

Spatial Asset Pricing: A First Step*

François Ortalo-Magné[†]
University of Wisconsin–Madison

Andrea Prat[‡]
Columbia University

Revised May 1, 2013

Abstract

People choose where to live and how much to invest in housing and financial assets, including whether to buy or to rent their home. Traditionally, the first decision has been the domain of spatial economics, while the second has been analyzed in finance. Spatial asset pricing combines the two sets of decisions in a tractable dynamic portfolio choice problem, which leads to an array of predictions on the geographical allocation of human capital, households portfolio over space and time, housing rents, housing and asset prices. Despite its simplicity, our framework provides a unified explanation for a number of known phenomena (e.g hedging role of housing, home bias, housing investment over the life cycle, link between prices and local productivity shocks) and makes novel testable predictions that span financial and spatial observables (e.g cross-sectional distribution of price-rent yields, relation between real estate returns and financial returns, deviations from production-maximizing allocation of human capital).

*We are grateful to Orazio Attanasio, Philippe Bracke, Morris Davis, Christian Julliard, Robert Kollmann, Alex Michaelides, Sven Rady, Stijn Van Nieuwerburgh, Dimitri Vayanos, and Nancy Wallace for helpful discussions; and to seminar participants at Brown University, the University of California–Berkeley, Columbia University, ECARES, ESSET, the Federal Reserve Bank of St. Louis, HEC Paris, Hong Kong University of Science and Technology, INCAE, the London School of Economics, the University of Mannheim, the NBER Summer Institute, the University of North Carolina, Northwestern University, and the Toulouse School of Economics for useful comments. We are grateful to CEPR, the Financial Markets Group at LSE, and the Toulouse School of Economics for their support.

[†]fom@bus.wisc.edu

[‡]ap3116@columbia.edu

1 Introduction

The urban economics literature provides insights into the allocation of households over space and the cross-sectional distribution of rents. Given dividends for housing and other assets, the finance literature provides insights into the demand and price of housing within standard asset pricing frameworks expanded to include one uniform housing commodity.

So far, these two literatures have evolved independently. The urban economics literature abstracts from the stochastic nature of returns to housing, the investment demand for housing, and any risk premium built into housing prices. The finance literature abstracts from the spatial heterogeneity of houses and the endogenous cross-sectional distributions of households' investments and rents.

We explore the gains from merging the two literatures. We design a tractable model with closed-form solutions that are comparable to well-established results in both fields. The exercise reveals meaningful interactions between spatial equilibrium and asset pricing, hence "spatial asset pricing."

We demonstrate that choosing a location amounts to solving an expanded portfolio problem with the requirement to choose one unit of one of the free securities representing each location. These securities deliver a stream of location specific net earnings, local labor income minus housing rent.

As in standard spatial models, rents are priced by agents who are indifferent across locations. The relevant attributes of each location include not only expected net earnings but also their variance, or rather, the cost of hedging this variance: a function of the covariance of the agent's earnings with local rents and asset prices.

The hedging motive for investment introduces home bias in portfolio choice. Because of this home bias, the covariances of agents' income within each location and the allocation of agents across locations affect the aggregate demands and prices for all assets, including houses in every location.

As such, spatial asset pricing brings to the fore new testable predictions spanning both financial and spatial variables. For example, we find the rate of homeownership in each location is relevant to the pricing of all assets. Spatial asset pricing also provides a unified conceptual framework to reconsider a variety of empirical questions concerning variations in households' portfolios over space and time, and the relationship between housing and financial asset prices.

The model. We assume four classes of assets: a risk-free bond, stocks, houses in a number of locations, and non-transferrable human capital. As in standard asset pricing models, agents may lend and borrow at the risk-free bond rate without any constraint. Agents may

also invest in stocks, defined as claims over exogenous stochastic streams of dividends. The dividend stream of houses, however, is determined endogenously. Houses provide access to a stochastic production technology that is specific to their location. An agent’s human capital determines the expected level of his or her earnings at each location and the covariance of earnings with the location-specific production technology. The distribution of individual characteristics across the population is expressed in a general form.¹

Houses differ only in their location. The supply of houses is fixed in every location but one, the countryside, where the supply is unlimited. Houses can be rented at the local equilibrium market rate. In our benchmark model, houses can be purchased or sold (even fractionally) at the local equilibrium price. Agents may buy houses in every market, including in their city.

There are no frictions on any of the asset markets; e.g., no credit constraints, no transaction costs for buying or renting, no limits to fractional ownership.

We want to obtain closed-form solutions and expressions that are comparable to standard results obtained from mean-variance asset pricing models, so we assume an overlapping generations structure with finite life and constant population size. Agents have constant-absolute risk-aversion preferences with infinite elasticity of intertemporal substitution, and both city-productivity and stock-dividends stochastic shocks are normally distributed.

Agents choose where to live at the beginning of their life. Living in a location requires consuming one unit of housing in this location. While all investment decisions can be revisited in every period, the location choice is irreversible (moving costs are infinite).²

Location security. Each location is represented as a free location security composed of two parts: (1) a stream of location-specific stochastic benefits that yield location and agent specific wages or enjoyment of local amenities, and (2) a unit of local housing, which requires a stream of endogenous stochastic rent payments, and satisfies the local housing consumption requirement.

The location decision and the portfolio allocation problem of an agent can therefore be examined within the same dynamic optimization framework. Besides choosing financial assets, each household must pick one unit of one location security. This characterization reflects the discreteness of the choice of location.

¹In most of this work, we interpret local productivity as labor-related and hence translated into labor earnings, but the model has an equivalent interpretation in terms of leisure, where productivity is understood as the ability of the agent to enjoy local amenities. We also assume that there are no spillover effects across agents; that is, the productivity of an agent depends on its location but not on who else lives in that location. Later, we show that our characterization extends to a model with generic economies of agglomeration.

²The assumption that people cannot move is useful for tractability of analysis, but the assumption is not necessary for results about the role of housing as a hedge. Sinai and Souleles (2009) provide evidence of the empirical relevance of this phenomenon, accounting explicitly for US household moving patterns.

Spatial allocation and rents. Rents are determined by the productivity of marginal residents, households that are indifferent between two or more locations. We demonstrate that our model admits a set of hyper-marginal residents: households indifferent among all locations, whether city or countryside. These households are all age one as location choice takes place at age one, once and for all.³

Determining the set of hyper-marginal households is a key step to characterizing the equilibrium with stationary allocation of households across locations, linear housing rents and asset prices. We conjecture a functional form for rents and prices and we verify the validity of our guess. We build our guess by adapting standard results obtained in CARA-normal portfolio choice frameworks.

Local rents respond to local productivity shocks so as to keep the hyper-marginal residents indifferent across all locations. The indifference condition of the hyper-marginal resident pins down the relative level of rents in different locations. The fact that one location (the countryside) has an unlimited supply of land determines the absolute level of rents.

The location decision of any agent is determined by comparing that agent's location-specific set of productivity parameters with that of the hyper-marginal residents. By aggregating the investment demand functions of all agents, we obtain the asset pricing formulas both for real estate in different locations and for stocks.

We are then in a position to verify that the initial conjecture about the hyper-marginal resident is correct and that it is indeed an equilibrium. We thus prove the existence of an equilibrium where prices can be expressed as linear functions of the underlying parameters, and the allocation of households across location remains constant over time. Uniqueness can be proven in specific cases.

Portfolio choice. Because we assume households choose their location at birth, marginal residents are newborn households. As households age, we do not put any restriction on the covariance of their current income with their income at birth. A changing covariance exposes households to the risk that shocks to their income may not provide full insurance for the local productivity shocks that affect their housing costs. This risk can be hedged away by an appropriate holding of local real estate.

The optimal investment portfolio of every agent contains two components: (1) An investment in local housing that depends on the agent's exposure to local productivity shocks, and (2) a portfolio of stocks and houses, with identical weights across agents. The first com-

³An ancillary contribution of our work is to show existence in a spatial general equilibrium model with an infinite number of types of agents. Available existence results in the literature use a different approach based on a finiteness (Grimaud and Laffont, 1989).

ponent is a manifestation of home bias. This hedging demand depends on the covariance between the agent's earnings and local productivity shocks. The second component is identical across agents because in our benchmark case, all agents fully hedge their exposure to local productivity shocks with a purchase of local housing.

Asset pricing. Conditional on their purchases of local housing for hedging purposes, all households are fully hedged against their idiosyncratic risk and thus identical with regard to aggregate risk. Hence they all have the same investment demand for the remaining securities in the economy: the portfolio made up of all stocks and residential properties in the economy minus the houses held for hedging purposes. Let us call this portfolio the *adjusted market portfolio*.

Equilibrium requires that the price of all assets in the economy be such that total investment demand (beyond the hedging demand for houses) equals the adjusted market portfolio. All assets are therefore priced in this adjusted market portfolio.

Implications. Our portfolio choice and asset pricing expressions specify only objects – such as prices and covariances – that are in principle observable. We can thus develop a number of empirical questions linking spatial and financial variables.

Some of our predictions are already known, but have so far been discussed mainly in isolation; others are, to the best of our knowledge, new. The main strength of spatial asset pricing is that all of them can be analyzed within a consistent and tractable conceptual framework.

Home bias. With no friction on the housing market and identical homes within each location, local homes are a perfect hedge for agents' exposure to income minus rent risks. This results in home bias that manifests itself entirely in terms of local housing purchases. In a less perfect world without fractional ownership of housing and with local consumption that extends beyond housing, we would expect agents to use both local housing and local financial assets for hedging purposes- i.e., stocks with strong covariance with the local economy. Our framework provides a complementary rationale for the preference for geographically proximate investments pointed out in Coval and Moskowitz (1999).

Homeownership over the life-cycle. Sinai and Souleles (2005), and Davidoff (2006) already pointed out the relevance of the hedging motive for household's housing investment. Our theory opens new empirical questions with regards to the measurement of the determinants of this hedging motive and provides a new rationale for the well-known hump-shaped life-cycle profile of homeownership.⁴ Suppose that, as agents, get older, the covariance of their

⁴For example, see Fernandez-Villaverde and Krueger (2011) and Attanasio et al. (2011).

income with the income of newcomers to their city decreases. Their income provides less insurance against rent shocks. This pushes them to purchase an increasing amount of local housing for hedging purposes as they get older. Counter to this effect is the fact that as agents get older, the time left to live is shortened and so their demand for insurance declines. We show by example how this combination of effects can yield a hump-shaped homeownership profile.

Geographical allocation of agents. The celebrated location choice model of Rosen (1979) and Roback (1982) determines jointly housing rents and the geographical distribution of agents, but it does so in a static, deterministic model. By moving to a dynamic, stochastic setting, we create a role for investment. When they choose where to live, agents trade off expected net earnings opportunities (expected wage minus expected rents) against risk exposure (volatility of income minus rent). Agents therefore do not necessarily choose the location that maximizes their net earnings. They may prefer a location with lower expected income minus rents if their income in that location is less correlated with rents. In such a location, the purchase of local housing provides insurance benefits. Nevertheless, they earn a risk premium on the local housing because it is priced by outsiders to whom the volatility of housing returns is a risk, not an insurance.

Inferences from the cross-sectional dispersion of rents. By the argument above, spatial asset pricing points out the limitations of any model that implicitly assumes location decisions based entirely on expected earnings and amenities and ignores variations in a household's risk exposure. The quantitative importance of the risk/hedging factor in location and investment decisions is of critical empirical importance. Should this factor matter in any significant way, it would call into question standard approaches to the valuation of local amenities that are based on a static frameworks.

Joint determination of housing and financial asset prices As a number of authors have noted, households financial and housing investments and the prices of housing and financial assets can only be determined jointly.⁵ Our paper offers a three-fold contribution. First, we consider a large number of assets in both asset classes and we allow a generic covariance matrix. Second, rather than assuming an exogenous dividend process for houses we derive it endogenously from geographical location equilibrium. Third, we incorporate the hedging role of local housing.

⁵See for example Grossman and Laroque (1991), Flavin and Nakagawa (2008), Henderson and Ioannides (1983), Goetzmann (1993), Brueckner (1997), Flavin and Yamashita (2002), Englund, Hwang, and Quigley (2002), Iacoviello and Ortalo-Magné (2003), and LeBlanc and Lagarenne (2004), Cocco (2004), Yao and Zhang (2005), Piazzesi, Schneider, and Tuzel (2007)

The fact that the quantity of homes in each location in the pricing-relevant adjusted market portfolio is determined endogenously, adds a channel whereby the spatial allocation of households affects the prices of all assets. In other words, the spatial allocation of households does more than determine the stochastic properties of the rents in each location; it also determines what assets are part of the portfolio that is relevant to pricing systematic risk in equilibrium. As a result, our asset pricing formulas include some spatial variables such as the covariance between productivity shocks in different locations and the proportion of owner-occupied housing in each location. While the use of such variables is new, they are all potentially observable.

