Winding Number Criterion for Existence and Uniqueness of Equilibrium in Linear Rational Expectations Models

A. Onatski
Department of Economics
Columbia University
e-mail: ao2027@columbia.edu

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Abstract

This paper develops a simple geometric criterion for the existence and uniqueness of equilibrium in linear rational expectations (LRE) models. The class of models considered is very general and includes models with infinitely many lags and/or leads of endogenous variables. I show that the criterion is tightly related to the familiar Blanchard and Kahn (1980), Hansen and Sargent (1981), and Whiteman (1983) conditions for the existence and uniqueness of equilibrium in LRE models. An application of the criterion to an analysis of the monetary policy robustness under the uncertainty about the standard New Keynesian model of the economy is given.

1. Introduction

This paper develops a simple geometric criterion for the existence and uniqueness of equilibrium in linear rational expectations (LRE) models of the form:

$$\sum_{j=-\infty}^{+\infty} A_j E_t x_{t-j} = \Gamma z_t. \tag{1}$$

Here x and z are n-dimensional and m-dimensional vectors of endogenous and exogenous variables respectively, E_t represents mathematical expectation given information available at time t including the model's structure and all past and present realizations of the exogenous and endogenous processes, A_j are element-by-element absolutely summable $n \times n$ matrix coefficients, and Γ is an $n \times m$ matrix.

Since the number of lags and leads of the endogenous variables in model (1) may be infinite, it does not, in general, reduce to any of the particular models studied in the literature. Therefore, criteria of solvability developed in Blanchard and Kahn (1980), Hansen and Sargent (1981), Whiteman (1983), Anderson and Moore (1985), Broze, Gourieroux, and Szafarz (1995), Sims (2002), Klein (2000), Binder and Pesaran (1997), King and Watson (1998), McCallum (1998), Zadrozny (1998), and Uhlig (1999) cannot be applied. On the contrary, equations (1) nest a variety of models examined in the literature so that solvability of these models may be analyzed with the help of the criterion developed below.

There exist several reasons why studying models with an infinite number of lags and leads of endogenous variables may be interesting to economists. From a purely methodological perspective, analyzing more general models gives new insights about methods developed under more restrictive assumptions and allows their deeper interpretation. In the paper, I discuss in detail links between the new criterion and those of Blanchard and Kahn (1980), Hansen and Sargent (1981), and Whiteman (1983). This discussion helps to look at the previously developed criteria from a new angle and to better see connections between them.

More substantively, models of form (1) may result from the first order conditions of optimization problems with non-parametric description of habit formation, adjustment costs, time discount schemes, and time-to-build constraints. Last but not least, the infinite LRE models occur in the research on model uncertainty in macroeconomics and finance (see Hansen and Sargent 2003).[1] This literature uses the methods of robust control to analyze optimal economic decisions when there is no single reliable model of the economy but a whole ball of potentially true alternatives in the model space. For this research, it is very convenient not to restrict the set of potentially true models to those that can be finitely parameterized.

The criterion developed below is based on the behavior of the graph of function:

$$\det A(\lambda) = \det \sum_{j=-\infty}^{+\infty} A_j e^{ij\lambda}, \tag{2}$$

where λ is a real variable and $i = \sqrt{-1}$. It is assumed that model (1) has no unit roots so that $\det A(\lambda) \neq 0$ for any real λ. Since $e^{ij\lambda}$ are periodic functions with period $\frac{2\pi}{j}$, function $\det A(\lambda)$ tracks a closed contour in the complex space when λ changes from 0 to 2π. Therefore, the number of times the graph of $\det A(\lambda)$ rotates around zero counter-clockwise when λ goes from 0 to 2π is well defined. It is called the winding number of $\det A(\lambda)$. A negative winding number corresponds to clockwise rotations of the graph around zero. The main result of the paper can be formulated as follows.

A generic LRE model (1) has a unique solution, multiple solutions, or, in general, no solutions if and only if the winding number of $\det A(\lambda)$ is equal to zero, less than zero, or greater than zero respectively.

Non-generic models are rare. If model (1) is not generic, it can be made generic by

[1]For the papers using the infinite models see Onatski and Stock (2002), Tetlow and von zur Muehlen (2001), Onatski and Williams (2003).

an arbitrarily small change in the coefficients A_j. The genericity property is formally defined and linked to conditions familiar from the previous literature in sections 2 and 3.

To illustrate the above criterion let me consider a simple scalar LRE model:

$$ax_{t-1} + bx_t + cE_t x_{t+1} = z_t. \tag{3}$$

In the paper, we analyze the following two scenarios. First, the model equations are assumed to hold for $t = 0, 1, 2, ...$, exogenous and endogenous processes $\{z_t\}$ and $\{x_t\}$ must satisfy asymptotic growth restrictions, and the endogenous process must comply with initial conditions. The exogenous and endogenous processes are unrestricted otherwise and, in particular, may be non-stationary. Second, the model equations are assumed to hold for $t = ... - 1, 0, 1, ...$, both the exogenous and endogenous processes are assumed to be weakly stationary, and a solution x_t must be representable as a square summable linear combination of past and present realizations of the process fundamental for $\{z_t\}$.

For the above example, we focus on the first scenario and look for a bounded in expectation solution to (3) satisfying an initial condition $x_{-1} = \bar{x}$, where $\bar{x}$ is arbitrary. As our first framework dictates, we assume that equations (3) hold for $t = 0, 1, 2, ...$ and consider only bounded in expectation exogenous processes z_t. As is shown below, all scalar LRE models are generic. Hence, the criterion reduces to checking the sign of the winding number of function $\det A(\lambda)$. The function is equal to $ae^{i\lambda} + b + ce^{-i\lambda}$. Three different choices of a, b, and c: $(a, b, c) = (0, 1, 2)$, $(a, b, c) = (1, 2, 0)$, and $(a, b, c) = (2, 1, 0)$, correspond to situations when model (3) has, respectively, multiple, unique and no bounded solutions satisfying the initial condition. The corresponding graphs of $\det A(\lambda)$ are shown in figures 1a, 1b and 1c. In accordance with the criterion, the graph rotates around zero clockwise in figure 1a, does not encircle zero in figure 1b, and rotates around zero counter-clockwise in figure 1c.

To see a less trivial example consider a scalar LRE model:

$$E_t x_{t+1} - (1 - L)^{\frac{1}{3}} x_t = z_t, \tag{4}$$

where we again focus on the bounded solution.[2] For this model, we assume that the whole bounded initial history $x_{-1}, x_{-2}, ...$ is given and must be satisfied. Clearly, function $\det A(\lambda)$ is equal to $e^{-i\lambda} - (1 - e^{i\lambda})^{\frac{1}{3}}$. Its graph is shown in figure 1d. It rotates around zero clockwise so that, according to the criterion, the model has multiple solutions.

In this paper, I consider an application of the criterion to an analysis of the robustness of monetary policy under the uncertainty about the standard New Keynesian model of the economy, described by Woodford (1998) and Clarida, Gali and Gertler (1999) among others. The robustness of a particular policy is measured by

[2] Operator $(1 - L)^{\frac{1}{3}}$ is defined in terms of its Taylor expansion as $(1 - L)^{\frac{1}{3}} = \sum_{n=0}^{\infty} c_n L^n$, where $c_0 = 1$ and $c_n = \frac{1}{n!} \left(-\frac{1}{3}\right)^n \prod_{k=0}^{n-1} (1 - 3k)$ for $n \geq 1$.

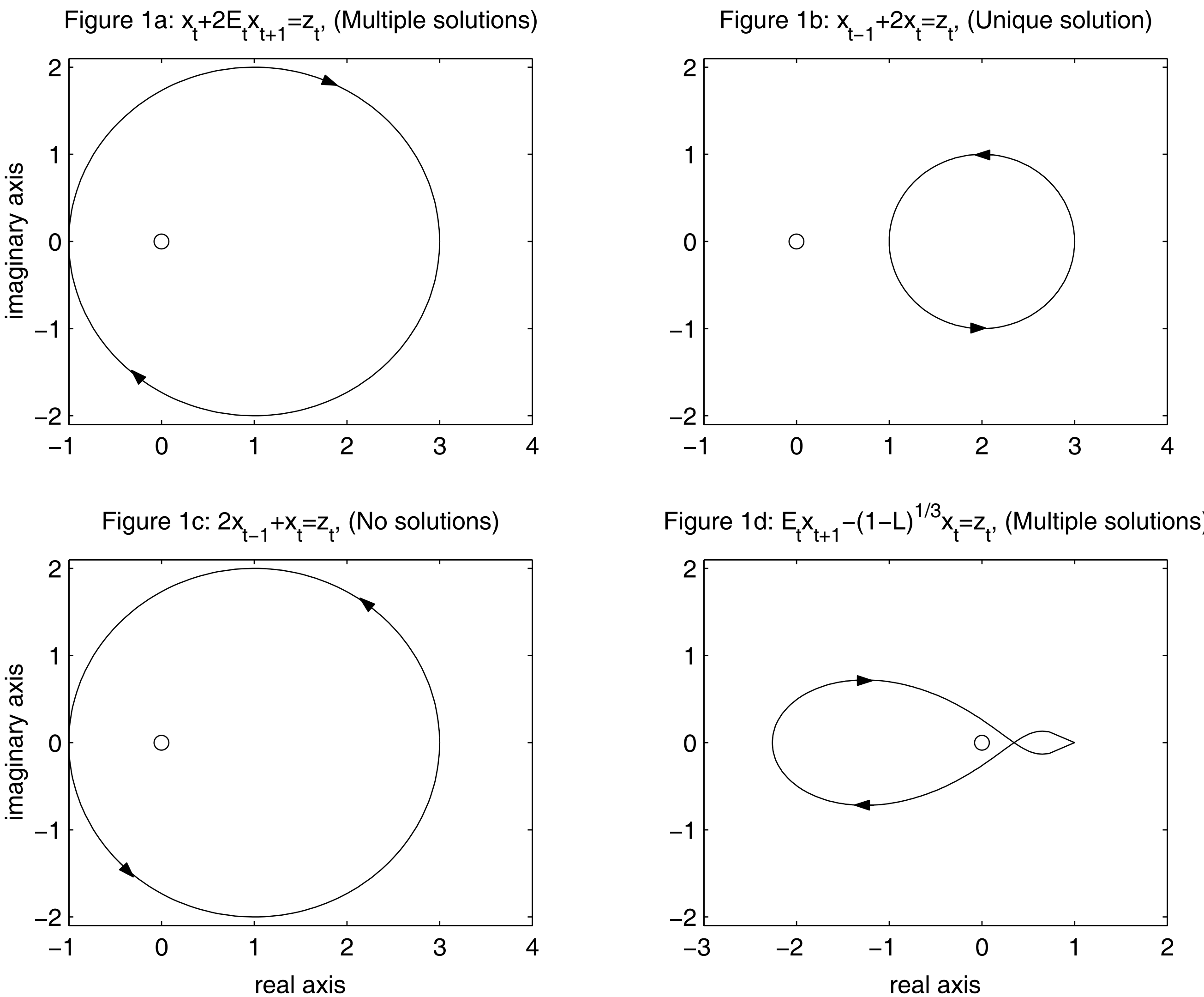
Figure 1a: $x_t+2E_tx_{t+1}=z_t$, (Multiple solutions)
Figure 1b: $x_{t-1}+2x_t=z_t$, (Unique solution)
Figure 1c: $2x_{t-1}+x_t=z_t$, (No solutions)
Figure 1d: $E_tx_{t+1}-(1-L)^{1/3}x_t=z_t$, (Multiple solutions)
imaginary axis
real axis

