cov}\left(\Sigma_{stj}^{(2)},\Sigma_{s_1t_1j_1}^{(2)}\right)$. (60) and (62) imply that the joint distribution of $\Sigma_{stj}^{(1)}+\Sigma_{stj}^{(2)}$ is equal to that of $\left\{N_{st}^{(1)}(x_j);(s,t,j)\in\Theta\right\}$.

Now we have to prove the tightness of all entries of $N_n^{(1)}(x)=\sum_{v=1}^{10}S^{(v)}(x)$. Since product and sum are continuous mappings from $C\left[\theta_1,\theta_2\right]^2$ to $C\left[\theta_1,\theta_2\right]$, it is enough to prove the tightness of every entry of each matrix entering definition of $S^{(v)}(x)$, $v=1,...,10$. Assumption 3 and (56) imply the tightness of every entry of each of the matrices $\left(\frac{F'F}{T}\right)^{1/2}$, $(L'L)^{1/2}$, $\sqrt{n}\left(L'L-D\right)$, $\left(\frac{\xi'\xi}{n}\right)^{-1/2}$, $\left(\frac{\xi'\xi}{n}\right)^{-1}$, $\sqrt{\frac{n}{T}}I$, $\sqrt{n}\left(\frac{n}{T}-c\right)I$, $\sqrt{T}\left(\left(\frac{F'F}{T}\right)^{1/2}-I_k\right)$, and $\sqrt{n}\left(I_k-\left(\frac{\xi'\xi}{n}\right)\right)$ considered as (constant) elements of $C\left[\theta_1,\theta_2\right]$. Therefore, we only need to prove the tightness of entries of

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\xi_{is}\xi_{it}-\delta_{st}}{h(x,\lambda_i)},\ \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\xi_{is}\eta_{it}}{h(x,\lambda_i)},\ \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\eta_{is}\eta_{it}-\delta_{st}}{h(x,\lambda_i)} \tag{63}$$

of $\sum_{i=1}^{n}\frac{1}{nh(x,\lambda_i)}$ and of $\sqrt{n}\left(\int\frac{d\mathcal{F}_c(\lambda)}{x-\lambda}-\sum_{i=1}^{n}\frac{1}{nh(x,\lambda_i)}\right)$.

Since ξ and η are, by definition, two independent $n\times k$ matrices with i.i.d. standard normal entries, to prove the tightness of the sequences of sums in (63), it is enough to prove the tightness of the first sum for all $1\le s\le t\le k$. We will use Theorem 12.3 of Billingsley (1968), p. 95. Condition i) of the theorem is equivalent in our context to the assumption of the tightness of the sum at $x=\theta_1$. Lemma 5 implies that this assumption is satisfied. We will verify condition ii) of Theorem 12.3 by proving the moment condition (12.51) of Billingsley (1968). We have $\frac{E\left(\sum_{i=1}^{n}\left(\frac{1}{h(x_1,\lambda_i)}-\frac{1}{h(x_2,\lambda_i)}\right)\frac{\xi_{is}\xi_{it}-\delta_{st}}{\sqrt{n}}\right)^2}{(x_1-x_2)^2}\le E\left(\sum_{i=1}^{n}\frac{1}{h(x_1,\lambda_i)h(x_2,\lambda_i)}\frac{\xi_{is}\xi_{it}-\delta_{st}}{\sqrt{n}}\right)^2\le\frac{16}{(\theta_1-b)^4}E\left(\sum_{i=1}^{n}\frac{\xi_{is}\xi_{it}-\delta_{st}}{\sqrt{n}}\right)^2=\frac{16}{(\theta_1-b)^4}(1+\delta_{st})$, where the first inequality follows from the fact that $\left|\frac{1}{h(x_1,\lambda_i)}-\frac{1}{h(x_2,\lambda_i)}\right|\le\frac{|x_2-x_1|}{h(x_1,\lambda_i)h(x_2,\lambda_i)}$. Hence, $\sup_{n;x_1,x_2\in[\theta_1,\theta_2]}E\left(\sum_{i=1}^{n}\left(\frac{1}{h(x_1,\lambda_i)}-\frac{1}{h(x_2,\lambda_i)}\right)\frac{\xi_{is}\xi_{it}-\delta_{st}}{\sqrt{n}}\right)^2/(x_1-x_2)^2$ is finite and the moment condition (12.51) of Billingsley (1968) is satisfied. In a more complete proof (in which the tightness of the elements of $N_n^{(2)}(x)$ is demonstrated), we also need to check Billingsley's moment condition when $h(\cdot,\cdot)$ is replaced by $h^2(\cdot,\cdot)$. We can use the above reasoning and inequality $\left|\frac{1}{h^2(x_1,\lambda_i)}-\frac{1}{h^2(x_2,\lambda_i)}\right|\le\frac{|x_2-x_1|(h(x_1,\lambda_i)+h(x_2,\lambda_i))}{h^2(x_1,\lambda_i)h^2(x_2,\lambda_i)}\le\frac{32\theta_2|x_2-x_1|}{(\theta_1-b)^4}$ to perform such a check.