The prices of stocks are therefore determined not only by how their dividends co-vary with those of other financial assets but also by how they co-vary with earnings of the hyper-marginal residents in each location. Note that information relevant for asset pricing related to the presence of housing in the economy cannot be represented by a single aggregate housing good.

Cross-sectional dispersion of rent-price ratios. Our model predicts the cross-sectional distribution of rent-price ratios. The local yield depends on the average within-location covariance of the income of each resident with the income of the current and future marginal residents. For instance, in a one-company town, wages of all cohorts are highly correlated with rents; residents there do not demand local housing for hedging purposes, and prices are depressed with respect to rents. Prices are higher in diversified towns where households invest in local housing for hedging purposes.

Extensions. The analytical results we obtain for the benchmark model support two useful extensions in the modeling of locations and housing assets.

First, the benchmark model can be extended to encompass economies of agglomeration and other forms of externalities among residents. The equilibrium characterization of the benchmark model remains valid, but the presence of direct externalities amplifies the possibility of equilibrium multiplicity. If the agglomeration economies are strong enough, there will be multiple linear stationary equilibria corresponding to different allocations of talent across cities. This means we may be able to create links between real estate finance and the vast literature on agglomeration effects.

Second, the benchmark model assumes the ownership of real estate is perfectly divisible. Households are allowed to buy exactly the amount of local housing they need to perfectly hedge their risk in income minus rent. A number of frictions are likely to lead household away from the perfect hedge investment; e.g., a preference for homeownership, preferential tax treatment of homeownership, housing transaction costs, housing property indivisibilities.

Any such impediment to obtaining a perfect hedge with local housing leads households to resort to exploiting the covariance between their local risk and each of the financial assets. Agents in different location therefore purchase a different portfolio of housing and stocks. This hedging demand for stocks ends up affecting stock prices. As a case in point, we propose explicit solutions for stock prices when all households are required to own their home. While in the baseline case the only asset used for hedging was local real estate, the presence of indivisibilities induces agents to use stocks to hedge against their exposure (or lack of exposure) to local real estate. This generates an additional set of testable portfolio implications.⁶

The paper is organized as follows: Section 2 reviews the related literature. Section 3 sets out the model. Section 4 presents the main equilibrium characterization result, through three propositions corresponding to: portfolio allocation (Proposition 1), asset pricing (Proposition 2), and location choice (Proposition 4). Section 5 uses the main result to discuss a number of related issues. Section 6 concludes. All proofs are in the Appendix.

2 Related Literature

This is – to the best of our knowledge – the first asset pricing model where location choices, housing rents, and asset prices are endogenous. While many of the phenomena that our model describes have been analyzed elsewhere, it is the first time that they can be explained within one model, thus providing a unified conceptual framework to think about spatial asset pricing.

Our work is partly inspired by an asset pricing model outside the real estate literature: DeMarzo, Kaniel, and Kremer (2004). They consider an economy with multiple communities and local goods as well as a global good. In this dynamic setting, some agents (the laborers) are endowed with human capital that will be used to produce local goods in future periods, but they are currently subject to borrowing constraints. Other agents (the investors) own shares in firms that produce the global good.

This approach yields a number of powerful results. Investors care about their relative wealth in the community because they bid for scarce local goods. This generates an externality in portfolio choice, which leads to the potential presence of multiple equilibria (in the stable equilibria, investors display a strong home bias). And, if there is a behavioral bias, this externality amplifies the bias through the portfolio decisions of rational investors.

⁶Other extensions yield similar predictions with regard to home bias. Suppose households enjoy more utility from the same property if they own it than if they rent it. Such an assumption implies that their investment in local housing is not driven purely by hedging considerations. Households are willing to “distort” their housing investment because of consumption benefits. It then becomes optimal to use stocks to deal with any residual risk in income minus rent not canceled with local housing investment.

Our model differs from DeMarzo et al. (2004) in a number of important dimensions: (1) Our local good does not produce utility directly, but it enables agents to realize their human capital potential; (2) there are no credit constraints; and (3) our spatial allocation is endogenous. We do share their goal of studying the properties of portfolio choice and asset pricing under uncertainty in the presence of community effects. As in their model, a home bias arises in equilibrium because of a hedging motive.⁷

Our contribution to the real estate finance literature, lies in endogenizing both housing prices and rents in a dynamic model with multiple locations.⁸ Our approach to the modeling of housing as access to a location is in the tradition of urban economics. Our location choice model follows the standard multi-cities framework of Rosen (1979) and Roback (1982), where residential properties provide access to the local labor market, and locations are differentiated by potential surplus. As in Rosen and Roback and the many more recent papers that build on this framework (e.g., Gyourko and Tracy, 1991, Kahn, 1995, Glaeser and Gyourko, 2005), we assume households face a unit housing consumption requirement and derive utility from consumption of numeraire only.

Lustig and Van Nieuwerburgh (2010) consider risk sharing across regions. Empirical evidence indicates that the amount of housing wealth in a region affects the sensitivity of local consumption to local income. This paper is particularly close to ours in that it considers multiple locations. Lustig and Van Nieuwerburgh, however, assume exogenous location choice and that housing supply is perfectly elastic in all locations (and hence rents depend only on aggregate shocks).

Because we are concerned with portfolio choice in a dynamic environment, we assume households are risk-averse. Risk aversion in the face of stochastic streams of income and rent provides a motivation for ownership of local residential properties – homeownership – in our model. This approach builds on the work of Ortalo-Magné and Rady (2002), Hilber (2005), Sinai and Souleles (2005, 2009), Davidoff (2006) and others who provide evidence of the relevance of such motivation for housing investment.

We do not here review the vast literature concerned with the determinants of housing prices. Typically in this literature, real estate prices are determined by a perfectly elastic supply function (Lustig and Van Nieuwerburgh, 2008) or by a perfectly elastic demand function (Davis and Heathcote, 2005, Davis and Ortalo-Magné, forthcoming, Gyourko, Mayer and Sinai, 2006, Kiyotaki, Michaelides and Nikolov, 2007, Van Nieuwerburgh and

⁷Our results on home bias are also related to the international finance literature on the home bias puzzle (Stockman and Dellas, 1989), but we differ in our focus on real estate and in that location choice is endogenous in our model.

⁸A review of the empirical literature on the cross-sectional dispersion of housing prices is beyond the scope of this paper. For recent evidence emphasizing variations in housing price premiums see Campbell et al. (2009).

Weil, 2007).

3 Model

Consider an overlapping generation economy where a mass 1 of agents is born in every period. Each agent in the t -cohort is born at the beginning of period t , lives for S periods, and dies at the beginning of period $t + S$. Hence, at every time t , there are a mass S of agents alive in the economy.

3.1 Geography

There are L cities, denoted by index $l = 1, \dots, L$ and a countryside denoted by index $l = 0$. City l has an exogenously given mass of houses. Let n^l be the mass of houses per cohort that will be active on the housing market so that total supply of housing in city l equals $S \times n^l$. We assume that housing supply is scarce in cities:

$$\sum_{l=1}^L n^l < 1$$

but it is abundant once we include the countryside:

$$\sum_{l=0}^L n^l > 1.$$

Each house accommodates exactly one agent.

3.2 Production

The income of a person who lives in the countryside is (normalized to) zero. Productivity in city l follows the process

$$y_t^l = y_{t-1}^l + \tau_t^l$$

where τ_t^l is a random variable, independently and identically distributed across time.

At birth, each agent draws:

- A vector of city-specific endowment surplus, $\boldsymbol{\varepsilon} = [\varepsilon^l]^{l=1, \dots, L}$, with $\varepsilon^l \in (-\infty, \infty)$.
- A matrix of city- and age-specific insulation parameters: $\boldsymbol{\rho} = [\rho_s^l]_{s=0, \dots, S}^{l=1, \dots, L}$, with $\rho_s^l \in [0, 1]$. Assume $\rho_0^l = 0$ for all l .

The parameters $(\boldsymbol{\varepsilon}, \boldsymbol{\rho})$ are i.i.d. across generations. Their joint distribution within a generation takes the general form $\phi(\boldsymbol{\varepsilon}, \boldsymbol{\rho})$, with the only requirement that it should be continuous and have full support.

At time $t + s$, the income of an agent living in city l , born at time t , with parameters (ε, ρ) is

$$y_{t,t+s}^l(\varepsilon^l, \rho^l) = y_{t-1}^l + \varepsilon^l + \sum_{m=0}^s (1 - \rho_m^l) \tau_{t+m}^l$$

for $s = 0, \dots, S - 1$ (note the difference between y_t^l , a city-wide variable, and $y_{t,t+s}^l(\varepsilon^l, \rho^l)$ an individual specific variable).

Hence, the income of each agent can be decomposed into a permanent part, which captures the initial productivity of the agent in a location and a time-dependent part, which is determined by the local productivity shocks in the city and that agent's sensitivity to the city's shocks. We call ε^l the *city-agent effect* and ρ_s^l the *shock insulation effect*.

We represent below the income earned by an agent born at time t , living in city l , for each of the first three years of life:

$$\begin{aligned} y_{t,t}^l(\varepsilon^l, \rho^l) &= \overbrace{\varepsilon^l}^{\text{city-agent effect}} + \overbrace{y_t^l}^{\text{city-cohort effect}} ; \\ y_{t,t+1}^l(\varepsilon^l, \rho^l) &= \overbrace{\varepsilon^l}^{\text{city-agent effect}} + \overbrace{y_t^l}^{\text{city-cohort effect}} + \overbrace{(1 - \rho_1^l) \tau_{t+1}^l}^{\text{year 1 innovation}} ; \\ y_{t,t+2}^l(\varepsilon^l, \rho^l) &= \overbrace{\varepsilon^l}^{\text{city-agent effect}} + \overbrace{y_t^l}^{\text{city-cohort effect}} + \overbrace{(1 - \rho_1^l) \tau_{t+1}^l}^{\text{year 1 innovation}} + \overbrace{(1 - \rho_2^l) \tau_{t+2}^l}^{\text{year 2 innovation}}. \end{aligned}$$

Similar formulations determine the agent's earnings until reaching age $S - 1$.⁹ At age S , the agent does not earn anything. It is mathematically convenient to set $\rho_S = 0$ for all agents even if it is irrelevant to the agents' earnings.

The city-agent effect, ε^l , is a standard object in multi-city models with heterogeneous agents. Depending on their human capital, agents face different earning opportunities in different locations.

The shock-insulation effect, ρ^l , captures two economic phenomena. First, agents may be exposed to a technological cohort-specific effect (documented by Goldin and Katz, 1998). The human capital of certain people, especially the young, may be more flexible. When a technological innovation appears, the income of certain agents will be more affected than the income of others.

Second, certain agents – like senior workers and public sector workers – may be part of an implicit labor insurance agreement. Their wages are more insulated from productivity shocks.

⁹The structure of ε and ρ could be more complex and still be amenable to analysis in our mean-variance set-up. For instance, we could say that the city-agent effect is not constant over the life of the agent but rather it follows a random walk. Also, we could assume that the extent to which a shock that occurs at age s affect future incomes depends on the age of the agent.

It is reasonable – but not necessary for the analysis – to assume that the insulation parameter, for a shock that occurs at a given age, increases with in the age of the agent: $\rho_{s+1}^l > \rho_s^l$. The two extreme cases are full insulation ($\rho_s^l = 1$) and full exposure ($\rho_s^l = 0$).¹⁰

For concreteness, we interpret $y_{t,t+s}^l$ as monetary income, but there is an alternative interpretation in terms of non-monetary benefits that is equivalent from a mathematical standpoint. The term $y_{t,t+s}^l$ can be viewed as a money-equivalent of the utility afforded by the amenities present in location l . The utility can be decomposed in turn into an agent-city effect (a preference for that particular location) and a shock component (perhaps an environmental or a social risk) multiplied by the agent’s sensitivity to that type of shock. Of course, the model can also be interpreted as a mix of monetary and non-monetary benefits.

An agent who lives and thus produces in city l , must rent exactly one unit of housing in city l .

3.3 Housing market

The housing market is frictionless. There are no transaction costs associated with renting, buying, or selling property. There is no difference between living in an owned or a rented house.

At birth, every agent chooses in what city (or the countryside) to live. The agent cannot move afterward. Living in city l at time t entails paying the market rent, on a unit of housing, r_t^l . Rents are determined in equilibrium.

Agents may invest in divisible shares of any city’s housing stock and revise their decision at every period. Let $a_{t,t+s}^l$ denote the amount of housing of city l owned by an agent born at time t of age s .