the smallest size of a potential deviation from the New Keynesian model that results in dynamic instability or indeterminacy of equilibrium under the policy analyzed. The criterion plays the key role in the analysis because the deviations considered are not necessarily finitely parameterized.

I find that simple policy rules that set the interest rate as a linear function of current inflation and the output gap are very robust to the model uncertainty for a wide range of the policy rules' coefficients. This stands in contrast to the robustness analysis of the policy rules under model uncertainty about Rudebusch and Svensson's (1999) purely backward-looking model of the economy performed by Onatski and Stock (2002). They showed that relatively more aggressive rules destabilize the economy for smaller deviations from Rudebusch and Svensson's model than relatively less aggressive rules. The difference in the results accords with the intuition that forward-looking models are in general more difficult to destabilize than models with adaptive expectations because rational agents react to would be instability by changing their actions so as to justify their beliefs in economic stability.

The remainder of this paper is organized as follows. In section 2, I describe the non-stationary and stationary frameworks of the analysis performed in the paper and formulate the new criterion for multivariate LRE models. Section 3 studies connections between the new criterion and those of Blanchard and Kahn (1980), Hansen and Sargent (1981), and Whiteman (1983). An application to an analysis of the robustness of monetary policy to the uncertainty about the standard New Keynesian model of the economy is considered in Section 4. Section 5 concludes. All proofs are given in the Appendix.

2. The new criterion for multivariate LRE models

As has been shown by Broze, Gourieroux, and Szafarz (1995) among others, if we do not restrict exogenous and endogenous processes $\{x_t\}$ and $\{z_t\}$ beyond requiring the existence of the first moments, then, at least for the conventional finite lags and leads LRE models, there typically exists an infinite number of solutions. Following most of the previous work, we therefore restrict our analysis to the processes that either satisfy asymptotic growth restrictions or are stationary. More formally, we consider the following two frameworks.

A non-stationary framework: Suppose that the flow of information over time is represented by an *a priori* given filtration $\{\mathcal{F}_t;\ t = 0, 1, ...\}$ of the underlying probability space $(\Omega, \mathcal{F}, P)$. We assume that the exogenous and endogenous processes are adapted to the filtration and must satisfy the following asymptotic growth restrictions for any $t = 0, 1, 2...$:

$$|E_t x_{t+i}| \leq X_t,\ \ |E_t z_{t+i}| \leq Z_t\ \forall i \geq 0. \tag{5}$$

Here X_t and Z_t are some positive n-dimensional and m-dimensional random vectors with finite unconditional expectation.[3] We assume that $\{x_t\}$ satisfies bounded initial

[3]These restrictions are relatively weak. In particular, they do not require processes $\{x_t\}$ and

conditions. That is values x_t are fixed for all $t<0$ and satisfy[4]:

$$\sup_{t<0} |x_t| < \infty. \tag{6}$$

The convergence on the left hand side of (1) is understood as the almost sure convergence. The model equations are assumed to hold for $t = 0, 1, 2, ...$

A stationary framework: We assume that the exogenous process is a weakly stationary process with known coefficients of Wold's representation and E_t denotes the operation of taking the best linear forecast based on information available at time t that consists of all past and present realizations (up to and including time t) of the exogenous and endogenous processes. Following Whiteman (1983), we restrict attention to those solutions $\{x_t\}$ to the model that can be represented as time-invariant square summable linear combination of past and present realizations of the process fundamental for $\{z_t\}$, where the square summability of the representation's matrix coefficients is understood in the element-by-element sense. The convergence of the infinite sum in the model's equation is understood in the mean square sense. The model equations are assumed to hold for $t = ... - 1, 0, 1, ...$ Finally, matrix Γ on the right hand side of (1) is assumed not to be identically equal to zero.

For the non-stationary framework, we will say that the model has a unique solution if for any admissible process $\{z_t\}$ and initial conditions, there exists one and only one solution to the model's equations. We will say that the model has no solutions if there exists an admissible process $\{z_t\}$ and initial conditions such that the model's equations do not have solutions. We will say that there are multiple solutions to the model if for any admissible $\{z_t\}$ and initial conditions there exists a solution and for some admissible $\{z_t\}$ and initial conditions there exist many solutions. For the stationary framework, the words "and initial conditions" in the above definition of solvability must be omitted.

Similar to many existent criteria such as those of Blanchard and Kahn (1980), Hansen and Sargent (1981), and Whiteman (1983), the new winding number criterion is based on a factorization of the LRE model into feedback and feedforward parts. The criterion is nothing but a summary of the properties of the factorization that are necessary and sufficient for the model to have no, unique, or multiple solutions.

We use the same factorization for both our non-stationary and stationary frameworks. It has the form:

$$A(\lambda) = A_-(\lambda)A_0(\lambda)A_+(\lambda), \tag{7}$$

where $A_+(\lambda)$ and $A_+^{-1}(\lambda)$, the matrix inverse of $A_+(\lambda)$, have absolutely summable Fourier expansions with zero coefficients on the negative powers of $e^{i\lambda}$; $A_-(\lambda)$ and $A_-^{-1}(\lambda)$ have absolutely summable Fourier expansions with zero coefficients on the positive powers of $e^{i\lambda}$; and $A_0(\lambda)$ has a diagonal matrix form:

$$A_0(\lambda) = \mathrm{diag}(e^{i\lambda k_1}, e^{i\lambda k_2}, ..., e^{i\lambda k_n}), \tag{8}$$

$\{z_t\}$ to have second moments, only the first moments must exist. The restrictions can easily be extended to allow exponentially growing processes with a bounded rate of growth.

[4] If the i-th columns of all matrix coefficients A_j, $j \geq \tau > 0$ are equal to zero, then values of the i-th components of $x_{-\tau}, x_{-\tau-1}, ...$ do not essentially enter model (1) and the corresponding initial conditions may be ignored or arbitrarily set.

where the integers $k_1 \leq k_2 \leq ... \leq k_n$ are uniquely determined and called *the partial indices* of $A(\lambda)$. $A_+(\lambda)$ represents the feedback part of the model, $A_-(\lambda)$ represents the feedforward part, and $A_0(\lambda)$ represents a pure time shift that plays the key role in the workings of the criterion.

In mathematical literature the above factorization is called the Wiener-Hopf factorization (see for example Bottcher and Silbermann 1999). It resembles the canonical spectral factorization (see for example Priestly 1981, p.761) of the spectral density function of a stationary vector process which is a basic ingredient of Kolmogorov and Wiener's solutions to the problem of predicting a future value of the process.[5] For the canonical spectral factorization, $A_-(\lambda)$ is the complex conjugate of $A_+(\lambda)$ and $A_0(\lambda)$ is the identity matrix. However, the spectral factorization is only valid for positive definite matrix functions, whereas factorization (7) holds for all complex-valued matrix functions with absolutely summable coefficients of their element-by-element Fourier expansions. The existence of such a factorization was established by Gohberg and Krein (1960).[6] It is relatively easy to construct for rational functions (see Clancey and Gohberg 1981) which correspond to conventional LRE models with a finite number of lags and leads of endogenous variables.

A proof of the following proposition for both the non-stationary and stationary frameworks can be found in the Appendix.