Similarly, conditions of Theorem 12.3 of Billingsley (1968) are satisfied for $\sum_{i=1}^{n}\frac{1}{nh(x,\lambda_i)}$. Condition i) is satisfied because, by (59), $\sum_{i=1}^{n}\frac{1}{nh(x,\lambda_i)}\xrightarrow{p}\int\frac{d\mathcal{F}_c(\lambda)}{x-\lambda}$ for any $x\in[\theta_1,\theta_2]$. Condition ii) is satisfied because $E\left(\sum_{i=1}^{n}\frac{1}{nh(x_1,\lambda_i)h(x_2,\lambda_i)}\right)^2\le\frac{16}{(\theta_1-b)^4}$.

To prove the tightness of $\sqrt{n}\left(\int\frac{d\mathcal{F}_c(\lambda)}{x-\lambda}-\sum_{i=1}^{n}\frac{1}{nh(x,\lambda_i)}\right)$, we adopt the argument on page 563 of Bai and Silverstein (2004). In notations of Bai and Silverstein (2004), $\hat{M}_n(\cdot)\to-\frac{1}{2\pi i}\int\frac{1}{x-z}\hat{M}_n(z)dz$ is a continuous mapping of $C\left(\mathcal{C},R^2\right)$ into $C[\theta_1,\theta_2]$. Since, $\hat{M}_n(\cdot)$ is tight, $-\frac{1}{2\pi i}\int\frac{1}{x-z}\hat{M}_n(z)dz$, and subsequently $n\left(\int\frac{d\mathcal{F}_{n/T}(\lambda)}{x-\lambda}-\sum_{i=1}^{n}\frac{1}{n}\frac{1}{x-\lambda_i}\right)$, form a tight sequence. But $\sup_{x\in[\theta_1,\theta_2]}\sqrt{n}\left(\int\frac{d\mathcal{F}_{n/T}(\lambda)}{x-\lambda}-\int\frac{d\mathcal{F}_c(\lambda)}{x-\lambda}\right)\xrightarrow{p}0$ because, by assumption, $n/T-c=o(1/\sqrt{n})$. Therefore, $\sqrt{n}\left(\int\frac{d\mathcal{F}_c(\lambda)}{x-\lambda}-\sum_{i=1}^{n}\frac{1}{n}\frac{1}{x-\lambda_i}\right)$ is tight too. Finally, the latter tightness and (58) imply that sequence $\sqrt{n}\left(\int\frac{d\mathcal{F}_c(\lambda)}{x-\lambda}-\sum_{i=1}^{n}\frac{1}{nh(x,\lambda_i)}\right)$ must be tight.□

Proof of Lemma 3:

Let $R(z,\varkappa)=(A(\varkappa)-zI_k)^{-1}$ be the resolvent of $A(\varkappa)$ defined for all complex z not equal to any of the eigenvalues of $A(\varkappa)$. We will denote $R(z,0)$ as $R(z)$. Let Γ be a positively oriented circle in the complex plain with center at a_j and radius r_0. The second Neumann series for the resolvent $R(z,\varkappa)=R(z)+\sum_{n=1}^{\infty}(-\varkappa)^n R(z)\left(A^{(1)}R(z)\right)^n$ (see Kato (1980), p.67, for a definition of the second Neumann series) is uniformly convergent on Γ for $\varkappa<\min_{z\in\Gamma}\frac{1}{\left\|A^{(1)}\right\|\|R(z)\|}=\frac{r_0}{\left\|A^{(1)}\right\|}$, where the last equality follows from the fact that $\|R(z)\| = \frac{1}{r_0}$ for any $z \in \Gamma$. Therefore, formula (1.19) of Kato (1980) implies that, for $|\varkappa| < \frac{r_0}{\|A^{(1)}\|}$, there is exactly one eigenvalue, $a_j(\varkappa)$, inside the circle Γ. Formulae (3.6)[5] and (2.32) of Kato (1980) imply the inequality stated in part i of Lemma 3.