The market price of a unit of housing in city l at time t is p_t^l . The agent revises his or her housing investment at the beginning of every period. For accounting purposes, imagine that the agent liquidates all housing assets and then buys the desired amount in each period. At the beginning of period $t + s$, the agent acquires $a_{t,t+s}^l$ units in city l at total cost $a_{t,t+s}^l p_{t+s}^l$. During period t , the agent collects rent on the housing investment for a total of $a_{t,t+s}^l r_t^l$. At the beginning of the next period, the agent liquidates the housing investment and receives $a_{t,t+s}^l p_{t+s+1}^l$. We denote $\mathbf{a}_{t,t+s}$ the vector of the agent’s housing investments, $\mathbf{a}_{t,t+s} = [a_{t,t+s}^l]_{l=1,\dots,L}$.¹¹

¹⁰We find it natural to restrict ρ_s^l to be between zero and one, but our mathematical analysis is valid even if $\rho_s^l > 1$ (the agent’s productivity is negatively correlated with local shocks) and $\rho_s^l < 0$ (the agent is overexposed to local shocks).

¹¹Given the frictionless nature of the housing market, derivative securities would be superfluous. In particular, Case-Shiller home price indices for our cities (a security bought at time t which pays a price p_{t+1}^l at time $t + 1$) would be equivalent to purchasing housing for one period, net of the “rent coupon.” Given the

3.4 Stock market

Besides housing, there is another class of securities called stocks. These are claims on productive assets, that – as in regular asset pricing models – produce an exogenous stochastic stream of income. There are Sz^k units of type- k asset, with $k \in \{1, \dots, K\}$ and $z^k > 0$. A unit of stock k produces dividend d_t^k at time t . The dividend follows the stochastic process:

$$d_t^k = d_{t-1}^k + \nu_t^k$$

where ν is i.i.d. across time with probability distribution as below.

As is the case for housing, every agent can buy units of every stock and revise portfolio allocations in every period. The market price of stock k at a particular time is q_t^k . At the beginning of period $t + s$, the agent acquires $b_{t,t+s}^k$ units of stock k at total cost $b_{t,t+s}^k q_{t+s}^k$. During period $t + s$, the agent receives dividends on investment in k for a total of $b_{t,t+s}^k d_{t+s}^k$. At the beginning of the next period, the agent liquidates the stock investment and receives $b_{t,t+s}^k q_{t+s+1}^k$. We denote $\mathbf{b}_{t,t+s}$ the vector of the agent's stock investments, $\mathbf{b}_{t,t+s} = [b_{t,t+s}^k]_{k=1, \dots, K}$.

3.5 Distribution of random shocks

There are two sources of exogenous shocks in our economy: a vector τ of local productivity shocks, and a vector ν of dividend shocks. The shocks are independently and identically distributed over time, according to a normal distribution with mean 0 and covariance matrix Σ : $(\tau_t, \nu_t) \sim N(0, \Sigma)$.

We do not impose any restriction on the correlation between local productivity shocks and dividends. One industry may be more affected by shocks in a certain market, and vice versa. We also do not impose any restriction on the correlation of productivity shocks across cities.

3.6 Consumption and savings

As our goal is to develop a closed-form expression for asset prices, we assume that agents derive CARA utility $-\exp(-\gamma w)$ from wealth at the end of their life, w , where γ is the standard risk-aversion parameter.

Agents face no credit constraints and can borrow and lend freely at discount rate $\beta \in (0, 1)$. For simplicity, we assume that agents are born with no wealth (this does not affect their decisions, given that they have CARA preferences).

random-walk nature of all our shocks, long-term securities are also redundant because they can be replicated by sequences of short-term investments. This includes long-term rentals or futures on real estate.

3.7 Non-negativity constraints

Asset pricing models with normally distributed shocks suffer from a well-known technical problem. As the value of dividends can become negative, agents may want to dispose of assets they own. If they could, the distribution of asset values would no longer be normal, and the model would not be tractable. Hence, all models in this class assume, implicitly or explicitly, that agents cannot dispose of assets. Typically, this assumption is unrealistic because in practice both agents and firms are protected by limited liability. Instead, in the model stocks can have negative prices, and their owners must pay to get rid of them.

Our CARA-normal framework inherits this non-negativity problem. That is, productivity in a city could become negative, and house prices there may be negative.¹²

The usual response to this criticism, which applies here as well, is that the unconstrained model should be viewed as an approximation of the model with non-negativity constraints, as long as the starting values are sufficiently far from zero.

3.8 Timing

The order of moves for an agent born at time t is as follows:

1. At birth, the agent chooses in which location l to spend the rest of his or her life.
2. At the beginning of each period $t + 0, \dots, t + S$, the agent learns the values of the random shocks for that period, ν_{t+s} and τ_{t+s} .
3. For $s = 0, \dots, S-1$, the agent revises housing and stock investments ($\mathbf{a}_{t,t+s}$ and $\mathbf{b}_{t,t+s}$), pays rent r_{t+s}^l for one unit of housing in the chosen location and collects dividends and rents on the assets owned.
4. At time $t+S$, the agent liquidates all investments ($\mathbf{a}_{t,t+S-1}$ and $\mathbf{b}_{t,t+S-1}$) and consumes all wealth before death.¹³

4 Analysis

An equilibrium is an allocation of households across cities, a vector of optimal portfolio holdings of housing and stocks for each agent, housing rents and prices for each city, and stock prices such that: (1) The location choice and portfolio holdings solve the agents' problem; (2) the housing markets (space and ownership) in each city clear; and (3) the stock markets clear.

¹²We assume homeowners have an obligation to rent their property (they pay a fine if it is vacant).

¹³The agent does not work or pay rent in the last period of life ($t + S$) but rather consumes all wealth at the beginning of the period before death.

A *stationary equilibrium* is an equilibrium where the mass of agents of a generation t who live in a given city l is the same across generations.¹⁴

A *linear equilibrium* is an equilibrium where stock prices, rents, and house prices can be expressed, respectively, as:

$$q_t^k = \frac{1}{1-\beta} d_t^k - \bar{q}^k \quad (1)$$

$$r_t^l = y_t^l + \bar{r}^l \quad (2)$$

$$p_t^l = \frac{1}{1-\beta} r_t^l - \bar{p}^l \quad (3)$$

where $\bar{q} = [\bar{q}^k]^{k=1,\dots,K}$ and $\bar{p} = [\bar{p}^l]^{l=1,\dots,L}$ are price discounts; and $\bar{r} = [\bar{r}^l]^{l=1,\dots,L}$ is a rent premium to be determined in equilibrium. The rent is equal to local productivity plus a local constant. House and stock prices are equal to the discounted value of a perpetuity that pays the current rent or dividend minus an asset-specific discount.

Price discounts can also be interpreted as expected returns of zero-cost portfolios.¹⁵ Throughout the analysis we describe \bar{p}^l and \bar{q}^k as price discounts or expected returns, depending on the context.

Our strategy for finding equilibria is as follows. We start by conjecturing that we are in a stationary linear equilibrium. We postulate a feasible allocation of agents to cities, and we solve the portfolio problem of a generic agent living in a given city. As it turns out, solving this agent problem is enough to characterize stock prices and house prices up to a vector of city-specific constants. With this information, we compute the expected utility of every agent, conditional on city choice. We determine aggregate location demand, given any price vector by comparing expected utilities across cities.

Finally, we consider the marginal residents. We show that for every vector of city-specific constants there are a set of agents who are indifferent among all locations (the hyper-marginal residents), while all others have strict preferences. The characteristics of the hyper-marginal residents are monotonic in the vector of city-specific constants, and we can identify the hyper-marginal residents so that the mass of agents who move to each city equals the local housing supply in each city. This proves that our initial conjecture on linear prices is correct.¹⁶

¹⁴A non-stationary equilibrium be structured as follows. As agents cannot move after they locate to city l , the stock of rented accommodation used by the t -cohort will not become available until members of the t -cohort die at the end of $t + S$. Hence, if the t -cohort is, say, overrepresented, the $t + S + 1$ -cohort will be equally overrepresented. The non-stationary equilibria are characterized by cycles of length $S + 1$.

¹⁵For instance, the expected return of a zero-cost one-unit portfolio invested in housing in city l (evaluated in today's dollars) is

$$E \left[\beta p_{t+1}^l - (p_t^l - r_t^l) \right] = \frac{\beta}{1-\beta} r_t^l - \beta \bar{p}^l - \frac{\beta}{1-\beta} r_t^l + \bar{p}^l = (1-\beta) \bar{p}^l.$$

¹⁶It is tempting to consider the two first parts of the analysis (portfolio choice and asset pricing) in

As agents have CARA preferences, their lifetime utility can be decomposed into:

$$E [w_t^l] = E [w_t^l] - \gamma V [w_t^l].$$

Proposition 1 re-writes the two components of the agents utility and uses them to compute his optimal portfolio choice and his expected utility.

In what follows we focus on one agent and we drop the argument representing the agent-specific characteristics: (ε, ρ) . All proofs are in appendix.

Proposition 1 (Portfolio Allocation) *Suppose that prices and rents are given by equations (1), (2), and (3), with given $\bar{\mathbf{r}}$, $\bar{\mathbf{q}}$ and $\bar{\mathbf{p}}$. Consider any allocation of agents to cities. Consider an agent born at period t characterized by a vector ε and a matrix ρ . If this agent lives in l and chooses investment profiles $[\mathbf{a}_{t,t+s}, \mathbf{b}_{t,t+s}]_{s=0,\dots,S-1}$, the expectation and the variance of the agent's end-of-life wealth can be written, respectively, as:*

$$E [w_t] = \sum_{s=0}^{S-1} \beta^{s-S} \left(\varepsilon^l - \bar{r}^l + (1 - \beta) \left((1 - \beta^{S-s-1}) \rho_{s+1}^l \bar{p}^l + \sum_{j=1}^L \tilde{a}_{t,t+s}^j \bar{p}^j + \sum_{k=1}^K b_{t,t+s}^k \bar{q}^k \right) \right)$$

$$Var [w_t] = \frac{\beta^2}{(1 - \beta)^2} \sum_{s=0}^{S-1} \beta^{2(s-S)} Var \left[\sum_{j=1}^L \tilde{a}_{t,t+s}^j \tau_{t+s+1}^j + \sum_{k=1}^K b_{t,t+s}^k \nu_{t+s+1}^k \right]$$

where

$$\tilde{a}_{t,t+s}^j = \begin{cases} a_{t,t+s}^l - (1 - \beta^{S-s-1}) \rho_{s+1}^l & \text{if } j = l \\ a_{t,t+s}^j & \text{otherwise} \end{cases}.$$

The agent's optimal investment profile is given by

$$\begin{bmatrix} \tilde{\mathbf{a}}_{t,t+s} \\ \mathbf{b}_{t,t+s} \end{bmatrix} = \frac{(1 - \beta)^3}{2\gamma\beta^{s+2}} \beta^S \Sigma^{-1} \begin{bmatrix} \bar{\mathbf{p}} \\ \bar{\mathbf{q}} \end{bmatrix},$$

for $s = 0, \dots, S - 1$, and expected log-utility is

$$U^l = \frac{1}{\beta^S} \sum_{s=0}^{S-1} \beta^s \left(\varepsilon^l - \bar{r}^l + (1 - \beta) (1 - \beta^{S-s-1}) \rho_{s+1}^l \bar{p}^l \right) + S \frac{(1 - \beta)^4}{4\gamma\beta^2} \begin{bmatrix} \bar{\mathbf{p}} \\ \bar{\mathbf{q}} \end{bmatrix}' \Sigma^{-1} \begin{bmatrix} \bar{\mathbf{p}} \\ \bar{\mathbf{q}} \end{bmatrix}.$$

Proposition 1 says that the optimal portfolio of any agent can be decomposed into:

isolation, but they are valid only if the third part is present too. If one assumes a different location model or an exogenous allocation of agents to cities, the three price processes in equations (1), (2), and (3) would be different, and Propositions 1 and 2 would no longer hold. For instance, if agents could move between cities during their lifetime, it is not clear that the rent the price in city l would depend only on productivity in city l .

We see this as both a weakness and a strength of spatial asset pricing. On the one hand, one cannot have a meaningful discussion about real estate prices in multiple locations without an underlying spatial model. On the other hand, this opens the door to a wealth of testable implications related to spatial and financial variables.