Proposition 1. *Model (1) has a unique solution if all partial indices of $A(\lambda)$ are equal to zero. It has no solutions if at least one of the partial indices of $A(\lambda)$ is positive. The model has many solutions if all partial indices of $A(\lambda)$ are non-positive and at least one of them is negative.*

To build intuition, let us consider the following example illustrating the above proposition. In our non-stationary framework, consider model:

$$x_t - \frac{1}{2}x_{t-1} = z_t, \quad t = 0, 1, 2, ..., \tag{9}$$

where values x_t for $t < 0$ are given and fixed. The function $A(\lambda)$ for this model is equal to $1 - \frac{1}{2}e^{i\lambda}$. It admits a trivial Wiener-Hopf factorization with $A_+(\lambda) = 1 - \frac{1}{2}e^{i\lambda}$, $A_0(\lambda) = 1$, and $A_-(\lambda) = 1$. Therefore, the partial index of $A(\lambda)$ is equal to zero and according to the proposition, there exists a unique bounded solution that can be obtained by the standard backward induction.

Now, shifting the left hand side of the model's equations by one period backwards is equivalent to multiplying $A(\lambda)$ by $e^{i\lambda}$, which does not affect neither $A_+(\lambda)$ nor $A_-(\lambda)$ but makes $A_0(\lambda)$ equal to $e^{i\lambda}$ and hence, makes the partial index of $A(\lambda)$ equal to 1. According to the above proposition, such a shifted model:

$$x_{t-1} - \frac{1}{2}x_{t-2} = z_t, \quad t = 0, 1, 2, ... \tag{10}$$

[5] I thank an anonymous referee for pointing out the relationship between the Wiener-Hopf factorization and the canonical spectral factorization.

[6] A full proof of the existence can also be found in Gohberg and Feldman (1974), Theorem 2.2, Chapter VIII.

in general has no solutions. Indeed, if we choose an exogenous process $\{z_t\}$ so that $z_0 \neq x_{-1} - \frac{1}{2}x_{-2}$, the equation of the model will not hold for $t = 0$ and the given initial conditions.

Similarly, if we shift the left hand side of the model one period forwards and take expectation conditional on information available at time t, $A_0(\lambda)$ becomes equal to $e^{-i\lambda}$ and the partial index of $A(\lambda)$ becomes equal to -1. According to the proposition, model:

$$E_t x_{t+1} - \frac{1}{2}x_t = z_t, \quad t = 0, 1, 2, ... \tag{11}$$

has multiple bounded solutions. It is not hard to see that processes $x_t = \sum_{i=1}^{t} \left(\frac{1}{2}\right)^{i-1} z_{t-i} + \sum_{i=0}^{t} \left(\frac{1}{2}\right)^{i} \varepsilon_{t-i}$, $t = 0, 1, 2, ...$, where $\{\varepsilon_t\}$ is any martingale difference sequence adapted to the flow of information are the solutions.

Note that to analyze the existence and uniqueness of solutions to model (1) we do not need to construct the factorization explicitly. All we need to know is whether all partial indices of $A(\lambda)$ are zero, non-positive, or neither of the above. As we show below, the partial indices of $A(\lambda)$ can be effectively analyzed with the help of the winding number of $A(\lambda)$.

Indeed, the sum of the partial indices of $A(\lambda)$ must be equal to the winding number of $\det A(\lambda)$. To see this, note that the natural logarithm of an arbitrary complex number $re^{i\varphi}$ is equal to the logarithm of its modulus, r, plus its argument, φ, multiplied by the imaginary unit i. Therefore, the winding number of $\det A(\lambda)$, $\operatorname{wind}\det A(\lambda)$, can alternatively be defined as $\frac{1}{2\pi i}\log\det A(\lambda)|_{\lambda=0}^{2\pi}$. This definition and the fact that the logarithm of $\det A(\lambda)$ is equal to the sum of the logarithms of $\det A_-(\lambda)$, $\det A_0(\lambda)$, and $\det A_+(\lambda)$ imply that the winding number of $\det A(\lambda)$ is equal to the sum of the winding numbers of $\det A_-(\lambda)$, $\det A_0(\lambda)$, and $\det A_+(\lambda)$. Appendix shows that $\operatorname{wind}\det A_-(\lambda) = \operatorname{wind}\det A_+(\lambda) = 0$. Hence,

$$\operatorname{wind}\det A(\lambda) = \operatorname{wind}\det A_0(\lambda) = \operatorname{wind} e^{i\lambda(k_1+k_2+...+k_n)} = k_1 + k_2 + ... + k_n. \tag{12}$$

Had all partial indices of $A(\lambda)$ been of the same sign or zero we would have been able to distinguish between the three cases described in Proposition 1 by computing $\operatorname{wind}\det A(\lambda)$. Indeed, in such a case, a zero winding number implies that all indices are zero, a positive winding number indicates that at least one partial index is positive, and a negative winding number corresponds to all partial indices being non-positive and at least one of them being negative.

It turns out that almost all functions $A(\lambda)$ have partial indices of the same sign or zero. More precisely, let us denote a normed space of $n \times n$ matrix functions over $\lambda \in [-\pi, \pi]$ with an absolutely summable Fourier series and the maximum of the element by element supremums norm as $W_{n\times n}$. Then, according to Theorem 2.1 and Corollary 2.2 in Chapter 10 of Clancy and Gohberg (1981), the set of functions $A(\lambda)$ whose partial indices are of the same sign or zero includes an open and everywhere dense set in $W_{n\times n}$. Hence, if we have an LRE model such that some of the corresponding partial indices are positive and some are negative then we can slightly perturb parameters A_j to get a new model with the corresponding partial indices of the same sign or zero. Moreover, partial indices associated with this new model will be stable under small perturbations of its parameters.

We will, therefore, define a generic LRE model as a model (1) that corresponds to $A(\lambda)$ whose partial indices are of the same sign or zero. Equation (12) then implies the following winding number criterion for the existence and uniqueness of equilibrium in multivariate LRE models:

Proposition 2. *A generic LRE model (1) has a unique solution, multiple solutions, or, in general, no solutions if and only if the winding number of* $\det A(\lambda)$ *is equal to zero, less than zero, or greater than zero respectively.*

Note that since, for the univariate models, $A(\lambda)$ has only one partial index, the univariate models are always generic. Unfortunately, there is no guarantee that non-generic models will not be encountered in practice. In fact, Dotsey and Boyd (1990) give examples of simple economic models that are not generic. In the next section, I show that the genericity of LRE models is equivalent to relatively non-restrictive familiar regulatory conditions for some special models studied in the literature.

In practice, the winding number criterion can be checked numerically in a straightforward manner. Computation of the winding number of a complex function $\det A(\lambda)$ amounts to counting the number of times that $\operatorname{Re}\det A(\lambda)$ changes its sign given that $\operatorname{Im}\det A(\lambda)$ is positive. The latter can be directly programmed as a function of the roots of $\operatorname{Re}\det A(\lambda)$ and $\operatorname{Im}\det A(\lambda)$ and the value of $\det A(\lambda)$ at zero frequency. In many relatively less complex cases, the criterion can be checked visually by inspecting the graph of $\det A(\lambda)$.

The criterion sheds some light on the validity of using finite lag/lead approximations to study the existence and uniqueness of equilibrium in (1). Restricting attention to generic models, the existence and uniqueness properties of equilibrium for the approximation will not change as long as the corresponding winding number is the same as that for the approximated model. Note that since the winding number $\operatorname{wind}\det A(\lambda)$ is a continuous function of A_j for any $A(\lambda) \in W_{n\times n}$, all sufficiently accurate finite lag/lead approximations will have the same winding number as the approximated model.

It can be shown that the winding number criterion remains true for LRE models:

$$\sum_{j=-\infty}^{+\infty} \sum_{s=0}^{r} A_{j,s} E_{t-s} x_{t-j} = \Gamma z_t \tag{13}$$

that may include lagged expectations.[7] For such models, we define function $A(\lambda)$ as $A(\lambda) = \sum_{j=-\infty}^{+\infty} \sum_{s=0}^{r} A_{j,s} e^{i(j-r)\lambda}$. Note that, in contrast to the finite lag/lead case, models (13) cannot be reduced to (1) by introduction of a finite number of additional endogenous variables.

[7] A proof of such an extension can be obtained from the author upon request.

3. Connections to previous literature

The winding number criterion derived in the previous section is intimately related to many other criteria proposed in the literature. In this section, we will discuss its connections to the Blanchard and Kahn (1980), Hansen and Sargent (1981), and Whiteman (1983) conditions for the existence and uniqueness of equilibrium.

3.1. Connections to Blanchard-Kahn

In their seminal article, Blanchard and Kahn (1980) study models of form:

$$\begin{pmatrix} y_{t+1} \\ {}_tp_{t+1} \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} y_t \\ p_t \end{pmatrix} + \Gamma z_t, \qquad t = 0, 1, 2, ..., \qquad y_{t=0} = y_0 \quad (14)$$

where y is a $k \times 1$ vector of variables predetermined at t, p is an $m \times 1$ vector of variables non-predetermined at t, z is an $r \times 1$ vector of exogenous variables, and ${}_tp_{t+1}$ is the agent's expectations of p_{t+1} held at t. They focus on the endogenous and exogenous processes that may be polynomially (but not exponentially) growing.

Matrix M from (14) has Jordan decomposition $M = B \times \text{diag}(J_1, J_2) \times B^{-1}$ where B is the matrix of the right eigenvectors of matrix M; J_1 is the $\bar{k} \times \bar{k}$ matrix of all those Jordan blocks that correspond to the eigenvalues of M on or inside the unit circle; and J_2 is the $\bar{m} \times \bar{m}$ matrix of the remaining Jordan blocks.