We now turn to the proof of part ii. According to Kato (1980), p.67, projection $P_j(\varkappa)$ can be represented as $P_j(\varkappa) = -\frac{1}{2\pi i}\int_\Gamma R(z,\varkappa)\,dz$. Substituting the second Neumann series for the resolvent in this formula, we obtain

$$P_j(\varkappa) = P_j - \frac{1}{2\pi i}\sum_{n=1}^{\infty}(-\varkappa)^n \int_\Gamma R(z)\left(A^{(1)}R(z)\right)^n dz \tag{64}$$

where $P_j \equiv P_j(0)$ and the series absolutely converges for $|\varkappa| < \frac{r_0}{\|A^{(1)}\|}$. Kato (1980), page 76, shows that

$$\frac{1}{2\pi i}\int_\Gamma R(z)\,A^{(1)}R(z)\,dz = -P_jA^{(1)}S_j - S_jA^{(1)}P_j. \tag{65}$$

This equality and (64) imply that $P_j(\varkappa) = P_j - \varkappa\left(P_jA^{(1)}S_j - S_jA^{(1)}P_j\right) - \frac{1}{2\pi i}\sum_{n=2}^{\infty}(-\varkappa)^n\int_\Gamma R(z)\left(A^{(1)}R(z)\right)^n dz$. Therefore, we have:

$$\left\|\frac{1}{\varkappa}\left(P_j(\varkappa) - P_j\right) + P_jA^{(1)}S_j + S_jA^{(1)}P_j\right\| \le \frac{|\varkappa|\left\|A^{(1)}\right\|^2}{r_0\left(r_0 - |\varkappa|\left\|A^{(1)}\right\|\right)} \tag{66}$$

for any $|\varkappa| < \frac{r_0}{\|A^{(1)}\|}$.

Since A is diagonal with decreasing elements along the diagonal, e_j is an eigenvector of A corresponding to the eigenvalue a_j. By definition of $P_j(\varkappa)$, $e_j(\varkappa) \equiv \frac{P_j(\varkappa)e_j}{\|P_j(\varkappa)e_j\|}$ must be an eigenvector of $A(\varkappa)$ corresponding to the eigenvalue $a_j(\varkappa)$. We have:

$$\frac{1}{\varkappa}\left(e_j(\varkappa) - e_j\right) + S_jA^{(1)}e_j = \left(\frac{1}{\varkappa}\left(P_j(\varkappa)e_j - e_j\right) + S_jA^{(1)}e_j\right) + \frac{1}{\varkappa}e_j(\varkappa)\left(1 - \|P_j(\varkappa)e_j\|\right). \tag{67}$$

For the first term on right hand side of (67), using (66) and the fact that $S_je_j = 0$, we have:

$$\left\|\frac{1}{\varkappa}\left(P_j(\varkappa)e_j - e_j\right) + S_jA^{(1)}e_j\right\| \le \frac{|\varkappa|\left\|A^{(1)}\right\|^2}{r_0\left(r_0 - |\varkappa|\left\|A^{(1)}\right\|\right)}. \tag{68}$$

Using the fact that $P_j(\varkappa)$ is a projection operator so that $\|P_j(\varkappa)e_j\| \le 1$ and $P_j(\varkappa)^2 = P_j(\varkappa)$, for the second term on right hand side of (67), we have

$$\left\|\frac{1}{\varkappa}e_j(\varkappa)\left(1 - \|P_j(\varkappa)e_j\|\right)\right\| \le \frac{1}{|\varkappa|}\left(1 - \|P_j(\varkappa)e_j\|^2\right) = |\varkappa|\left\|\frac{1}{\varkappa}\left(P_j(\varkappa)e_j - e_j\right)\right\|^2. \tag{69}$$

[5] Note the difference in notations. Kato's r_0 is ours $r_0/\left\|A^{(1)}\right\|$.