- Demand for real estate in the city where the agent lives, $(a_{t,t+s}^l - \tilde{a}_{t,t+s}^l)$, driven by a desire to hedge shocks to disposable income due to rent fluctuations. As the price of a house is linear in the rent, a house in a certain city is a perfect hedge against rent fluctuations in that city. The hedging demand is given by $(1 - \beta^{S-s-1}) \rho_{s+1}^l$. Hence, the hedging demand depends on how well the agent is insulated from local productivity shocks at time t . The hedging demand varies across agents and across time for a given agent, but it does not depend on the expected return of real estate in that city (if a city has a high return, that will be reflected in the mutual fund share only).¹⁷
- Investment in a mutual fund includes all stocks and houses in all cities, with weights $(\tilde{\mathbf{a}}, \mathbf{b})$. The mutual fund is the same for all agents. All agents within a cohort buy the same amount of mutual fund shares (but older agents buy more shares, purely because of the discount rate β). Given a vector of expected returns (which for now is still exogenous), the weights $(\tilde{\mathbf{a}}, \mathbf{b})$ that the mutual fund puts on various stocks and real estate assets are given by a standard CAPM allocation. The portfolio puts more weight on an asset if its returns are less correlated with other assets and have a higher expected value.

Now that we have solved the portfolio allocation problem for any given vector of premiums, we solve for the equilibrium expected returns. Denote any (measurable) allocation of agents to cities with the indicator function $I_{\varepsilon, \rho}^l$, which takes a value of 1 if agents with personal characteristics ε and ρ locate to city l , and zero otherwise (such that $\sum_{l=0}^L I_{\varepsilon, \rho}^l = 1$ for all ε and ρ).

Proposition 2 (Asset Pricing) *Suppose that rents are given by equation (2), with given $\bar{\mathbf{r}}$. Consider any allocation of agents over space so that all cities are populated. Then, prices are given by equations (1) and (3) with discounts:*

$$\begin{bmatrix} \bar{\mathbf{p}} \\ \bar{\mathbf{q}} \end{bmatrix} = 2\gamma S \frac{\beta}{(1-\beta)^2 (1-\beta^S)} \boldsymbol{\Sigma} \begin{bmatrix} \mathbf{n} - \mathbf{R} \\ \mathbf{z} \end{bmatrix},$$

where $\mathbf{R} = [R^1, \dots, R^l, \dots, R^L]'$ and

$$R^l = \frac{1}{S} \sum_{s=0}^{S-1} (1 - \beta^{S-s-1}) \int_{\varepsilon} \int_{\rho} I_{\varepsilon, \rho}^l \rho_{s+1}^l \phi(\varepsilon, \rho) d\varepsilon d\rho.$$

¹⁷Davis and Willen (2000) obtain a related result (their Proposition 1) in decomposition of the optimal portfolio of agents who face labor risk into a speculative component and a hedging component.

Houses and stocks are priced according to their contribution to systematic risk by a classic CAPM formula. Proposition 2 finds the correct definition of systematic risk for this model. The weights of stocks in the market portfolio correspond to the quantity of stocks available, as in the regular CAPM. The weights of real estate, however, are reduced by the total hedging demand. Namely, the weight of houses in city l is equal to the mass of homes n^l minus the integral of the hedging demand by residents of l : R^l .

To explore the pricing expressions in Proposition 2 further, define the *adjusted market portfolio* M as a portfolio allocation that includes

$$\begin{aligned} & \frac{n^l - R^l}{Q} \text{ units of housing in city } l \text{ for every city } l, \text{ and} \\ & \frac{z^k}{Q} \text{ units of stock } k \text{ for every stock } k \end{aligned}$$

where $Q = \sum_{l=1}^L (n^l - R^l) + \sum_{k=1}^K z^k$. The mutual fund that all agents buy is the adjusted market portfolio.

Denote the expectation and the variance of the adjusted market portfolio, respectively, by \bar{p}^M and $Var(M)$. Define $Cov(l, M)$ as the covariance between the return of real estate in city l and the return of M . For every stock k , define $Cov(k, M)$ similarly. Then:

Corollary 3 *The expected return of real estate in city l is given by*

$$\bar{p}^l = \frac{Cov(l, M)}{Var(M)} \bar{p}^M$$

and the expected return of stock k is

$$\bar{p}^k = \frac{Cov(k, M)}{Var(M)} \bar{p}^M.$$

The expression in the Corollary is akin to the classic CAPM pricing formula where $\frac{Cov(l, M)}{Var(M)}$ is a beta-factor for housing in city l . The main innovation lies in identification of the adjusted market portfolio, for which this formula is true.¹⁸

Propositions 1 and 2 are really intermediate results. They rest on a specific conjecture about the stochastic process that determines local market rents, described in equation (2). But rents are not primitives, and we must now check that for the location model used here the conjecture is in fact correct. It is useful to reiterate that the conjecture would in general not extend to other location models, implying that Propositions 1 and 2 are valid only if accompanied by the specific spatial allocation model that we have chosen.

Besides closing the fixed-point argument, we also need to determine the vector of rent premiums $\bar{\mathbf{r}}$, and to find the vector of hedging demands \mathbf{R} .

¹⁸For instance, if one defines the market portfolio without the $-R$ correction, such a beta representation would not be valid.

For an agent with personal characteristics $(\boldsymbol{\varepsilon}, \boldsymbol{\rho})$, the log-utility of locating in city l is given by U in Proposition 1, where now $\bar{\boldsymbol{p}}$ and $\bar{\boldsymbol{q}}$ are defined in terms of primitives through Proposition 2. For every $(\boldsymbol{\varepsilon}, \boldsymbol{\rho})$, let

$$\bar{u}^l(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) \equiv \varepsilon^l + \frac{(1-\beta)^2}{1-\beta^S} \bar{p}^l \sum_{s=1}^S (1-\beta^{S-s-1}) \rho_s^l. \quad (4)$$

with the utility of being in the countryside: $\bar{u}^0(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) = \bar{u}^0$.¹⁹

Also let

$$\bar{U} = S \frac{(1-\beta)^4}{4\gamma\beta^2} \begin{bmatrix} \bar{\boldsymbol{p}} \\ \bar{\boldsymbol{q}} \end{bmatrix}' \boldsymbol{\Sigma}^{-1} \begin{bmatrix} \bar{\boldsymbol{p}} \\ \bar{\boldsymbol{q}} \end{bmatrix}.$$

Then, we can write the utility of locating in city l as:²⁰

$$U^l = \bar{U} + \frac{1}{\beta^S} \left(\frac{1-\beta^S}{1-\beta} (\bar{u}^l(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) - \bar{r}^l) \right).$$

Namely, the agent's utility can be decomposed into a component that is common to all agents (and depends on investment in the mutual fund) and an agent-specific component that depends on the city-agent effect ε^l and the shock-insulation vector $\boldsymbol{\rho}^l$ that the agent faces if the choice is to locate in city l .

A given agent locates in city l if and only if $U^l = \max_m U^m$. For every L -vector $\hat{\boldsymbol{r}}$, we can write the aggregate demand for location l as

$$\nu^l(\hat{\boldsymbol{r}}) = \int_{(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) : \bar{u}^l(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) - \bar{r}^l = \max_m (\bar{u}^m(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) - \hat{r}^m)} \phi(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) d(\boldsymbol{\varepsilon}, \boldsymbol{\rho}).$$

We obtain:

Proposition 4 (Location Choice) *A linear stationary equilibrium exists. In it, an agent with personal characteristics $(\boldsymbol{\varepsilon}, \boldsymbol{\rho})$ locates in city l if and only if*

$$\bar{u}^l(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) - \bar{r}^l = \max_{j=0, \dots, L} (\bar{u}^j(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) - \bar{r}^j)$$

and $\bar{\boldsymbol{r}}$ is the unique value of the vector $\hat{\boldsymbol{r}}$ such that $\nu^l(\hat{\boldsymbol{r}}) = n^l$ in all cities.

The equilibrium rent in city l is

$$r_t^l = y_t^j + \bar{r}^l.$$

¹⁹ Assuming that $\varepsilon^0 = 0$ is without loss of generality. If it was not, one could redefine all the ε as differences with ε^0 .

²⁰ To see this, note that:

$$\sum_{s=0}^{S-1} \beta^s (\varepsilon^l + (1-\beta) (1-\beta^{S-s}) \rho_{s+1}^l \bar{p}^l) = \frac{1-\beta^S}{1-\beta} \left(\varepsilon^l + \frac{(1-\beta)^2}{1-\beta^S} \bar{p}^l \sum_{s=1}^S (1-\beta^{S-s+1}) \rho_s^l \right)$$

Proposition 4 validates the conjectures that allowed us to obtain Propositions 1 and 2. The most important step in Proposition 4 is determination of the identity of hyper-marginal residents (the agents who are indifferent among all locations including the countryside). As we argue below, a key property of our set-up is that the characteristics of the hyper-marginal residents are constant over time. The indifference conditions for these agents determine market rents. This means that the local rent processes are the same, but for a constant term, as the local productivity processes. This validates the linearity assumption for the rent process built into equation (2).

Let us retrace, at an intuitive level, the steps that lead to Proposition 4. Despite the fact that the payoff of an agent in a given city is determined by $S+1$ parameters (ε^l plus the vector ρ^l), the expected utility U^l of the agent in that city can be condensed into a simple expression including $\bar{u}^l(\varepsilon, \rho)$. For any possible vector of rents \hat{r} , the demand function $\nu^l(\hat{r})$ establishes how many agents will live in each location.

Hence, for every vector of rent constants \hat{r} , we identify a set of measure zero of hyper-marginal residents such that their expected utility is the same in every city and in the countryside:

$$\bar{u}^l(\varepsilon, \rho) - \hat{r}^l = \bar{u}^0 \text{ for all } l.$$

Note that this correspond to multiple personal characteristic profiles: all the vectors (ε, ρ) that yield the same $\bar{u}^l(\varepsilon, \rho)$. One can show that the vector of expected utilities of the hyper-marginal resident in different location is monotonic in the rent constant vector \hat{r} . This means that the mapping can be inverted. Given the identity of the hyper-marginal resident, there is only one vector of rents that guarantees that the hyper-marginal residents are indeed indifferent across all locations.

The assumption that the distribution of individual characteristics $\phi(\varepsilon, \rho)$ has full support guarantees that the demand function is continuous. As the hyper-marginal residents determine the vector of location demands, one can find a set of hyper-marginal residents that guarantees that demand equals supply in every location. This set is associated with the rent constant vector \bar{r} .

In equilibrium, we have that housing demand equals housing supply on the space market in every city:

$$\nu(\bar{r}) = n;$$

and that the identity of the hyper-marginal residents is given by the set of values (ε, ρ) so that, given the equilibrium rent vector, their expected utility is the same in every city and in the countryside:

$$\bar{u}^l(\varepsilon, \rho) - \bar{r}^l = 0 \text{ for all } l.$$

A key feature of our location equilibrium is that the characteristics of the hyper-marginal resident are cohort-invariant. It is this feature that guarantees that the rent process is linear and that our equilibrium characterization is valid. If, for instance, agents could change city, the time-invariance property would not hold, and the rent process would not be linear. As a result, the properties of portfolio allocation and asset prices would differ.

We view this as a strength of spatial asset pricing models. The underlying geographic model – which we can potentially observe through demographic and labor data – affects equilibrium in the asset market.

The issue of uniqueness is complex. Obviously, there can be non-linear and/or non-stationary equilibria. Given an allocation of residents to cities, there is only one linear stationary equilibrium. There could be multiple spatial equilibria, however. The agent’s expected utility in equation (4) includes a multiplicative term $\bar{p}^l \rho_s^l$. As Proposition 2 shows, the real estate return \bar{p}^l depends on R^l and hence on who lives in city l , which creates a non-trivial fixed-point problem. The (economically interesting) possibility remains that there are multiple allocations of residents to cities that give rise to linear stationary equilibrium.

Uniqueness can be achieved under certain functional assumptions, as some examples illustrate.

While we obtained closed-form solutions for portfolio decisions and asset premiums, Proposition 4 does not express rents in closed form. This is natural as the probability distribution over individual characteristics, $\phi(\boldsymbol{\varepsilon}, \boldsymbol{\rho})$, is left in a general form. By making specific assumptions over personal characteristics and geography, one can obtain closed-form expressions for all variables, as the following example illustrates.

Assume that:

- Agents in each cohort draw city-specific endowments $\boldsymbol{\varepsilon}$ from a uniform distribution defined over $[0, 1]^L$.
- At each age, all agents face the same city-specific insulation parameter $[\tilde{\rho}_s^l]_{s=0, \dots, S}^{l=1, \dots, L}$.
- All cities are of the same size: $n^l = \frac{1}{L}N$ for every l , with $N \in (0, 1)$.