Blanchard and Kahn formulate the following conditions for the existence and uniqueness of equilibrium in (14). Assume that a sub-matrix, B_{11}, corresponding to the first k rows and the first $\bar{k}$ columns of the matrix of the right eigenvectors, B, has full rank. Then if the number of the eigenvalues of M outside the unit circle, $\bar{m}$, is equal to the number of the non-predetermined variables, m, then there exists a unique solution to (14). If the number of the eigenvalues outside the unit circle is greater than the number of the non-predetermined variables, then, in general, there is no solution. Finally, if the number of the eigenvalues outside the unit circle is less than the number of the non-predetermined variables, then there are many solutions.

Let us now assume that an LRE model described by Blanchard and Kahn's canonical representation (14) does not have unit roots, and that we consider the endogenous and exogenous processes that have bounded expectations conditional on the information available at time t. Then, we can cast the problem in the non-stationary format described in the previous section.

Define x_t as $(y'_{t+1}, p'_t)'$. Note that the first part of the vector of the endogenous variables x_t is defined as the next period's realization of y. With these notations, model (14) can be rewritten in form (1) as follows:

$$A_{-1}E_t x_{t+1} + A_0 x_t + A_1 x_{t-1} = \Gamma z_t, \qquad t = 0, 1, 2, 3, ... x^i_{-1} = y^i_0, i = 1, ..., k \quad (15)$$

where $A_{-1} = \begin{pmatrix} 0 & 0 \\ 0 & I_m \end{pmatrix}$, $A_0 = \begin{pmatrix} I_k & -M_{12} \\ 0 & -M_{22} \end{pmatrix}$, $A_1 = \begin{pmatrix} -M_{11} & 0 \\ -M_{21} & 0 \end{pmatrix}$, and a superscript i over a vector denotes the i-th component of the vector. The corresponding function $A(\lambda)$ has the following form:

$$A(\lambda) = \begin{pmatrix} I_k - M_{11}e^{i\lambda} & -M_{12} \\ -M_{21}e^{i\lambda} & I_m e^{-i\lambda} - M_{22} \end{pmatrix}. \tag{16}$$

Proposition 3. *Matrix B_{11} has full rank if and only if all partial indices of $A(\lambda)$ are of the same sign or zero.*

The proof of this proposition is given in the Appendix. Recall that an LRE model is generic, in the sense described in the previous section, if and only if the function $A(\lambda)$ associated with the model has all partial indices of the same sign or zero. Hence, Proposition 3 implies that Blanchard and Kahn's (1980) requirement that B_{11} is of full rank is equivalent to the requirement that model (15) is generic.

Now let us show that the Blanchard-Kahn conditions on the number of the unstable eigenvalues vs. the number of the non-predetermined variables can be reformulated in terms of the behavior of the graph of $\det A(\lambda)$. Formula (16) implies that $A(\lambda)$ can be represented as the following product:

$$A(\lambda) = B\left[\begin{pmatrix} I_k e^{-i\lambda} & 0 \\ 0 & I_m e^{-i\lambda} \end{pmatrix} - \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}\right] B^{-1} \begin{pmatrix} I_k e^{i\lambda} & 0 \\ 0 & I_m \end{pmatrix}.$$

Therefore, $\det A(\lambda) = e^{ik\lambda} \prod\limits_{|\rho_j|<1} \left(e^{-i\lambda} - \rho_j\right) \prod\limits_{|\rho_j|>1} \left(e^{-i\lambda} - \rho_j\right)$, where ρ_j are the eigenvalues of M. As should be clear from the discussion in the previous section, such a decomposition implies that $\operatorname{wind}\det A(\lambda) = \operatorname{wind} e^{ik\lambda} + \sum_{|\rho_j|<1} \operatorname{wind}\left(e^{-i\lambda} - \rho_j\right) + \sum_{|\rho_j|>1} \operatorname{wind}\left(e^{-i\lambda} - \rho_j\right)$. Clearly, $\operatorname{wind} e^{ik\lambda} = k$ and the winding number of $\left(e^{-i\lambda} - \rho_j\right)$ is equal to -1 if $|\rho_j| < 1$, and it is equal to zero if $|\rho_j| > 1$. Hence, the winding number of $\det A(\lambda)$ is equal to the number of the predetermined variables minus the number of the eigenvalues of M that lie outside the unit circle.

Therefore, the graph of $\det A(\lambda)$ does not rotate around zero, rotates around zero clockwise, or rotates around zero counterclockwise if and only if the number of the stable eigenvalues of M coincides, is larger than, or is less than the number of the predetermined variables respectively. This is the same as to say that $\operatorname{wind}\det A(\lambda)$ is zero, negative, or positive if and only if the number of the unstable eigenvalues of M coincides, is less than, or is larger than the number of the non-predetermined variables respectively, which constitutes the Blanchard and Kahn conditions for the model to have a unique, multiple or no solutions respectively.

Although the Blanchard and Kahn (1980) framework substantially overlaps with the non-stationary framework described in the previous section, there exist some important differences. Blanchard-Kahn cannot accommodate infinite leads or lags in an LRE model. However, in contrast to our framework, that of Blanchard and Kahn can fit models with unit roots and polynomially growing processes. Only for those LRE models that can be cast in both frameworks, the two criteria for the existence and uniqueness of equilibrium are equivalent.

3.2. Connections to Hansen-Sargent

Hansen and Sargent (1981) study LRE models of the form:

$$E_t\left[H + D\left(\beta L^{-1}\right) D\left(L\right)\right] x_t = z_t, \tag{17}$$

where $\{x_t\}$ and $\{z_t\}$ are n-dimensional endogenous and exogenous processes respectively, β is a positive number less than 1, $D(L) = D_0 + D_1 L + ... + D_m L^m$, D_0 is a full rank matrix, and H is a positive definite matrix. They focus on the solutions satisfying the asymptotic growth restriction:

$$E_0 \sum_{t=0}^{\infty} \beta^t x_t' H x_t < +\infty. \tag{18}$$

It is also assumed that a solution must satisfy $m \cdot n$ initial conditions $x_{-1}, ..., x_{-m}$.

Model (17) can be interpreted as an Euler equation arising from a linear quadratic stochastic optimal control problem. Hansen and Sargent (1981) show that the model has a unique solution satisfying the initial conditions and asymptotic growth restriction (18).

Hansen and Sargent (1981) use the canonical spectral factorization in their method of solution of (17). As has been mentioned above, the canonical spectral factorization is a special case of the Wiener-Hopf factorization that can be applied to positive definite matrix functions. For the case $\beta = 1$, model (17) correspond to $A(\lambda) = H + D\left(e^{-i\lambda}\right) D\left(e^{i\lambda}\right)$, which is positive definite for all real λ. The canonical spectral factorization has $A_0(\lambda)$ equal to the identity matrix, and hence, the winding number of $A(\lambda)$ is equal to zero. Then, according to the winding number criterion, the model indeed has a unique equilibrium with bounded expectations.[8]

3.3. Connections to Whiteman

In chapter 4 of his book, Whiteman (1983) studies LRE models with n-dimensional endogenous and exogenous variables and no unit roots:

$$\sum_{j=-l}^{m} A_j E_t x_{t-j} = z_t. \tag{19}$$

Similar to our stationary framework, Whiteman restricts attention to the stationary endogenous and exogenous processes. He claims[9] that if A_{-l} is of full rank, the roots of det $\left[y^l \sum_{j=-l}^{m} A_j y^j\right]$ are distinct, and $r \cdot n$ of these roots are inside the unit circle while the others lie outside the unit circle, then (i) if $r < l$, there are many solutions to the model; (ii) if $r = l$, there is one solution to the model; and (iii) if $r > l$, there is no solution to the model.

[8] As has been already mentioned, the winding number criterion can be easily extended to allow exponentially growing solutions. Such an extension would handle cases with $\beta < 1$.

[9] As we show below, the conditions of Whiteman's criterion are not sufficient.

Let us denote *inverses* of the roots of polynomial $\det\left[y^l \sum_{j=-l}^{m} A_j y^j\right]$ that appears in the above criterion as ρ_k, $k = 1, ..., p$, where $p \leq (l+m)\cdot n$. Then, $\det A(\lambda) = \det \sum_{j=-l}^{m} A_j e^{ij\lambda} = (\det A_{-l})\left(\det e^{-il\lambda} I_n\right) \prod_{j=1}^{p}\left(1-\rho_j e^{i\lambda}\right)$, and since the winding number of a product is equal to the sum of the winding numbers of the components, we have: $\operatorname{wind}\det A(\lambda) = \operatorname{wind}\det A_{-l} + \operatorname{wind} e^{-il\lambda} I_n + \sum_{j=1}^{p} \operatorname{wind}\left(1-\rho_j e^{i\lambda}\right)$. Clearly, the winding number of a constant is equal to zero, the winding number of $e^{-il\lambda} I_n$ is equal to $-l\cdot n$, and the winding number of $1-\rho_j e^{i\lambda}$ is equal to zero if $\left|\rho_j\right| < 1$ and one if $\left|\rho_j\right| > 1$.

Hence, the winding number of $\det A(\lambda)$ is equal to the number of *roots* (not inverses of the roots) of $\det\left[y^l \sum_{j=-l}^{m} A_j y^j\right]$ inside the unit circle minus $l\cdot n$. Therefore, the statements (i), (ii), and (iii) of the Whiteman criterion can be reformulated as: (i) if $\operatorname{wind}\det A(\lambda) < 0$, there are many solutions to the model; (ii) if $\operatorname{wind}\det A(\lambda) = 0$, there is one solution to the model; and (iii) if $\operatorname{wind}\det A(\lambda) > 0$, there is no solution to the model.