But, form (68),

$$\left\| \frac{1}{\varkappa} \left(P_j \left(\varkappa \right) e_j - e_j \right) \right\|^2 \leq 2 \left\| S_j A^{(1)} e_j \right\|^2 + \frac{2 \left| \varkappa \right|^2 \left\| A^{(1)} \right\|^4}{r_0^2 \left(r_0 - \left| \varkappa \right| \left\| A^{(1)} \right\| \right)^2} \leq \frac{\left\| A^{(1)} \right\|^2}{2 r_0^2} + \frac{2 \left| \varkappa \right|^2 \left\| A^{(1)} \right\|^4}{r_0^2 \left(r_0 - \left| \varkappa \right| \left\| A^{(1)} \right\| \right)^2} \quad (70)$$

Combining (67), (68), (69), and (70), we obtain: $\left\| \frac{1}{\varkappa} \left(e_j \left(\varkappa \right) - e_j \right) + S_j A^{(1)} e_j \right\| \leq \frac{|\varkappa| \left\| A^{(1)} \right\|^2 \left(3r_0^2 - 4 r_0 |\varkappa| \left\| A^{(1)} \right\| + 5 |\varkappa|^2 \left\| A^{(1)} \right\|^2 \right)}{2 r_0^2 \left(r_0 - |\varkappa| \left\| A^{(1)} \right\| \right)^2} \leq \frac{2 |\varkappa| \left\| A^{(1)} \right\|^2}{\left(r_0 - |\varkappa| \left\| A^{(1)} \right\| \right)^2}$, where the last inequality follows from the fact that $r_0 > |\varkappa| \left\| A^{(1)} \right\|$. This proves statement ii) of the lemma.□

Proof of Lemma 4:

Since $f_n(x) \xrightarrow{d} f_0(x)$, $\{f_n(x)\}$ is tight and, hence, for any $\varepsilon > 0$, we can choose a compact K such that $P\left(f_n(x) \in K\right) > 1 - \frac{\varepsilon}{2}$ for all n. By the Arzelà-Ascoli theorem (see, for example, Billingsley (1999), p.81), for any positive ε_1, we have $K \subset \{f : |f(\theta_1)| \leq r\}$ for large enough r and $K \subset \{f : w_f\left(\delta(\varepsilon_1)\right) \leq \varepsilon_1\}$ for small enough $\delta(\varepsilon_1)$, where $w_f(\delta)$ is the modulus of continuity of function f, defined as $w_f(\delta) = \sup_{|s-t| \leq \delta} |f(s) - f(t)|, 0 < \delta \leq \theta_2 - \theta_1$. Let us choose $N(\varepsilon, \varepsilon_1)$ so that for any $n > N(\varepsilon, \varepsilon_1)$, $P\left(|x_n - x_0| > \delta(\varepsilon_1)\right) < \frac{\varepsilon}{2}$. Then, for $n > N(\varepsilon, \varepsilon_1)$, we have:

$$\begin{aligned} P\left(|f_n(x_n) - f_n(x_0)| > \varepsilon_1\right) &= P\left(|f_n(x_n) - f_n(x_0)| > \varepsilon_1 \text{ and } |x_n - x_0| \leq \delta(\varepsilon_1)\right) \\ &\quad + P\left(|f_n(x_n) - f_n(x_0)| > \varepsilon_1 \text{ and } |x_n - x_0| > \delta(\varepsilon_1)\right) \\ &\leq P\left(f_n(x) \notin K\right) + P\left(|x_n - x_0| > \delta(\varepsilon_1)\right) < \varepsilon, \end{aligned}$$

which proves the lemma.□

References

[1] Anderson, T.W. (1971) The statistical analysis of time series, John Wiley & Sons, New York, Chichister, Brisbane, Toronto.

[2] Anderson, T.W. (1984) An introduction to multivariate statistical analysis, second edition, John Wiley & Sons, New York, Chichister, Brisbane, Toronto, Singapore.

[3] Bai, J. (2003) “Inferential Theory for Factor Models of Large Dimensions”, Econometrica 71, 135-171.