Proposition 5 *An agent with human capital $\boldsymbol{\varepsilon}$ locates in city l if: (1) $\varepsilon^l = \max_m \varepsilon^m$; and (2) $\varepsilon^l \geq (1 - N)^{\frac{1}{L}}$. The equilibrium rent in city l is*

$$r^l = (1 - N)^{\frac{1}{L}} + \frac{(1 - \beta)^2}{1 - \beta^S} \bar{p}^l \sum_{s=1}^S (1 - \beta^{S-s-1}) \tilde{\rho}_s^l.$$

If there are only two cities ($L = 2$), we can provide a two-dimensional representation of the equilibrium allocation. If, for instance, we assume that $n^1 = n^2 = \frac{1}{3}$ (and hence $n^0 \geq \frac{1}{3}$),

we have the situation depicted in Figure 1. The agents who locate in the countryside are those with a low ε^1 and a low ε^2 (the bottom left square region). Those who locate in city 1 have $\varepsilon^1 \geq (1 - N)^{\frac{1}{2}}$ and $\varepsilon^1 \geq \varepsilon^2$ (bottom right trapezoid). Those who locate in city 2 have $\varepsilon^2 \geq (1 - N)^{\frac{1}{2}}$ and $\varepsilon^2 \geq \varepsilon^1$. The marginal resident is found at the intersection of those three regions.

We can also see what happens when cities have different sizes. The general characterization is more complex than the one in Proposition 5, but one can work out examples. For instance, if $n^1 = \frac{2}{9}$ and $n^2 = \frac{4}{9}$ (and a measure $\frac{1}{3}$ of agents still locate in the countryside), the allocation is depicted in Figure 2. The city-1 region is now smaller and the city-2 region is larger. The hyper-marginal residents are now to the southeast of the hyper-marginal residents of Figure 1.

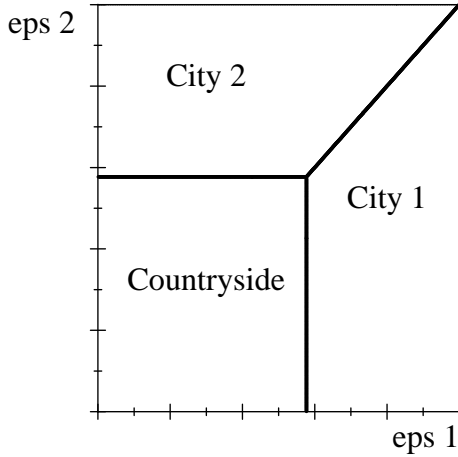


Figure 1

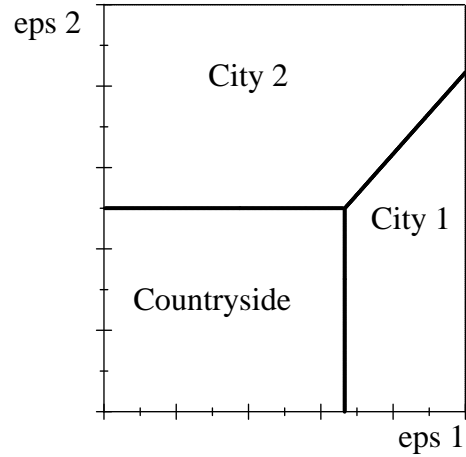


Figure 2

Another simple case assumes there is only one city, and agents differ on two dimensions: ε and ρ . Suppose both types are uniformly (and independently) distributed on $[0, 1]$. This

equilibrium allocation is represented in Figure 3.

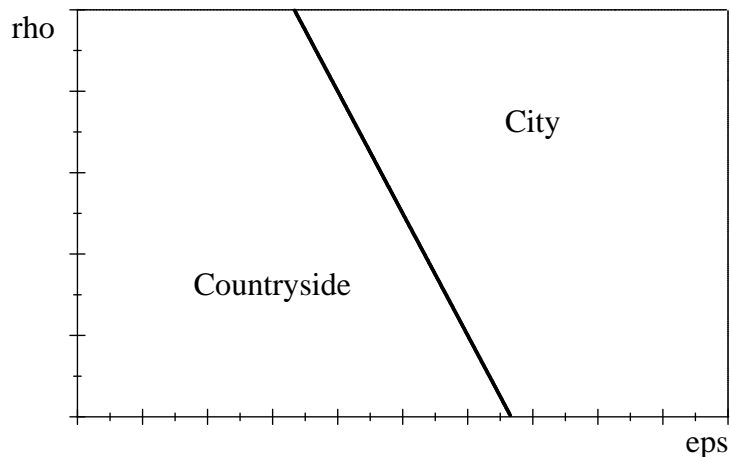


Figure 3

In Figure 3, there are values for ϵ so that agents with a low ρ locate in the countryside, and agents with a high ρ locate in the city. Agents with a high ρ are more insulated from city-level technology shocks. They buy more housing for hedging purposes than agents with a low ρ . They earn the risk premium on their housing investment although for them it provides insurance. This benefit that comes from living in the city is not available to agents with a low ρ who purchase less housing for hedging purposes.

5 Discussion

Our spatial asset pricing model yields a rich set of implications linking spatial and financial variables. We first discuss cross-sectional and life-cycle implications, and then talent allocation across cities. We explore the pricing of portfolios of stocks and portfolios of real estate. We conclude with a short discussion of how the model can be extended to include economies of agglomeration and frictions in the housing market.

5.1 Returns on housing across cities

Our model yields predictions about the cross-sectional differences in real estate returns (Proposition 2 and Corollary 3). To get some understanding of those predictions, consider a simple benchmark. Assume that shocks across cities are uncorrelated, and suppose there are no stocks. Let $Var(\tau^l) = \sigma_l^2$. Proposition 2 yields

$$\bar{p}^l = 2\gamma \frac{\beta}{(1-\beta)^2 (1-\beta^S)} \sigma_l^2 (n^l - R^l).$$

The expected return of housing in city l is an increasing function of the variance of shocks in that city and of the outstanding real estate stock $n^l - R^l$. In turn, the latter is a declining function of the average housing demand for hedging purposes, R^l , in that city. The value of R^l is determined in equilibrium.

If a location specializes in an industry and thus offers low shock-insulation parameters. All residents, whether old or young, are affected by industry productivity shocks in the same way. The residents have a low demand for housing for hedging purposes. The city's homeownership rate is low, and so are prices. The opposite, a city centered around an industry with high shock-insulation parameters – perhaps a high-tech industry where older workers struggle to keep up with innovation or a highly protected sector, where older workers face implicit insurance – will display a high hedging demand for housing, high homeownership rates, and high housing prices compared to rents.

5.2 Home ownership over the life-cycle

The model yields intertemporal predictions on individual home ownership rates. We know from Proposition 1 that housing demand for hedging purposes depends on the shock-insulation parameter, which in turn varies with age. The hedging demand by someone at age s anticipating a shock-insulation parameter the following period of ρ_{s+1}^l is

$$D_s^l = (1 - \beta^{S-s-1}) \rho_{s+1}^l.$$

Suppose the parameters ρ_s^l are determined by a differentiable function $g(s)$ defined over the positive real line. Suppose that the covariance of an agent's earnings with the earnings of the young marginal newcomers in that city declines with age; i.e., $g'(s) < 0$. Then, abusing notation for ease of exposition, we write the change in hedging demand for local home ownership with age as follows:

$$\frac{d}{ds} D_s^l = (1 - \beta^{S-s-1}) g'(s) + \log \beta \cdot \beta^{S-s-1} g(s)$$

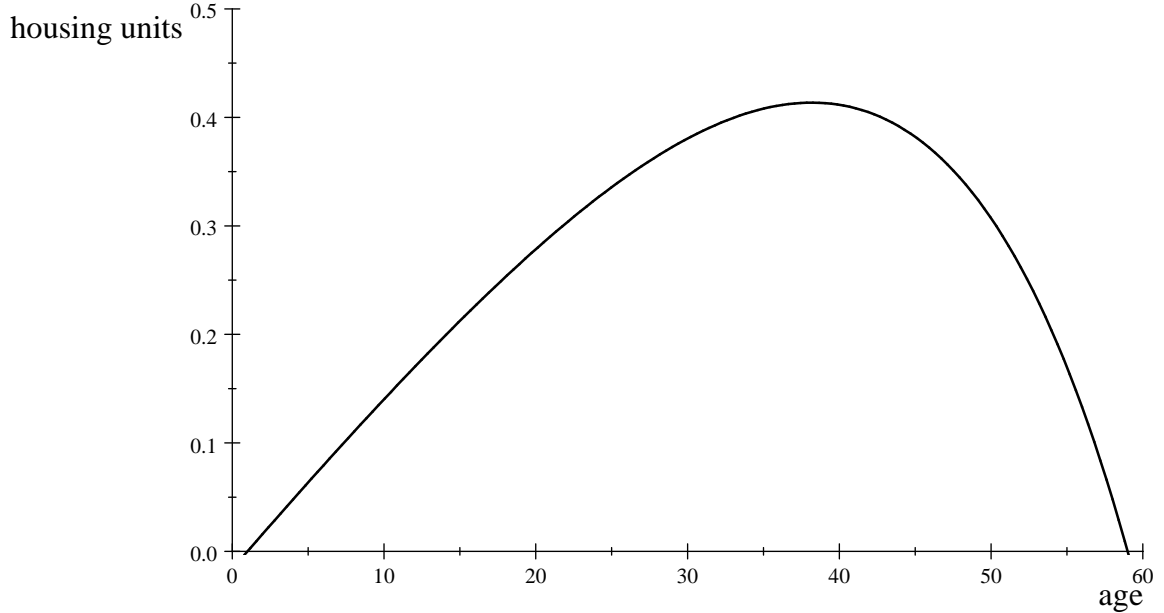
The first term represents the increasing insurance demand as the agent gets older due to ρ declining with age. The second term is the effect of the declining number of periods of life as the agent gets older. The first term increases with age; the second term is declines with age.

Next we ask under what conditions do the two effects generate a hump-shaped pattern for homeownership by looking at the second derivative:

$$\frac{d^2}{ds^2} D_s^l = (1 - \beta^{S-s-1}) g''(s) + 2 \log \beta \cdot \beta^{S-s-1} g'(s) - (\log \beta)^2 \cdot \beta^{S-s-1} g(s)$$

A sufficient condition for this expression to be negative is that $g''(s)$ be negative.

Assume that the shock-insulation parameter can be written as $\rho_s^l = k \frac{s-1}{S-1}$, with $k \in [0, 1]$ (implying $\rho_1^l = 0$ and ρ_s^l linearly increasing with age). If $\beta = 0.95$, $S = 60$, and $k = 1$, the hedging demand over the life-cycle is plotted in Figure 4.



This result offers another explanation – complementary to credit constraints – for why home-ownership rates should be lower for younger people. Younger households do not need much insurance against rent shocks because their earnings provides such insurance. As they get older, earnings provide less insurance, and their hedging demand for home ownership increases. Against this force is the fact that as an agent gets older, there are fewer remaining periods of life, reducing the demand for insurance; this last point is made by Sinai and Souleles (2005) who provide evidence of its empirical relevance.

5.3 Talent allocation

Does our market equilibrium have the potential to attain productive inefficiency?

Let us begin by defining and characterizing productive efficiency. The economy's total product at time t is

$$Y_t = \sum_{l=1}^L \int_{(\varepsilon, \rho): \bar{u}^l(\varepsilon, \rho) - \bar{r}^l = \max_m (\bar{u}^m(\varepsilon, \rho) - \bar{r}^m)} y_{t,t+s}^l(\varepsilon^l, \rho^l) \phi(\varepsilon, \rho) d(\varepsilon, \rho).$$

Suppose a planner wishes to maximize the expected discounted sum of future total products:

$$Y = \sum_{s=0}^{\infty} \beta^s E[Y_{t+s}].$$

We begin by characterizing the solution of the production maximization problem:

Proposition 6 *The allocation of agents to cities that maximizes Y depends only on ε , not on ρ . An agent with ε locates in city l if $\varepsilon^l - \bar{\varepsilon}^l = \max_m \varepsilon^m - \bar{\varepsilon}^m$, where $\bar{\varepsilon}$ is the unique vector that guarantees that the mass of agents in every city equals the housing supply.*

We can show that productive efficiency is typically not achieved, except in very special circumstances:

Proposition 7 *Exactly one of the following statements is true:*

- (1) *For all cities, $\bar{p}^l = 0$.*
- (2) *The linear stationary equilibrium does not maximize Y .*

Proposition 7 says that productive efficiency is reached if and only if the expected return on real estate is zero in every city. In that case, insurance against rent risk is available at a cost of zero instead of a negative cost if the return is positive. Agents base their location decisions exclusively on ε , and output is maximized.

Expected returns on real estate are zero when: (1) The covariance matrix Σ is such that there is no systematic risk; (2) the local productivity shocks are uncorrelated and the number of cities goes to infinity (there is still systematic risk coming from stocks). Outside these restrictive conditions, the distribution of ρ matters for location choices, and the equilibrium allocation does not maximize expected product.