Our criterion would require genericity of the model for the above statements to be true. Interestingly, Whiteman's requirements that A_{-l} is of full rank, the roots of $\det\left[y^l \sum_{j=-l}^{m} A_j y^j\right]$ are distinct, and $r\cdot n$ of these roots are inside the unit circle while the others lie outside the unit circle do not imply genericity. Vice versa, the genericity does not imply these requirements. In particular, the winding number criterion does handle models with singular leading coefficient matrices. Further investigation shows that Whiteman's conditions are insufficient for his criterion to hold.

Indeed, consider LRE model:

$$\begin{pmatrix} -\frac{1}{4} & 0 \\ 0 & 1 \end{pmatrix} E_t x_{t+1} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x_{t-1} = z_t. \tag{20}$$

Note that A_{-1} is of full rank and that $\det\left[y \sum_{j=-1}^{1} A_j y^j\right] = y^2 - \frac{1}{4}$ has two distinct roots (hence, $r\cdot n = 2$ and $r = 1$), both of which are inside the unit circle, so that the model satisfies Whiteman's conditions. Since r is equal to $l = 1$, according to Whiteman the model must have a unique solution. However, it is easy to see that the model separates into two equations for the two components of the endogenous and exogenous vectors. The first of these equations, $-\frac{1}{4} E_t x_{t+1}^1 + x_{t-1}^1 = z_t^1$, has no solutions (for example according to Whiteman's univariate criterion or the winding number univariate criterion). Hence, the whole model cannot have solutions and Whiteman's criterion is incorrect.

Note that the Wiener-Hopf factorization of $A(\lambda)$ corresponding to the above model is

$$A(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{i\lambda} & 0 \\ 0 & e^{-i\lambda} \end{pmatrix} \begin{pmatrix} 1-\frac{1}{4}e^{-i2\lambda} & 0 \\ 0 & 1 \end{pmatrix},$$

and the two partial indices of $A(\lambda)$ are 1 and -1. Therefore, the model is non-generic. Of course, according to the winding number criterion, any example considering a generic model would not falsify Whiteman's criterion, because under the genericity it becomes a special case of the winding number criterion.

4. Analysis of monetary policy under model uncertainty

In this section we will apply the criterion developed above to analyze the robustness of simple Taylor-type rules to the model uncertainty about the standard new Keynesian model of the economy described, for example, in Woodford (1998) and Clarida, Gali, and Gertler (1999). The model consists of two equations

$$\pi_t = \beta E_t \pi_{t+1} + \alpha y_t + \varepsilon_t \tag{21}$$

$$y_t = E_t y_{t+1} - \varphi(i_t - E_t \pi_{t+1}) + \eta_t \tag{22}$$

Here π_t is inflation at an annual percentage rate, y_t is the output gap, measured as 100 times the log ratio of actual real output to potential output, and i_t is the federal funds rate at an annual rate. Parameters β, α, and φ represent the time discount factor of the private sector, a measure of the speed of price adjustment, and the intertemporal elasticity of substitution respectively. In the numerical analysis below I assume $\alpha = 0.024$, $\varphi = 6.37$, and $\beta = 0.99$ as in Woodford (1998).

We will assume that the above model is a reference model for a monetary authority, say Fed, that is able to control the interest rate. The Fed views the model as only an approximation of the reality and wants to make sure that its policy works well not only for the reference model, but also for some deviations from it. Denote the right hand side of the reference model equations (21) and (22) as ref_π and ref_y respectively. Following Onatski and Stock (2002), we model the potential deviations as:

$$\begin{aligned} \pi_t &= ref_\pi + w_1 E_t \Delta_1(L)(\pi_{t+1} - \pi_t) + w_2 E_t \Delta_2(L) y_t & (23) \\ y_t &= ref_y + w_3 E_t \Delta_3(L) y_{t+1} + w_4 E_t \Delta_4(L)(i_t - E_t \pi_{t+1}), & (24) \end{aligned}$$

where $\Delta_1, \Delta_2, \Delta_3$, and Δ_4 are potentially infinite series in the positive and negative powers of the lag operator, L, with absolutely summable coefficients. The deltas are meant to represent the economic dynamics that was left unmodeled by the reference model. The weights w_i are supposed to reflect the relative importance of the uncertainties.

The Fed is assumed to restrict attention only to those deviations (23, 24) that have some empirical or theoretical value. We will model this restriction by considering only those deviations that correspond to relatively small Δ_i. The size of Δ_i can be defined as $\|\Delta_i\| = \sup_{\lambda \in [0,2\pi]} \left|\Delta_i\left(e^{i\lambda}\right)\right|$. We can, for example, focus only on the deviations with $\|\Delta_i\| < r, i = 1, ..., 4$, where r is some real number that describes the size of the model uncertainty. For convenience of notation, we introduce $\Delta = \{\Delta_1, ..., \Delta_4\}$ and define $\|\Delta\|$ as the maximum of $\|\Delta_i\|, i = 1, ..., 4$.

We take $w_1 = 0.6, w_2 = 0.01, w_3 = 0.01, w_4 = 3$. The weight w_1 was chosen so that $\Delta_1 = L$, which has size 1, introduces an empirically plausible amount of backward-lookingness into the Phillips curve. The weights w_2 and w_4 were chosen to be half as large as the corresponding structural parameters, α and φ. Finally, w_3 was chosen so that "small" deviations from the unit root in the output gap's own dynamics in the IS curve are allowed. Such a choice of weights is arbitrary. For a more careful calibration of the model uncertainty see Onatski and Williams (2003).

We assume that the Fed wants to study the robustness to the model uncertainty of policy rules:

$$i_t = g_\pi \pi_t + g_y y_t. \tag{25}$$

Onatski and Stock (2002) analyze the model uncertainty about purely backward-looking models of the economy. They describe the robustness of a given policy rule (25) by the minimal size of the model uncertainty, which includes potential deviations from the reference model such that, controlled by the policy rule, they lead to the dynamic instability of the economic processes. If such a minimal size is relatively large, the rule does not lead to the instability under a variety of deviations from the reference model and, therefore, is relatively robust. This measure of robustness can be generalized to models that include rational expectations by requiring that relatively more robust rules should lead neither to the dynamic instability nor to the indeterminacy of equilibrium for model deviations of relatively large size. In the rest of this section, we, therefore, will be interested in the question: How, for a given policy rule (25), to find the minimal size of the model uncertainty that includes LRE models that either have no stationary solutions or have multiple solutions?

Let us assume that the reference model controlled by a given policy rule has a unique solution. If this is not so, then the rule is not robust to an arbitrarily small amount of the model uncertainty and should be rejected. According to our criterion for the existence and uniqueness of equilibrium in LRE models, the graph of the function $\det A(\lambda)$ corresponding to the reference model controlled by the rule does not encircle zero. Let the function $A_\Delta(\lambda)$ correspond to the deviation from the reference model (23, 24). The graph of $\det A_\Delta(\lambda)$ is a continuous function of Δ. Therefore, as long as the size of Δ is small, and the deviation remains a generic LRE, the graph of $\det A_\Delta(\lambda)$ will not encircle zero and the deviation will still have a unique solution. The uniqueness or existence of equilibrium will be lost when the graph "hits" zero or when the deviation becomes a non-generic model.

The potential non-genericity of the deviation creates a problem for our analysis of robustness because we cannot detect the non-genericity using the behavior of the graph of $\det A(\lambda)$. However, as we said above, the non-genericity of a model is not stable under small model perturbations. We will, therefore, ignore non-generic deviations and measure the robustness of a given policy rule by an index of robustness, which is equal to the minimum of the uncertainty size $\|\Delta\|$ such that $\det A_\Delta(\lambda) = 0$ for some $\lambda \in [0, 2\pi]$.

There exist effective computational methods that estimate $\min\{\|\Delta\| : \det A_\Delta(\lambda) = 0\}$ from below and from above. The methods are described in detail, for example, in Packard and Doyle (1993). The commercially available mu-analysis and synthesis toolbox for MATLAB contains the relevant computer codes. We compute the upper and lower bounds on the index of robustness for Taylor-type policy rules with parameters $g_\pi \in [0, 10]$ (grid of 0.2), $g_y \in [0, 10]$ (grid of 0.25). The upper and lower bounds turn out to be very close to each other. Figure 2 reports the contour plot of the upper bound on the index.

We see that the rules with a smaller than 1 reaction to inflation and a relatively weak response to the output gap perform very poorly. Such rules destabilize even

Figure 1:

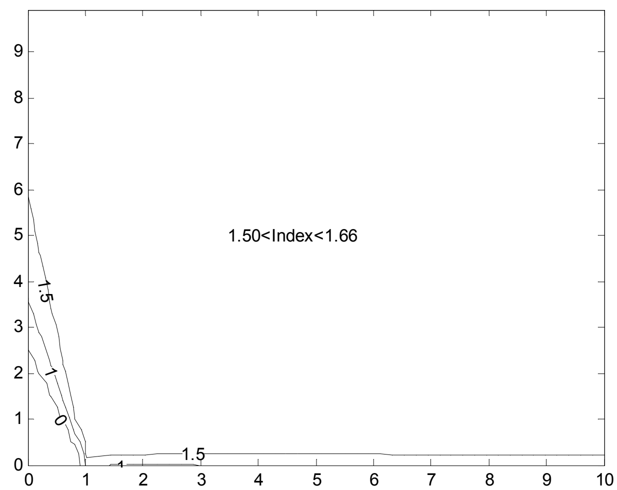

Figure 2: The index of robustness for Taylor-type rules to the model uncertainty about the standard New Keynesian model of the economy.

the reference model of the economy. The other rules have about the same index of robustness. This situation is drastically different from that for purely backward-looking models. Onatski and Stock (2002) showed that the index of robustness is relatively high for the rules with a low (but higher than 1) reaction to inflation and a relatively high (about 2) reaction to the output gap. More aggressive rules are relatively less robust and may even destabilize the reference model (the Rudebusch and Svensson model (1999) for that study).