[4] Baik, J., G. Ben Arous, and S. Péché (2004) "Phase transition of the largest eigenvalue for non-null complex sample covariance matrices”, http://xxx.lanl.gov/abs/math.PR/0403022

[5] Bai, J., and S. Ng (2005) "Confidence intervals for diffusion index forecasts and inference for factor augmented regressions", *Econometrica*, forthcoming.

[6] Bai, Z.D., and J.W. Silverstein (2004) "CLT of linear spectral statistics of large dimensional sample covariance matrices", The Annals of Probability 32(1A) pp. 553-605

[7] Baik, J., and J. W.. Silverstein (2004) "Eigenvalues of Large Sample Covariance Matrices of Spiked Population Models", manuscript available at http://www4.ncsu.edu/~jack/pub.html

[8] Bai, Z., Silverstein, J., and Yin, Y. (1988) "A note on the largest eigenvalue of a large dimensional sample covariance matrix", Journal of Multivariate Analysis, 26, pp 166-168.

[9] Bernanke, B., Boivin, J., and Eliasz, P. (2004) "Measuring the Effects of Monetary Policy: A Factor-Augmented Vector Autoregressive (FAVAR) Approach", NBER Working Paper 10220

[10] Billingsley, P. (1968) Convergence of probability measures, John Wiley & Sons, New York, Chichester, Brisbane, Toronto.

[11] Billingsley, P. (1999) Convergence of probability measures, second edition, Wiley, New York.

[12] Chamberlain, G. and Rothschild, M. (1983) "Arbitrage, factor structure, and mean-variance analysis on large asset markets", Econometrica, 51, pp.1281-1304.

[13] Connor, G. and R. Korajczyk (1988) "Risk and return in an equilibrium APT", Journal of Financial Econometrics 21, pp. 255-289.

[14] Forni, M. and Reichlin, L. (1998) "Let's get real: a factor analytical approach to disaggregated business cycle dynamics", Review of Economic Studies, 65, pp 453-473.

[15] Hall, P., and C.C. Heyde (1980) Martingale limit theory and its application, New York : Academic Press.

[16] Horn, R.A., and C.R.Johnson (1991) Matrix analysis. Cambridge University Press, Cambridge, New York, Port Chester, Melbourne, Sydney.

[17] Johnstone, I.M. (2001) "On the distribution of the largest eigenvalue in principal components analysis", Annals of Statistics, 29, 295–327

[18] Johnstone, I.M., and A. Y. Lu (2004) “Sparce principal components analysis”, manuscript available at http://www-stat.stanford.edu/people/faculty/johnstone/WEBLIST/publist.html

[19] Kato, T (1980) Perturbation theory for linear operators, Springer-Verlag, Berlin, New York.

[20] Marčenko, V.A., and L.A. Pastur (1967) “Distribution of eigenvalues for some sets of random matrices”, Math. USSR-Sbornik, vol. 1, no. 4, 457-483

[21] McLeish, D.L. (1974) "Dependent Central Limit Theorems and Invariance Principles", Annals of Probability, Vol. 2, No. 4, p.620-628.

[22] Paul, D. (2004) “Asymptotics of the leading sample eigenvalues for a spiked covariance model”, manuscript available at http://www-stat.stanford.edu/~debashis/papers.html

[23] Péché, S. (2003) “Universality of local eigenvalue statistics for random sample covariance matrices“, PhD thesis, Ecole Polytechnique Federale de Lausanne.

[24] Silverstein, J. (1995) “Strong convergence of the empirical distribution of large dimensional random matrices”, Journal of Multivariate Analysis, 55, pp 331-339.

[25] Stock, J. and Watson, M. (2002) “Macroeconomic Forecasting Using Diffusion Indexes”, Journal of Business and Economic Statistics, 20, pp. 147-162.

[26] Stock, J., and M. Watson (2005) “Implications of Dynamic Factor Models for VAR Analysis”, manuscript available at http://www.wws.princeton.edu/mwatson/wp.html

[27] Vasilchuk, V. (2001) “On the Law of Multiplication of Random Matrices”, Mathematical Physics, Analysis and Geometry Vol. 4, pp.1-36.

[28] White, H. (1999) Asymptotic theory for econometricians, revised edition, Academic press, San Diego, San Francisco, New York, Boston, London, Sydney, Tokyo.