Of course, productive inefficiency does not imply overall inefficiency. Our market equilibrium may be constrained-efficient, given the insurance options available in the model. Full insurance is offered only if local labor shocks – and hence local house prices – are uncorrelated with systematic risk. Beyond that special case, local real estate prices carry systematic risk, and location choices are affected by the desire of agents to cash in on risk premiums.²¹

To reinforce the point of Proposition 7, we fully solve an example in closed-form. For ease of exposition, we let $S = 2$, and restrict the stock market to a single stock. We assume agents enjoy a constant insulation parameter ρ over life. Each cohort is equally divided into two types of agent: Type 0 agents have no insulation ($\rho = 0$), and type 1 agents have full insulation, $\rho = 1$.²² The distribution of agent-city match parameter is independent of agent type, ε , and uniform over the unit interval.

²¹Proving welfare theorems in our case is difficult because the allocation space includes a discrete variable, the allocation of agents to cities.

²²This example is not, strictly speaking, included in our model because it violates the assumption that the distribution of types is continuous and has full support. It demonstrates that the full-support assumption is sufficient but not necessary for equilibrium existence.

An agent (ε, ρ) locates in the city if and only if:

$$\frac{1}{\beta^2} \sum_{s=0}^1 \beta^s (\varepsilon - \bar{r} + (1 - \beta) (1 - \beta^{1-s}) \rho \bar{p}) \geq 0.$$

The marginal city dwellers of type 0, $\hat{\varepsilon}^0$, and type 1, $\hat{\varepsilon}^1$, satisfy:

$$\begin{cases} \hat{\varepsilon}^0 = \bar{r} \\ \hat{\varepsilon}^1 = \bar{r} - \frac{(1-\beta)^2 \bar{p}}{1+\beta}. \end{cases}$$

The market clearing condition on the spatial market is $\frac{(1-\hat{\varepsilon}^1+1-\hat{\varepsilon}^0)}{2} = n$, which yields a solution for the rent premium as a function of the housing price discount:

$$\bar{r} = 1 - n + \frac{(1 - \beta)^2}{2(1 + \beta)} \bar{p}.$$

The asset market clearing conditions are

$$\begin{aligned} \begin{bmatrix} 2n \\ 2z \end{bmatrix} &= \left(1 + \frac{1}{\beta}\right) \frac{(1 - \beta)^3}{2\gamma} \Sigma^{-1} \begin{bmatrix} \bar{p} \\ \bar{q} \end{bmatrix} + \begin{bmatrix} (1 - \beta) \left(n - \frac{1}{2} + \frac{\bar{r}}{2}\right) \\ 0 \end{bmatrix}, \\ \left(1 + \frac{1}{\beta}\right) \frac{(1 - \beta)^3}{2\gamma} \Sigma^{-1} \begin{bmatrix} \bar{p} \\ \bar{q} \end{bmatrix} &= \begin{bmatrix} 2n \\ 2z \end{bmatrix} - \begin{bmatrix} (1 - \beta) \left(n - \frac{1}{2} + \frac{\bar{r}}{2}\right) \\ 0 \end{bmatrix}, \\ \begin{bmatrix} \bar{p} \\ \bar{q} \end{bmatrix} &= \frac{2\gamma}{\left(1 + \frac{1}{\beta}\right) (1 - \beta)^3} \Sigma \begin{bmatrix} 2n - (1 - \beta) \left(n - \frac{1}{2} + \frac{\bar{r}}{2}\right) \\ 2z \end{bmatrix}. \end{aligned}$$

Let $\Sigma = \begin{bmatrix} \sigma_h^2 & \sigma_{hs} \\ \sigma_{hs} & \sigma_s^2 \end{bmatrix}$. We now have

$$\bar{p} = \frac{2\gamma}{\left(1 + \frac{1}{\beta}\right) (1 - \beta)^3} \left(\left(2n - (1 - \beta) \left(n - \frac{1}{2} + \frac{\bar{r}}{2} \right) \right) \sigma_h^2 + 2z\sigma_{hs} \right)$$

which yields

$$\bar{p} = \frac{4\gamma\beta(1 + \beta)}{(1 - \beta)^3 \left(2(1 + \beta)^2 + \gamma\beta \right)} \frac{\left(\left(2 - \frac{(1-\beta)}{2} \right) n\sigma_h^2 + 2z\sigma_{hs} \right)}{\sigma_h^2}$$

The last equation provides a solution for \bar{p} as a function of parameters only. It is then easy to obtain \bar{r} and \bar{q} from the equations above. In particular,

$$\bar{q} = \frac{2\gamma}{\left(1 + \frac{1}{\beta}\right) (1 - \beta)^3} \left(\left(2n - (1 - \beta) \left(n - \frac{1}{2} + \frac{\bar{r}}{2} \right) \right) \sigma_{hs} + 2z\sigma_s^2 \right).$$

We therefore obtain a full characterization of the equilibrium.

Assume numerical values $\beta = n = z = -\sigma_{hs} = 0.5$ and $\sigma_h = \sigma_s = \gamma = 1$; the equilibrium solution is $\bar{r} = \hat{\varepsilon}^0 = 0.65$; $\hat{\varepsilon}^1 = 0.35$; $\bar{p} = 1.8$; and $\bar{q} = 3.1$. Maximizing output would have required $\hat{\varepsilon}^0 = \hat{\varepsilon}^1 = 0.5$; i.e., not enough type 0 agents locate in the city (agents with $\varepsilon^0 \in [0.5, 0.65]$ are in the countryside instead of the city) and too many type 1 agents locate in the city (agents with $\varepsilon^1 \in [0.35, 0.5]$ are in the city instead of the countryside).

5.4 Housing and stock indices

As in the CAPM, one can price any portfolio with respect to the market. In this model, the relevant market is defined by the adjusted market portfolio M , discussed in Corollary 3.

We can price a housing-only index with weights $\frac{n-R}{\mathbf{1} \cdot [n-R]}$ (called H) and a stock-only index with weights $\frac{z}{\mathbf{1} \cdot z}$ (called S). We have:

$$\begin{aligned}\bar{p}^H &= \frac{Cov(H, M)}{Var(M)} \bar{p}^M, \\ \bar{p}^S &= \frac{Cov(S, M)}{Var(M)} \bar{p}^M.\end{aligned}$$

Note that H can be interpreted as an index tracking the market portfolio of REITs. It is the housing demand vector that is the same for all agents. It includes all houses that are not owned by local residents for hedging purposes. The result in Corollary 8 is immediate (by putting together the two return expressions above):

Corollary 8 *The relative returns of the housing index and the stock index are given by*

$$\bar{p}^H = \frac{Cov(H, M)}{Cov(S, M)} \bar{p}^S.$$

The corollary implies that, ceteris paribus, the difference between real estate returns and stock returns is related to home-ownership rates. The higher the percentage of residential property owned by local residents, the lower the returns on real estate.

Our model can also be used for predictions on stock returns. Often, the return of a stock is computed according to a CAPM formula that takes into account stocks only. Namely, the return of stock k is assumed to be

$$\tilde{q}^k = \frac{Cov(k, S)}{Var(S)} \bar{p}^S.$$

In our setting, this expression is of course incorrect, because it does not take into account the presence of housing. The correct expression is $\bar{q}^k = \frac{Cov(k, M)}{Var(M)} \bar{p}^M$. The ratio between the wrong expression and the correct one is

$$\frac{\tilde{q}^k}{\bar{q}^k} = \frac{Cov(k, S)}{Cov(k, M)} \frac{Var(M)}{Var(S)} \frac{\bar{p}^S}{\bar{p}^M}.$$

5.5 Economies of agglomeration

So far we have assumed no production externalities (or amenity externalities, if one embraces the amenity interpretation of our model). The model can be easily extended to incorporate externalities. Most results still hold, except possibly uniqueness.

Assume that the income of an agent if he or she locates in l is now given by

$$y_{t,t+s}^l(\varepsilon^l, \rho^l) = y_{t-1}^l + \varepsilon^l(\mathbf{E}^l) + \sum_{m=0}^s (1 - \rho_m^l) \tau_{t+m}^l,$$

where \mathbf{E}^l is the collection of ε^l of other agents living in city l .

It is easy to see that Propositions 1 and 2 hold as stated. Proposition 4 can be restated as follows. For every $(\varepsilon, \rho, \mathbf{E}^l)$, let

$$\bar{u}^l(\varepsilon, \rho, \mathbf{E}^l) = \varepsilon^l(\mathbf{E}^l) + \frac{(1 - \beta)^2}{1 - \beta^S} \bar{p}^l \sum_{s=1}^S (1 - \beta^{S-s-1}) \rho_s^l.$$

As before, an agent locates in city l if and only if $U^l = \max_m U^m$.

An allocation of agents to cities is described by $\mathbf{E} = (\mathbf{E}^1, \dots, \mathbf{E}^L)$. Hold \mathbf{E} constant. For every L -vector \hat{r} , the aggregate demand for location l is

$$\nu^l(\hat{r}, \mathbf{E}) = \int_{(\varepsilon, \rho): \bar{u}^l(\varepsilon, \rho, \mathbf{E}^l) - \hat{r}^l = \max_m (\bar{u}^m(\varepsilon, \rho, \mathbf{E}^m) - \hat{r}^m)} \phi(\varepsilon, \rho) d(\varepsilon, \rho).$$

Proposition 9 *An allocation \mathbf{E} is part of a linear stationary equilibrium if and only if:*
(1) *for all (ε, ρ) , an agent with personal characteristics (ε, ρ) locates in city l if and only if*

$$\bar{u}^l(\varepsilon, \rho, \mathbf{E}^l) - \hat{r}^l = \max_m (\bar{u}^m(\varepsilon, \rho, \mathbf{E}^m) - \hat{r}^m)$$

and (2) $\bar{\mathbf{r}}$ is the unique value of the vector $\hat{\mathbf{r}}$ such that $\nu^l(\hat{\mathbf{r}}, \mathbf{E}) = n^l$ in all cities.

Thus, the equilibrium characterization part of Proposition 4 is still valid. Existence of an equilibrium will depend on the properties of the functions $\varepsilon^l(\cdot)$. Moreover, as is well known, agglomeration economies tend to lead to multiple equilibria.

5.6 Ownership only

In our frictionless model, there are no intrinsic advantages to owning or renting, and households are free to own any divisible amount of their home. Consider instead the extreme case where renting is impossible. An agent can move to city l only if he or she buys one house there. In this world, all houses are owned by residents, and all residents own exactly one house. Agents can still invest in stocks.

Note that the covariance matrix can be written as

$$\Sigma = \begin{bmatrix} \Sigma_{\tau\tau} & \Sigma_{\tau\nu} \\ \Sigma_{\nu\tau} & \Sigma_{\nu\nu} \end{bmatrix}.$$

We first characterize the optimal portfolio allocation:

Proposition 10 *Given a vector of stock premiums \bar{q} , the optimal portfolio allocation for an agent with parameters (ε, ρ) is*

$$\begin{bmatrix} b_{t,t+s}^1 \\ \vdots \\ b_{t,t+s}^K \end{bmatrix} = \Sigma_{\nu\nu}^{-1} \left(\frac{1}{H} \begin{bmatrix} \bar{q}^1 \\ \vdots \\ \bar{q}^K \end{bmatrix} - \begin{bmatrix} \text{cov}(\nu^1, \tau^l) \\ \vdots \\ \text{cov}(\nu^k, \tau^l) \end{bmatrix} \omega_s(\rho_{s+1}^l) \right)$$

where $H = 2\gamma \frac{\beta^{s-S+2}}{(1-\beta)^3}$ and

$$\omega_s(\rho_{s+1}^l) = (1 - \rho_{s+1}^l) (1 - \beta^{S-s-1}) + \beta^{S-s-1}.$$

The expected utility of an agent with parameters (ε, ρ) if he or she locates in city l can be expressed as

$$U^l(\varepsilon, \rho) = \kappa_0 + \kappa_1 \bar{p}^l + \kappa_2 \varepsilon^l + \sum_{s=1}^S \xi_s \rho_s^l + \varsigma_s (\rho_s^l)^2,$$

where $\kappa_0, \kappa_1, \kappa_2, \xi_s$, and ς_s do not depend on (ε, ρ) or on \bar{p}^l , and $\kappa_1 > 0, \kappa_2 > 0$.

The optimal portfolio allocation is different from the allocation in the frictionless case. Agents can no longer choose their real estate investment. They resort to stocks to insure against the risk created by local productivity shocks. The extent to which stocks are helpful in providing insurance depends on the covariance matrix Σ .

The amount of stock k that a certain agent demand is determined by two components:

- A classic speculative element (the same as in Proposition 1).
- A hedging element, which is a function of $-\text{cov}(\nu^k, \tau^l) \omega(\rho_{s+1}^l)$, where $\omega_s(\rho_{s+1}^l)$ is a measure of hedging demand and $-\text{cov}(\nu^k, \tau^l)$ determines the value of stock k as a hedge for homes in city l . If dividend shocks are positively correlated with local productivity shocks, the hedging demand is negative.