The fact that the robustness of the policy rules in the forward-looking new Keynesian model does not deteriorate with a rise in the aggressiveness of the rules agrees with the intuition that forward-looking models are much more difficult to destabilize than backward-looking models. Note, however, that in forward-looking models, there exists an additional potential source of non-robustness corresponding to the multiplicity of equilibria. It turns out that the potential of the multiplicity of equilibria does not add enough to the variability of the index of robustness to differentiate between more and less aggressive rules on our grid.

In the above analysis, our criterion for the existence and uniqueness of equilibrium in "infinite" LRE models played an instrumental role. Since the deviations from the reference model were formulated in terms of $\Delta(L)$, which were allowed to be infinite series in the positive and negative powers of L, we could not analyze the existence and uniqueness of equilibrium using conventional criteria. Of course, we could have restricted attention to $\Delta(L)$, which are sums of a finite number of positive and negative powers of L. However, given the current state of knowledge, it would become much more difficult, if not impossible, to compute accurate bounds on the minimal size of the deviations from the restricted class, which result in the dynamic instability or indeterminacy of equilibrium for a particular policy rule.

5. Conclusion

This paper develops the following simple winding number criterion for the existence and uniqueness of equilibrium in LRE models of form:

$$\sum_{j=-\infty}^{+\infty} A_j E_t x_{t-j} = \Gamma z_t.$$

Assuming that the matrix coefficients A_j are absolutely summable and that the model is generic, the model has a unique equilibrium, multiple equilibria or no equilibrium at all if and only if the graph of $\det A(\lambda)$, where $A(\lambda) = \sum_{j=-\infty}^{+\infty} A_j e^{ij\lambda}$, does not rotate around zero, rotates around zero clockwise, or rotates around zero counter-clockwise, respectively.

We consider two different frameworks of the analysis: a non-stationary and stationary one. For the non-stationary framework, it is assumed that the model equations hold for $t \geq 0$, initial conditions $x_t, t < 0$ are given and bounded, both exogenous and endogenous processes $\{z_t\}$ and $\{x_t\}$ are adapted to an a priori given filtration of a probability space and have bounded conditional expectations. For the stationary framework, it is assumed that the model equations hold for both negative and positive t, the exogenous process is weakly stationary, and that the solutions are sought among time-invariant, square summable linear combinations of the past and current realizations of the process fundamental for $\{z_t\}$.

We explore the connections between the new criterion and those of Blanchard and Kahn (1980), Hansen and Sargent (1981), and Whiteman (1983). We show that our non-stationary framework of the analysis and Blanchard and Kahn's framework overlap considerably. However, there exist important differences. Our approach allows infinite-dimensional LRE models, whereas theirs allows unit roots and polynomially growing processes. For models that can be accommodated by both the present and Blanchard and Kahn's frameworks, the two criteria are equivalent.

We find that the winding number criterion can be applied to the models studied by Hansen and Sargent (1981) when the discount factor β is equal to 1. The criterion guarantees the existence of a unique bounded solution to such models. We show that Whiteman's (1983) criterion is, in general, incorrect but becomes a special case of the winding number criterion for generic models.

This paper applies the new criterion to a study of the robustness of monetary policy rules under the model uncertainty about the standard New Keynesian model of the economy. We show that, in contrast to the uncertainty about purely backward-looking models of the economy, the index of robustness of the policy rules is relatively insensitive to the aggressiveness of the rule.

6. Appendix

Proof of Proposition 1 for the non-stationary framework:

Let us transform (1) into an equivalent system of equations suitable for the analysis below. First, take the terms involving the initial conditions to the right hand side to get:

$$\sum_{j=-\infty}^{t} A_j E_t x_{t-j} = \Gamma z_t - \sum_{j=t+1}^{\infty} A_j E_t x_{t-j},\ t = 0, 1, 2, ..., \tag{26}$$

or, changing the index of summation and denoting the left hand side of the above equation as $f_t(z, x_{ini})$,

$$\sum_{k=0}^{\infty} A_{t-k} E_t x_k = f_t(z, x_{ini}),\ t = 0, 1, 2, ... \tag{27}$$

Applying the conditional expectation operation E_s to both sides of (27) we have:

$$E_s \sum_{k=0}^{\infty} A_{t-k} E_t x_k = E_s f_t(z, x_{ini}),\ t = 0, 1, 2, ..., \tag{28}$$

and therefore:

$$\sum_{k=0}^{\infty} A_{t-k} E_s x_k = E_s f_t(z, x_{ini}),\ s = 0, 1, 2, ...;\ t \geq s. \tag{29}$$

The latter equations follow from the former ones because of the law of iterated expectations and the fact that the order of the conditional expectation and the infinite sum on the left hand side of the former equation is interchangeable.

Finally, taking the terms with $k < s$ to the right hand side and using the fact that $E_s x_k = E_k x_k$ for $s \geq k$, transform (29) into:

$$\sum_{k=s}^{\infty} A_{t-k} E_s x_k = E_s f_t(z, x_{ini}) - \sum_{k=0}^{s-1} A_{t-k} E_k x_k,\ s = 0, 1, 2, ...;\ t \geq s. \tag{30}$$

Consider now a system of equations to determine n-dimensional random vectors ξ_{sk}, $s = 0, 1, 2, ...$; $k \geq s$ given n-dimensional random vectors η_{st}, $s = 0, 1, 2, ...$; $t \geq s$ having the same form as (30):

$$\sum_{k=s}^{\infty} A_{t-k} \xi_{sk} = \eta_{st} - \sum_{k=0}^{s-1} A_{t-k} \xi_{kk},\ s = 0, 1, 2, ...;\ t \geq s. \tag{31}$$

Below, we always assume that ξ_{sk} and η_{st} are measurable with respect to $\mathcal{F}_s$ and the sequences ξ_{sk}, $k = s, s+1, ...$ and η_{st}, $t = s, s+1, ...$, which we denote as $\{\xi_{s\cdot}\}$ and $\{\eta_{s\cdot}\}$ respectively, are bounded almost surely for $s = 0, 1, 2, ...$ In what follows, we will first analyze the solvability properties of (31) and then show that (31) and (30), and hence (1), have the same solvability properties.

Consider (31) for $s = 0$:

$$\sum_{k=0}^{\infty} A_{t-k}\xi_{0k} = \eta_{0t},\ t \geq 0. \tag{32}$$

According to Theorem 4.2, Chapter VIII of Gohberg and Feldman (1974), for any bounded sequence $\{\eta_{0\cdot}\}$ there exists a unique bounded sequence $\{\xi_{0\cdot}\}$ satisfying equations (32) if and only if all partial indices of $A(\lambda)$ are equal to zero. There exists no solution to (32) at least for some bounded sequences $\{\eta_{0\cdot}\}$ if and only if at least one of the partial indices of $A(\lambda)$ is positive. Finally, there exist infinitely many solutions to (32) if and only if all partial indices of $A(\lambda)$ are non-positive and at least one of them is negative.[10]

Now consider (31) for $s = 1$:

$$\sum_{k=1}^{\infty} A_{t-k}\xi_{1k} = \eta_{1t} - A_t\xi_{00},\ t \geq 1. \tag{33}$$

Assume that ξ_{00} is determined by the solution to (32). Using the theorem from Goldberg and Feldman (1974) one more time, we can show that the solvability of (33) depends on the partial indices of $A(\lambda)$ in the same way the solvability of (32) does.

In general, we can show that system (31) has a unique solution for any set of bounded sequences $\{\eta_{s\cdot}\}$, $s = 0, 1, 2, ...$ if all partial indices of $A(\lambda)$ are zero, has no solutions at least for some sets $\{\eta_{s\cdot}\}$, $s = 0, 1, 2, ...$ if at least one partial index of $A(\lambda)$ is positive, and has multiple solutions for any $\{\eta_{s\cdot}\}$, $s = 0, 1, 2, ...$ if all partial indices of $A(\lambda)$ are non-positive and at least one of them is negative.

It remains to show that: 1) if (31) has no solution, then (30) has no solution, 2) if (31) has a unique solution, then (30) has a unique solution, and 3) if (31) has many solutions, then (30) has many solutions. Let me turn to the proof of the above three statements.