Proposition 11 characterizes asset pricing in the ownership only economy:

Proposition 11 *For a given allocation of agents to cities, the excess return on stocks is given by*

$$\bar{q} = H \Sigma_{\nu\nu} \mathbf{z} + \frac{H}{S} \Sigma_{\nu\tau} \mathbf{\Omega},$$

where $\mathbf{\Omega} = [\Omega^1 \ \dots \ \Omega^L]'$ and

$$\Omega^l = \sum_{s=0}^{S-1} \int_{\hat{l}(\varepsilon, \rho)=l} \omega_s(\rho_{s+1}^l) \phi(\varepsilon, \rho) d(\varepsilon, \rho).$$

Our asset pricing characterization now refers only to stocks. As real estate investment is fully determined by location decisions, nothing can be said about house prices until location decisions are discussed. Stock prices have two components: a classic beta-pricing element, $H\Sigma_{\nu\nu}\mathbf{z}$, and an additional part that depends on their use for hedging against local productivity risk. This hedging component is proportional to $\Sigma_{\nu\tau}\mathbf{\Omega}$.

To understand the hedging component of the stock price, note that $\mathbf{\Omega}$ is a vector of aggregate hedging demands, one for every city. The total hedging demand $\mathbf{\Omega}^l$ in city l depends on the size of the city and how low the average shock-insulation parameter is for residents of that city. The price of stock k depends on how its dividend shocks covary with productivity shocks in all cities, weighted by the total hedging demand in every city.

To discuss the optimal location, let

$$\bar{u}^l(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) = \frac{1}{\kappa_1} \left(\kappa_2 \varepsilon^l + \sum_{s=1}^S \xi_s \rho_s^l + \varsigma_s (\rho_s^l)^2 \right).$$

For every L -vector $\hat{\mathbf{p}}$, we can write the aggregate demand for location l as

$$\nu^l(\hat{\mathbf{p}}) = \int_{\bar{u}^l(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) - \bar{p}^l = \max_{j=0, \dots, L} (\bar{u}^j(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) - \bar{p}^j)} \phi(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) d(\boldsymbol{\varepsilon}, \boldsymbol{\rho}).$$

Then, we have

Proposition 12 *There is a linear stationary equilibrium. An agent with characteristics $(\boldsymbol{\varepsilon}, \boldsymbol{\rho})$ locates in city l if and only if*

$$\bar{u}^l(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) + \bar{p}^l = \max_{j=0, \dots, L} (\bar{u}^j(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) + \bar{p}^j)$$

and $\bar{\mathbf{p}}$ is the unique value of the vector $\hat{\mathbf{p}}$ such that $\nu^l(\hat{\mathbf{p}}) = n^l$ in all cities.

The equilibrium price in city l is

$$p_t^l = \frac{1}{1 - \beta} y_t^j - \bar{p}^l.$$

As in the frictionless case, the equilibrium housing price is ultimately determined by the preferences of the hyper-marginal residents. As before, the expected utility of an agent who locates in city l depends only on the value of parameters for city l (i.e. ε^l and ρ_s^l , for all s).²³ As in Proposition 4, there is a unique price vector for which aggregate demand equals aggregate supply.

²³However, now the expected utility of an agent who locates in city l takes a different form (quadratic in ρ_s^l).

6 Conclusion

Choosing where to live amounts to choosing a zero-net price combination of a positive stream of dividends (income, access to local amenities) and a negative stream of dividends (housing costs). With this insight in mind, we approach the pricing of residential properties with a model that combines a standard spatial equilibrium framework with a standard portfolio choice and asset pricing framework. Housing rents are determined by market clearing in the space market; every home is occupied in all populated cities. The pricing of homes in each city and the pricing of all other assets in the economy are determined by market clearing in the asset market; all assets are held by investors.

Our model highlights significant interactions between the space market and the asset market. For one, the location choice of households depends not only on expected income minus rent (as in standard spatial equilibrium models) but also on the risk premium embedded in the price of local homes and the risk each household faces as measured by the covariance of its income with that of other city residents.

The pricing of assets depends on the location choices of the households. Who lives where determines (1) the expected rents for residential properties everywhere, (2) their volatility and covariance with other assets, and (3) the weight of residential properties from each location in the adjusted market portfolio that is relevant for the pricing of all assets in the economy.

Our results show the cost of ignoring the spatial nature of the economy when one studies the allocation of households over space and the pricing of all assets. The theory also generates new empirical questions with regard to the cross-sectional and time series variations of households' portfolios and returns to housing, and invites further refinements to standard practices in the pricing of financial assets.

This represents just a first step toward a theory of spatial asset pricing. Our goal has been to obtain a simple, tractable set-up to illustrate the links between location decisions on the one hand and investment decisions and asset prices on the other. Future research should explore, analytically or numerically, richer models of spatial asset pricing. Clearly, it would be interesting to move beyond CARA utility functions. It would also be useful to study real estate prices when moving costs are finite. Finally, it would be useful to allow for an elastic housing supply, perhaps even one that is determined endogenously through the political process (see Ortalo-Magné and Prat, 2009, for a first step in this direction).

References

- [1] Brueckner, Jan K., 1997, "Consumption and Investment Motives and the Portfolio Choices of Homeowners," *Journal of Real Estate Finance and Economics*, 16:159–180.
- [2] Campbell, Sean D., Morris A. Davis, Joshua Gallin, and Robert F. Martin, 2009, "What Moves Housing Markets: A Variance Decomposition of the Rent-Price Ratio," *Journal of Urban Economics*, 66:90–102.
- [3] Cocco, João F., 2004, "Portfolio Choice in the Presence of Housing," *The Review of Financial Studies*, 18:535–567.
- [4] Davidoff, Thomas, 2006, "Labor Income, Housing Prices and Homeownership," *Journal of Urban Economics*, 59:209–235.
- [5] Davis, Morris, and Jonathan Heathcote, 2005, "Housing and the Business Cycle," *International Economic Review*, 46:751–784.
- [6] Davis, Morris, and François Ortalo-Magné, forthcoming, "Household Expenditures, Wages, Rents," *Review of Economic Dynamics*.
- [7] Davis, Steven J., and Paul Willen, 2000, "Using Financial Assets to Hedge Labor Income Risk: Estimating the Benefits," Working paper, University of Chicago, March 2000.
- [8] DeMarzo, Peter M., Ron Kaniel, and Ilan Kremer, 2004, "Diversification as a Public Good: Community Effects in Portfolio Choices," *Journal of Finance*, 54:1677–1715.
- [9] Englund, Peter, Min Hwang, and John M. Quigley, 2002, "Hedging Housing Risk," *Journal of Real Estate Finance and Economics*, 24:163–197.
- [10] Flavin, Marjorie, and Shinobu Nakagawa, 2008, "A Model of Housing in the Presence of Adjustment Costs: A Structural Interpretation of Habit Persistence," *American Economic Review*, 98:474–495.
- [11] Flavin, Marjorie, and Takashi Yamashita, 2002, "Owner-Occupied Housing and the Composition of the Household Portfolio over the Life Cycle," *American Economic Review*, 92:345–362.
- [12] Glaeser, Edward, and Joseph Gyourko, 2005, "Urban Decline and Durable Housing," *Journal of Political Economy*, 113:345–375.

- [13] Goetzmann, W. N., 1993, “The Single Family Home in the Investment Portfolio,” *Journal of Real Estate Finance and Economics*, 6, 201–222.
- [14] Goldin Claudia, and Lawrence Katz, 1998, “The Origins of Technology-Skill Complementarity,” *The Quarterly Journal of Economics*, 113:693–732.
- [15] Grimaud, Andre, and Jean-Jacques Laffont, 1989, “Existence of a Spatial Equilibrium,” *Journal of Urban Economics*, 25:213–218.
- [16] Grossman, Sanford J., and Guy Laroque, 1991, “Asset Pricing and Optimal Portfolio Choice in the Presence of Illiquid Durable Consumption Goods,” *Econometrica*, 58:25–51.
- [17] Gyourko, Joseph, and Joseph Tracy, 1991, “The Structure of Local Public Finance and the Quality of Life,” *Journal of Political Economy* 99:774–806.
- [18] Henderson, J. Vernon, and Yannis M. Ioannides, 1983, “A Model of Housing Tenure Choice,” *American Economic Review*, 73:98–111.
- [19] Hilber, Christian A.L. , 2005, “Neighborhood Externality Risk and the Homeownership Status of Properties,” *Journal of Urban Economics*, 57:213–241.
- [20] Iacoviello, Matteo, and François Ortalo-Magné, 2003, “Hedging Housing Risk in London,” *Journal of Real Estate Finance and Economics*, 27:191–209.
- [21] Kahn, Matthew E., 1995, “A Revealed Preference Approach to Ranking City Quality of Life,” *Journal of Urban Economics* 38:221–235.
- [22] Kiyotaki, Nobuhiro, Alexander Michaelides, and Kalin Nikolov, 2007, “Winners and Losers in Housing Markets,” *Journal of Money, Credit and Banking*, April 2011, Vol. 43, 2-3, pp. 255-296.
- [23] LeBlanc, David, and Christine Lagarenne, 2004, “Owner-Occupied Housing and the Composition of the Household Portfolio: The Case of France,” *The Journal of Real Estate Finance and Economics*, 29:259–275.
- [24] Lustig, Hanno, and Stijn Van Nieuwerburgh, 2005, “Housing Collateral, Consumption Insurance and Risk Premia: an Empirical Perspective,” *Journal of Finance*, 60:1167–1219.
- [25] Lustig, Hanno, and Stijn Van Nieuwerburgh, 2007, “Can Housing Collateral Explain Long-Run Swings in Asset Returns?” New York University, mimeo.

- [26] Lustig, Hanno, and Stijn Van Nieuwerburgh, 2010, “How Much Does Household Collateral Constrain Regional Risk Sharing,” *Review of Economic Dynamics*, 13:265–294.
- [27] Ortalo-Magné, François, and Andrea Prat, 2009, “The Political Economy of Housing Supply,” London School of Economics and Political Science, mimeo.
- [28] Ortalo-Magné, François, and Sven Rady, 2002, “Tenure Choice and the Riskiness of Non-Housing Consumption,” *Journal of Housing Economics*, 11:266–279.
- [29] Piazzesi, Monika, Martin Schneider, and Selale Tuzel, 2007, “Housing, Consumption, and Asset Pricing,” *Journal of Financial Economics*, 83:531–569.
- [30] Roback, Jennifer, 1982, “Wages, Rents, and the Quality of Life,” *Journal of Political Economy*, 90:1257–1278.
- [31] Rosen, Sherwin, 1979, “Wage Based Indices of Urban Quality of Life,” in *Current Issues in Urban Economics*, edited by P. Mieszkowski and M. Straszheim, Johns Hopkins University Press, Baltimore, MD.
- [32] Sinai, Todd, and Nicholas Souleles, 2005, “Owner-Occupied Housing as a Hedge Against Rent Risk,” *Quarterly Journal of Economics*, 120:763–789.
- [33] Sinai, Todd, and Nicholas Souleles, 2009, “Can Owning a Home Hedge the Risk of Moving?” University of Pennsylvania, mimeo.
- [34] Stockman, Alan C., and Harris Dellas, 1989, “International Portfolio Diversification and Exchange Rate Variability,” *Journal of International Economics*, 26:271-290.
- [35] Van Nieuwerburgh, Stijn, and Pierre-Olivier Weill, 2006, “Why Has House Price Dispersion Gone Up?” *Review of Economic Studies*, October 2010, Vol. 77 (4), pp. 1567-1606.
- [36] Yao, Rui, and Harold H. Zhang, 2005, “Optimal Consumption and Portfolio Choices with Risky Housing and Borrowing Constraints,” *The Review of Financial Studies*, 18:197–239.