1) Suppose that (31) has no solutions for η_{st}. We can choose

$$\eta_{0t} = A_{t+1}c \tag{34}$$

for some $c \in R^n$ and set $\eta_{st} = \eta_{0t}$ for any s, $0 \leq s \leq t$. Indeed, below we check that those equations from (31) that correspond to $s = 0$ do not have solutions for such a choice. To see this, note that (31) for $s = 0$, together with (34), is equivalent to:

$$\sum_{r=0}^{\infty} A_{-,t-r} \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} A_{0,r-j}A_{+,j-k}\xi_{0k} = \sum_{r=0}^{\infty} A_{-,t-r} \sum_{j=0}^{\infty} A_{0,r+1-j}A_{+,j}c,\ t = 0, 1, 2, ..., \tag{35}$$

[10] Although Gohberg and Feldman's theorem is formulated for deterministic sequences, it remains true for stochastic sequences. Indeed, it can be applied to the realizations of such sequences for any elementary event. Measurability of the solution ξ_{0k} follows from the fact that, as shown by Gohberg and Feldman (1974), ξ_{0k} can be represented as an absolutely summable linear combination of η_{0t}, each of which is measurable with respect to $\mathcal{F}_0$.

where $A_{-,j}, A_{0,j}$, and $A_{+,j}$ are the coefficients on $e^{ij\lambda}$ in the Fourier expansions of the Wiener-Hopf factors $A_{-}(\lambda), A_{0}(\lambda)$, and $A_{+}(\lambda)$ respectively. System (35) in turn is equivalent to:

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} A_{0,r-j}A_{+,j-k}\xi_{0k} = \sum_{j=0}^{\infty} A_{0,r+1-j}A_{+,j}c,\ r = 0,1,2,... \tag{36}$$

Let k_n be the largest partial index of $A(\lambda)$. It is positive because by assumption, (31) has no solutions. Consider the equation from (36) that corresponds to $r = k_n - 1$. (8) implies that the n-th row of $A_{0,r-j}$ is equal to zero for any $j \geq 0$. Therefore, the n-th component of the vector defined by the left hand side of the equation is equal to zero for any $\{\xi_{0\cdot}\}$. On the other hand, the right hand side is equal to $A_{0,r+1}A_{+,0}c$. Since $A_{+,0}$ is non-singular, we can always choose $c \in R^n$ so that the vector defined by the right hand side has a non-zero n-th component.

Assume now that (30) always has a solution. Let $\{x_t\}$ be a solution to (30) for $z_t \equiv 0$ and $x_{ini} = \{-c, 0, 0, ...\}$. Consider $\tilde{\xi}_{sk} = E_s x_k$ and $\tilde{\eta}_{st} = E_s f_t(0, x_{ini})$. Then, $\tilde{\xi}_{sk}$ is a solution to (31) for $\tilde{\eta}_{st}$. But $\tilde{\eta}_{st} = E_s f_t(0, x_{ini}) = E_s\eta_{0t} = \eta_{0t} = \eta_{st}$. Therefore, (31) has a solution for η_{st} which contradicts our assumption.

2) Suppose that (31) has a unique solution. Let ξ_{sk} be the solution corresponding to:

$$\eta_{st} = E_s f_t(z, x_{ini}). \tag{37}$$

Consider a stochastic process $\{x_t\}$ defined by $x_t = \xi_{tt}$, $t \geq 0$. We will prove that $E_s x_t = \xi_{st}$ for $s = 0, 1, 2, ...$ and $t \geq s$, and therefore, $\{x_t\}$ is a solution to (30). First, we will prove the following lemma.

Lemma 4. *Suppose that for any η_{st} (31) has a solution ξ_{sk}, which is not necessarily unique. Then there exists a solution to (31), $\tilde{\xi}_{sk}$, such that $E_s\tilde{\xi}_{kk} = \tilde{\xi}_{sk}$ for any $s < k$.*

Proof: Define $\tilde{\xi}_{sk}$ recursively:

$$\begin{aligned}
\tilde{\xi}_{0k} &= \xi_{0k}, k \geq 0, \\
\tilde{\xi}_{1k} &= \xi_{1k} - \left[E_0\xi_{1k} - \tilde{\xi}_{0k}\right], k \geq 1, \\
&\ \ \ ... \\
\tilde{\xi}_{ik} &= \xi_{ik} - \left[E_{i-1}\xi_{ik} - \tilde{\xi}_{i-1,k}\right], k \geq i, \\
&\ \ \ ...
\end{aligned}$$

The fact that $\tilde{\xi}_{sk}$ is a solution to (31) is trivially true for $s = 0$ and can be proven for general s by the mathematical induction.

We have: $E_{k-1}\tilde{\xi}_{kk} = E_{k-1}\xi_{kk} - E_{k-1}\xi_{kk} + \tilde{\xi}_{k-1,k} = \tilde{\xi}_{t-1,t}$. Noting that $E_s\tilde{\xi}_{kk} = E_sE_{s+1}...E_{k-1}\tilde{\xi}_{kk}$, and applying the above equality as many times as needed, we get the desired result.$\square$

Returning to the proof of our second statement, since by assumption there is only one solution to (31), the above lemma implies that $\mathrm{E}_s x_t = E_s\xi_{tt} = \xi_{st}$ for $s = 0, 1, 2, ...$ and $t \geq s$ and hence, $\{x_t\}$ is a solution to (30).

Finally, note that if (30) has many solutions, then (30) has a nontrivial solution $\{\varphi_t\}$ for $z_t \equiv 0$ and zero initial conditions. Solution $\{\varphi_t\}$ generates a non-trivial solution $\phi_{sk} = E_s\varphi_k$ to (31) for $\eta_{st} \equiv 0$. Now if ξ_{sk} is any solution to (31) with arbitrary η_{st}, then $\xi_{ks} + \gamma\phi_{sk}$ is also a solution, where γ is any real number. But we assumed that (31) has only one solution. Therefore, (30) has only one solution and our second statement has been proven.

3) Suppose that (31) has many solutions. Then, for $\eta_{st} \equiv 0$, there exists ξ_{st}, a solution to (31), such that, for some k, $\xi_{0k} \neq 0$. The random variables ξ_{sk} can be chosen to be constant almost everywhere. Define $\varphi_t = \xi_{0t}$ for $t \geq 0$. Clearly, $\{\varphi_t\}$ is a non-trivial solution to (30) for $z_t \equiv 0$ and zero initial conditions. Hence, if for arbitrary admissible $\{z_t\}$ and initial conditions there exists at least one solution $\{x_t\}$ to (30), then all processes $x_t + \gamma\varphi_t$ are solutions to (30). That is, the number of solutions to (30) is infinite.

It remains to be proven that there exists at least one solution to (30) for arbitrary admissible $\{z_t\}$ and initial conditions. One such solution is given by $x_t = \tilde{\xi}_{tt}$, $t \geq 0$, where $\tilde{\xi}_{sk}$ is a solution to (31) for η_{st} defined by (37), which is described by the lemma.

<u>Proof of Proposition 1 for the stationary framework:</u>

Let $z_t = \sum_{k=0}^{\infty} C_k\varepsilon_{t-k}$ be a Wold decomposition of the exogenous process, and let $\varepsilon_t = \displaystyle\int_{-\pi}^{\pi} e^{it\lambda}\Phi(d\lambda)$ be the spectral representation of the fundamental process ε_t. Then z_t can be represented as:

$$z_t = \int_{-\pi}^{\pi} e^{it\lambda} C(\lambda)\Phi(d\lambda), \tag{38}$$

where $C(\lambda) = \sum_{k=0}^{\infty} C_k e^{ik\lambda}$ (see for example Rozanov 1967).

Recall that we are looking for solutions of the form $x_t = \sum_{k=0}^{\infty} D_k\varepsilon_{t-k}$, where $\{D_k\}$ is an element-by-element square summable series of matrices. Let us define:

$$D(\lambda) = \sum_{k=0}^{\infty} D_k e^{ik\lambda},\ \ D(\lambda) \in L^2[-\pi,\pi].^{11} \tag{39}$$

Then a potential solution has a representation $x_t = \displaystyle\int_{-\pi}^{\pi} e^{it\lambda}D(\lambda)\Phi(d\lambda)$. Furthermore, since $\{A_j\}$ is absolutely summable, the linear transformation, $\sum_{j=-\infty}^{\infty} A_j x_{t-j}$, must have form:

$$\sum_{j=-\infty}^{\infty} A_j x_{t-j} = \int_{-\pi}^{\pi} e^{it\lambda} A(\lambda) D(\lambda)\Phi(d\lambda). \tag{40}$$

[11] All propertiers stated for matrix functions must be understood is the element-by-element sense.

Since the order of the linear projection E_t and the infinite sum in (1) can be interchanged, equations (40) and (38) imply that model (1) is equivalent to:

$$\mathrm{E}_t \int_{-\pi}^{\pi} e^{it\lambda} A(\lambda) D(\lambda) \Phi(d\lambda) = \int_{-\pi}^{\pi} e^{it\lambda} \Gamma C(\lambda) \Phi(d\lambda), \tag{41}$$

which, in its turn, is equivalent to:

$$[A(\lambda)D(\lambda)]_+ = \Gamma C(\lambda) \tag{42}$$

almost everywhere with respect to the Lebesgue measure on $[-\pi, \pi]$. Here the annihilation operator $[\cdot]_+$ is defined as follows. For any $R(\lambda)$ with the Fourier expansion $R(\lambda) = \sum_{j=-\infty}^{\infty} R_j e^{ij\lambda}$ we have $[R(\lambda)]_+ = \sum_{j=0}^{\infty} R_j e^{ij\lambda}$.

Substituting the Wiener-Hopf factorization (7) into (42) we have:

$$[A_-(\lambda)A_0(\lambda)A_+(\lambda)D(\lambda)]_+ = \Gamma C(\lambda), \tag{43}$$

which is equivalent to the existence of a function $F(\lambda)$ that has a one-sided Fourier expansion:

$$F(\lambda) = \sum_{j=-\infty}^{-1} F_j e^{ij\lambda}, \tag{44}$$

such that:

$$A_-(\lambda)A_0(\lambda)A_+(\lambda)D(\lambda) = \Gamma C(\lambda) + F(\lambda). \tag{45}$$

Multiplying both sides of (45) by $A_-^{-1}(\lambda)$, we have:

$$A_0(\lambda)A_+(\lambda)D(\lambda) = A_-^{-1}(\lambda)\Gamma C(\lambda) + A_-^{-1}(\lambda)F(\lambda). \tag{46}$$

Suppose that the largest partial index of $A(\lambda)$, k_n, is positive. Then, in general, there is no $D(\lambda)$ of form (39) that solves (46) and therefore there is no solution to model (1). Indeed, below we will show that there exists $C(\lambda) = \sum_{k=0}^{\infty} C_k e^{ik\lambda}$ from $L^2[-\pi, \pi]$[12] such that for any $F(\lambda)$ of form (44), at least one element in the n-th row of the matrix on the right hand side of equation (46) has the Fourier expansion with a non-zero coefficient on the constant. On the other hand, since $k_n > 0$, any element in the n-th row of the matrix on the left hand side of (46) has zero Fourier coefficient corresponding to the constant.

Denote the coefficient on the constant in the Fourier expansion of $A_-(\lambda)$ as K. Let the first element in the n-th row of $K\Gamma$ be different from zero. This is without loss of generality because factorization (7) can be chosen so that K is a pre-specified non-singular matrix. Let us choose $C(\lambda)$ equal to a constant matrix with the first column having all zero components except the first entry. Then, the element in the n-th row and the first column of $A_-^{-1}(\lambda)\Gamma C(\lambda)$ has a non-zero coefficient on the constant in its Fourier expansion. Finally, note that $A_-^{-1}(\lambda)F(\lambda)$ has zero coefficients

[12] The square summability of $C(\lambda)$ insures that the exogenous process defined by $z_t = \sum_{k=0}^{\infty} C_k \varepsilon_{t-k}$ is weakly stationary.

on $e^{ij\lambda}$, $j \geq 0$, and therefore the Fourier coefficient on the constant for the right hand side of (46) coincides with that for $A_-^{-1}(\lambda)\Gamma C(\lambda)$.

Suppose now that all partial indices of $A(\lambda)$ are non-positive, and some are negative. Then there exists an infinite number of solutions to (46) and, hence, an infinite number of solutions to model (1). Indeed, let us define a matrix function $M(\lambda)$ such that all elements of this matrix except the one in the first row and the first column are equal to zero. Let the upper left element of $M(\lambda)$ be $e^{-i(k_1+1)\lambda}$, where k_1 is the smallest (and hence negative) partial index of $A(\lambda)$. Then, for *any* real number k the following is a solution to (46) of form (39):

$$D(\lambda) = A_+^{-1}(\lambda)\left\{A_0^{-1}(\lambda)\left[A_-^{-1}(\lambda)\Gamma C(\lambda)\right]_+ + kM(\lambda)\right\}. \tag{47}$$

Finally, if all partial indices of $A(\lambda)$ are zero, then $A_0(\lambda)$ is the $n \times n$ identity matrix and $D(\lambda)$ must be equal to $A_+^{-1}(\lambda)\left[A_-^{-1}(\lambda)C(\lambda) + A_-^{-1}(\lambda)F(\lambda)\right]_+$. But for any $F(\lambda)$ satisfying (44), all Fourier coefficients of $A_-^{-1}(\lambda)F(\lambda)$ on the non-negative powers of $e^{i\lambda}$ must be equal to zero. Therefore, $D(\lambda) = A_+^{-1}(\lambda)\left[A_-^{-1}(\lambda)C(\lambda)\right]_+$ is the unique solution to (46) and thus the original model has a unique solution.

Proof of the fact that $\operatorname{wind}\det A_+(\lambda) = \operatorname{wind}\det A_-(\lambda) = 0$:

Let $A_+(\lambda) = \sum_{j=0}^{\infty} F_j e^{ij\lambda}$ be the absolutely summable Fourier expansion of $A_+(\lambda)$. The function of a complex variable $f_+(z) = \det\sum_{j=0}^{\infty} F_j z^j$ is clearly analytic and bounded inside the unit circle. Furthermore, this function does not have zeros inside the unit circle. This follows from the fact that the Fourier expansion of $A_+^{-1}(\lambda)$, $A_+^{-1}(\lambda) = \sum_{j=0}^{\infty} G_j e^{ij\lambda}$, is absolutely summable, so that $f_+^{-1}(z) = \det\sum_{j=0}^{\infty} G_j z^j$ is analytic and bounded inside the unit circle. Hence, $f_+(z)$ transforms the unit disk into a compact subset of the complex plane that does not include zero. As z goes along the unit circle, $f_+(z)$ encircles the subset and, therefore, does not rotate around zero. This implies that $\operatorname{wind}\det A_+(\lambda) = 0$. Similar argument proves $\operatorname{wind}\det A_-(\lambda) = 0$.

Proof of Proposition 3:

Let us consider a function

$$L(\lambda) = e^{i\lambda}A(\lambda) = B\left[I_{k+m} - \begin{pmatrix} J_1 e^{i\lambda} & 0 \\ 0 & J_2 e^{i\lambda}\end{pmatrix}\right]B^{-1}\begin{pmatrix} I_k e^{i\lambda} & 0 \\ 0 & I_m\end{pmatrix}.$$

According to Gohberg, Lerer, and Rodman (1978), the following equalities hold for the partial indices $l_1 \leq l_2 \leq ... \leq l_{k+m}$ of $L(\lambda)$:

$$l_i = \#\left\{j | k+m+r_{j-1} - r_j \leq i-1,\ j = 1,2\right\}, \qquad i = 1,2,...,k+m, \tag{48}$$

where $r_1 = \operatorname{rank}\begin{bmatrix} C_{-1} \\ C_{-2}\end{bmatrix}$, $r_2 = \operatorname{rank}\begin{bmatrix} C_{-1} & C_{-2} \\ C_{-2} & C_{-3}\end{bmatrix}$, and $C_{-j} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ij\lambda}L^{-1}(\lambda)d\lambda$.

From the definition of $L(\lambda)$, it is clear that the partial indices of $A(\lambda)$ are equal to those of $L(\lambda)$ minus one. Therefore, the partial indices of $A(\lambda)$ are not all of the same sign or zero if and only if $l_1 - 1 < 0$ and $l_{k+m} - 1 > 0$. Since $0 \leq l_i \leq 2$, the former inequalities are equivalent to $l_1 = 0$, $l_{k+m} = 2$ and (48) implies

$$1 \leq r_1 < k + m, \qquad 1 \leq r_2 - r_1 < k + m. \tag{49}$$

Below, we will show that these inequalities hold if and only if B_{11} does not have full rank, which will prove the proposition.

Indeed, using the above formula for C_{-j}, we get

$$C_{-1} = \begin{pmatrix} B_{11} & 0 \\ 0 & B_{22}J_2^{-1} \end{pmatrix}, C_{-2} = \begin{pmatrix} 0 & B_{12}J_2^{-1} \\ 0 & B_{22}J_2^{-2} \end{pmatrix}, C_{-3} = \begin{pmatrix} 0 & B_{12}J_2^{-2} \\ 0 & B_{22}J_2^{-3} \end{pmatrix}.$$

Therefore,

$$\begin{aligned} r_1 &= \operatorname{rank} \begin{bmatrix} B_{11} & 0 \\ 0 & B_{22}J_2^{-1} \\ 0 & B_{12}J_2^{-1} \\ 0 & B_{22}J_2^{-2} \end{bmatrix} = \operatorname{rank} B_{11} + \operatorname{rank} \begin{bmatrix} B_{12} \\ B_{22} \end{bmatrix} = \operatorname{rank} B_{11} + \bar{m}, \\ r_2 &= \operatorname{rank} \begin{bmatrix} B_{11} & 0 & 0 & B_{12}J_2^{-1} \\ 0 & B_{22}J_2^{-1} & 0 & B_{22}J_2^{-2} \\ 0 & B_{12}J_2^{-1} & 0 & B_{12}J_2^{-2} \\ 0 & B_{22}J_2^{-2} & 0 & B_{22}J_2^{-3} \end{bmatrix} = \operatorname{rank} \begin{bmatrix} B_{11} & 0 & B_{12}J_2^{-1} \\ 0 & B_{22}J_2^{-1} & 0 \\ 0 & B_{12}J_2^{-1} & 0 \\ 0 & B_{22}J_2^{-2} & 0 \end{bmatrix} \\ &= \operatorname{rank} \begin{bmatrix} B_{11} & B_{12} \end{bmatrix} + \operatorname{rank} \begin{bmatrix} B_{12} \\ B_{22} \end{bmatrix} = k + \bar{m}. \end{aligned}$$

Hence, inequalities (49) can be rewritten as:

$$1 \leq \operatorname{rank} B_{11} + \bar{m} < k + m, \qquad 1 \leq k - \operatorname{rank} B_{11} < k + m, \tag{50}$$

which is the same as

$$-\bar{m} < \operatorname{rank} B_{11} < \bar{k}, \qquad -m < \operatorname{rank} B_{11} < k, \tag{51}$$

or, finally,

$$-\min\{m, \bar{m}\} < \operatorname{rank} B_{11} < \min\left\{k, \bar{k}\right\}. \tag{52}$$

The right-most inequality is equivalent to the statement that B_{11} does not have full rank. We, however, have an "extra" inequality $-\min\{m, \bar{m}\} < \operatorname{rank} B_{11}$ that must be satisfied for some partial indices of $A(\lambda)$ to have different signs. This inequality is only non-trivial if $\min\{m, \bar{m}\} = 0$. But in such a case, B_{11} either contains at least one full row of B if $\bar{m} = 0$, or contains at least one full column of B if $m = 0$. Therefore, since B cannot be singular, $\operatorname{rank} B_{11}$ must be non-zero and the inequality $-\min\{m, \bar{m}\} < \operatorname{rank} B_{11}$ always holds and does not bring any information in addition to the requirement that B_{11} is not of full rank.

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