Appendix

Proof of Proposition 1

The cash flow at period $t + s$ for agent born at t , living in city l is

$$\begin{aligned} v_{t,t+s} &= y_{t,t+s}^l - r_{t+s}^l \\ &\quad - \sum_j \left((p_{t+s}^j - r_{t+s}^j) a_{t,t+s}^j - p_{t+s}^j a_{t,t+s-1}^j \right) \\ &\quad - \sum_k \left((q_{t+s}^k - d_{t+s}^k) b_{t,t+s}^k - q_{t+s}^k b_{t,t+s-1}^k \right) \end{aligned}$$

for $s = 0, \dots, S - 1$ and

$$v_{t,t+S} = \sum_j p_{t+S}^j a_{t,t+S-1}^j + \sum_k q_{t+S}^k b_{t,t+S-1}^k.$$

The end-of-life wealth of an agent born in t (evaluated at the beginning of his or her life) is:

$$w_t = \frac{1}{\beta^S} \sum_{s=0}^S \beta^s v_{t,t+s}$$

Plug in the income process and the linear prices:

$$\begin{aligned} w_t &= \frac{1}{\beta^S} \sum_{s=0}^{S-1} \beta^s \left(y_{t-1}^l + \varepsilon^l + \sum_{m=0}^s (1 - \rho_m^l) \tau_{t+m}^l - y_{t-1}^l - \sum_{m=0}^s \tau_{t+m}^l - \bar{r}^l \right) \\ &\quad + \frac{1}{\beta^S} \sum_j \sum_{s=0}^{S-1} \beta^s a_{t,t+s}^j \left(\frac{\beta}{1-\beta} \tau_{t+s+1}^j + (1-\beta) \bar{p}^j \right) \\ &\quad + \frac{1}{\beta^S} \sum_k \sum_{s=0}^{S-1} \beta^s b_{t,t+s}^k \left(\frac{\beta}{1-\beta} \nu_{t+s+1}^k + (1-\beta) \bar{q}^k \right) \\ &= \frac{1}{\beta^S} \sum_{s=0}^{S-1} \beta^s (\varepsilon^l - \bar{r}^l) - \frac{1}{\beta^S} \sum_{s=0}^{S-1} \beta^s \sum_{m=0}^s \rho_m^l \tau_{t+m}^l \\ &\quad + \frac{1}{\beta^S} \sum_j \sum_{s=0}^{S-1} \beta^s a_{t,t+s}^j \left(\frac{\beta}{1-\beta} \tau_{t+s+1}^j + (1-\beta) \bar{p}^j \right) \\ &\quad + \frac{1}{\beta^S} \sum_k \sum_{s=0}^{S-1} \beta^s b_{t,t+s}^k \left(\frac{\beta}{1-\beta} \nu_{t+s+1}^k + (1-\beta) \bar{q}^k \right) \end{aligned}$$

because $\rho_0^l = 0$. Note that

$$\begin{aligned}
\sum_{s=0}^{S-1} \beta^s \sum_{m=0}^s \rho_m^l \tau_{t+m}^l &= \sum_{s=0}^{S-1} \rho_s^l \tau_{t+s}^l \sum_{m=s}^{S-1} \beta^m = \sum_{s=0}^{S-1} \rho_s^l \tau_{t+s}^l \beta^s \sum_{m=0}^{S-1-s} \beta^m \\
&= \sum_{s=0}^{S-1} \rho_s^l \tau_{t+s}^l \frac{\beta^s (1 - \beta^{S-s})}{1 - \beta} \\
&= \sum_{s=1}^{S-1} \rho_s^l \tau_{t+s}^l \frac{\beta^s (1 - \beta^{S-s})}{1 - \beta} \text{ because } \rho_0 = 0 \\
&= \sum_{s=0}^{S-2} \beta^s (1 - \beta^{S-s-1}) \rho_{s+1}^l \frac{\beta}{1 - \beta} \tau_{t+s+1}^l \\
&= \sum_{s=0}^{S-1} \beta^s (1 - \beta^{S-s-1}) \rho_{s+1}^l \frac{\beta}{1 - \beta} \tau_{t+s+1}^l \text{ because } \rho_S \equiv 0
\end{aligned}$$

Then,

$$\begin{aligned}
w_t &= \sum_{s=0}^{S-1} \beta^{s-S} \left((\varepsilon^l - \bar{r}^l) - (1 - \beta^{S-s-1}) \rho_{s+1}^l \frac{\beta}{1 - \beta} \tau_{t+s+1}^l \right) \\
&\quad + \sum_{s=0}^{S-1} \beta^{s-S} \sum_j a_{t,t+s}^j \left(\frac{\beta}{1 - \beta} \tau_{t+s+1}^j + (1 - \beta) \bar{p}^j \right) + \sum_{s=0}^{S-1} \beta^{s-S} \sum_k b_{t,t+s}^k \left(\frac{\beta}{1 - \beta} \nu_{t+s+1}^k + (1 - \beta) \bar{q}^k \right)
\end{aligned}$$

Hence,

$$\begin{aligned}
E[w_t] &= \sum_{s=0}^{S-1} \beta^{s-S} \left(\varepsilon^l - \bar{r}^l + (1 - \beta) \left((1 - \beta^{S-s-1}) \rho_{s+1}^l \bar{p}^l + \sum_{j=1}^L \tilde{a}_{t,t+s}^j \bar{p}^j + \sum_{k=1}^K b_{t,t+s}^k \bar{q}^k \right) \right) \\
Var[w_t] &= \frac{\beta^2}{(1 - \beta)^2} \sum_{s=0}^{S-1} \beta^{2(s-S)} Var \left[\sum_{j=1}^L \tilde{a}_{t,t+s}^j \tau_{t+s+1}^j + \sum_{k=1}^K b_{t,t+s}^k \nu_{t+s+1}^k \right]
\end{aligned}$$

where $\tilde{a}_{t,t+s}^l = a_{t,t+s}^l - (1 - \beta^{S-s-1}) \rho_{s+1}^l$ and $\tilde{a}_{t,t+s}^j = a_{t,t+s}^j$ for all $j \neq l$.

In a matrix form, this is rewritten as

$$\begin{aligned}
E[w_t] &= \sum_{s=0}^{S-1} \beta^{s-S} \left(\varepsilon^l - \bar{r}^l + (1 - \beta) \left((1 - \beta^{S-s-1}) \rho_{s+1}^l \bar{p}^l + \begin{bmatrix} \tilde{\mathbf{a}}_{t,t+s} \\ \mathbf{b}_{t,t+s} \end{bmatrix}' \begin{bmatrix} \bar{\mathbf{p}} \\ \bar{\mathbf{q}} \end{bmatrix} \right) \right) \\
Var[w_t] &= \frac{\beta^2}{(1 - \beta)^2} \sum_{s=0}^{S-1} \beta^{2(s-S)} \begin{bmatrix} \tilde{\mathbf{a}}_{t,t+s} \\ \mathbf{b}_{t,t+s} \end{bmatrix}' \Sigma \begin{bmatrix} \tilde{\mathbf{a}}_{t,t+s} \\ \mathbf{b}_{t,t+s} \end{bmatrix}
\end{aligned}$$

The first-order conditions yield

$$\begin{bmatrix} \tilde{\mathbf{a}}_{t,t+s} \\ \mathbf{b}_{t,t+s} \end{bmatrix} = \frac{(1 - \beta)^3}{2\gamma\beta^{s+2}} \beta^S \Sigma^{-1} \begin{bmatrix} \bar{\mathbf{p}} \\ \bar{\mathbf{q}} \end{bmatrix},$$

Plug back into the utility function:

$$\begin{aligned}
U &= \frac{1}{\beta^S} \sum_{s=0}^{S-1} \beta^s (\varepsilon^l - \bar{r}^l) + \frac{1}{\beta^S} \sum_{s=0}^{S-1} \beta^s (1 - \beta^{S-s-1}) \rho_{s+1}^l (1 - \beta) \bar{p}^l \\
&\quad + \frac{1}{\beta^S} \sum_{s=0}^{S-1} \beta^s (1 - \beta) \left(\frac{(1 - \beta)^3}{2\gamma\beta^{s+2}} \beta^S \left[\begin{array}{c} \bar{p} \\ \bar{q} \end{array} \right]' \Sigma^{-1} \left[\begin{array}{c} \bar{p} \\ \bar{q} \end{array} \right] \right) \\
&\quad - \gamma \frac{1}{\beta^{2S}} \frac{\beta^2}{(1 - \beta)^2} \sum_{s=0}^{S-1} \beta^{2s} \frac{(1 - \beta)^3}{2\gamma\beta^{s+2}} \frac{(1 - \beta)^3}{2\gamma\beta^{s+2}} \beta^{2S} \left[\begin{array}{c} \bar{p} \\ \bar{q} \end{array} \right]' \Sigma^{-1} \Sigma \Sigma^{-1} \left[\begin{array}{c} \bar{p} \\ \bar{q} \end{array} \right] \\
&= \frac{1}{\beta^S} \sum_{s=0}^{S-1} \beta^s (\varepsilon^l - \bar{r}^l + (1 - \beta) (1 - \beta^{S-s-1}) \rho_{s+1}^l \bar{p}^l) + S \frac{(1 - \beta)^4}{4\gamma\beta^2} \left[\begin{array}{c} \bar{p} \\ \bar{q} \end{array} \right]' \Sigma^{-1} \left[\begin{array}{c} \bar{p} \\ \bar{q} \end{array} \right]
\end{aligned}$$

■

Proof of Proposition 2

The demands for assets beyond a hedging motive can be written as

$$\left[\begin{array}{c} \tilde{\mathbf{a}}_{t-s,t} \\ \mathbf{b}_{t-s,t} \end{array} \right] = \frac{(1 - \beta)^3}{2\gamma\beta^{s+2}} \beta^S \Sigma^{-1} \left[\begin{array}{c} \bar{p} \\ \bar{q} \end{array} \right]$$

for $s = 0, \dots, S - 1$. Since all agents have the same portfolio and there is a measure one of agents in each cohort, the aggregate portfolio demand for assets (excluding the hedging motive), is

$$\sum_{s=0}^{S-1} \left[\begin{array}{c} \tilde{\mathbf{a}}_{t-s,t} \\ \mathbf{b}_{t-s,t} \end{array} \right] = \left(1 + \frac{1}{\beta} + \frac{1}{\beta^2} + \dots + \frac{1}{\beta^{S-1}} \right) \left[\begin{array}{c} \tilde{\mathbf{a}}_{t,t} \\ \mathbf{b}_{t,t} \end{array} \right] = \frac{1 - \beta^S}{(1 - \beta) \beta^{S-1}} \left[\begin{array}{c} \tilde{\mathbf{a}}_{t,t} \\ \mathbf{b}_{t,t} \end{array} \right].$$

The housing demand in city l by people with age s due to the hedging motive is

$$(1 - \beta^{S-s-1}) \int_{\varepsilon} \int_{\rho} I_{\varepsilon, \rho}^l \rho_{s+1}^l \phi(\varepsilon, \rho) d\varepsilon d\rho.$$

It is then easy to see that the total housing demand in city l due to the hedging motive is SR^l , where R^l is defined as in the proposition.

The supply of houses minus the hedging demand in every city is $S(\mathbf{n} - \mathbf{R})$. The housing market clearing condition is therefore

$$\frac{1 - \beta^S}{(1 - \beta) \beta^{S-1}} \tilde{\mathbf{a}}_{t,t} = S(\mathbf{n} - \mathbf{R}).$$

Hence

$$\tilde{\mathbf{a}}_{t,t} = S \frac{(1 - \beta) \beta^{S-1}}{(1 - \beta^S)} (\mathbf{n} - \mathbf{R})$$

and by analogy

$$\mathbf{b}_{t,t} = S \frac{(1 - \beta) \beta^{S-1}}{(1 - \beta^S)} \mathbf{z}.$$

Plugging in the demand function yields a solution to the housing and stock risk premiums:

$$\begin{aligned}
S \frac{(1 - \beta) \beta^{S-1}}{(1 - \beta^S)} \left[\begin{array}{c} (\mathbf{n} - \mathbf{R}) \\ \mathbf{z} \end{array} \right] &= \frac{(1 - \beta)^3}{2\gamma\beta^2} \beta^S \Sigma^{-1} \left[\begin{array}{c} \bar{p} \\ \bar{q} \end{array} \right] \\
2\gamma S \frac{\beta}{(1 - \beta)^2 (1 - \beta^S)} \Sigma \left[\begin{array}{c} \mathbf{n} - \mathbf{R} \\ \mathbf{z} \end{array} \right] &= \left[\begin{array}{c} \bar{p} \\ \bar{q} \end{array} \right].
\end{aligned}$$

■

Proof of Corollary 3

Note that

$$\begin{aligned} Cov(l, M) &= \frac{1}{Q} \left(\sum_{m=1}^L (n^m - R^m) Cov(\tau^l, \tau^m) + \sum_{k=1}^K z^k Cov(\tau^l, \nu^k) \right) \\ Var(M) &= \frac{1}{Q^2} \begin{bmatrix} \mathbf{n} - \mathbf{R} \\ \mathbf{z} \end{bmatrix}' \Sigma \begin{bmatrix} \mathbf{n} - \mathbf{R} \\ \mathbf{z} \end{bmatrix} \end{aligned}$$

The expected return of a zero-cost market portfolio containing one unit of M is given by

$$\begin{aligned} \bar{p}^M &= \frac{\sum_{l=1}^L (n^l - R^l) \bar{p}^l + \sum_{k=1}^K z^k \bar{q}^k}{\sum_{l=1}^L (n^l - R^l) + \sum_{k=1}^K z^k} = \frac{1}{Q} \begin{bmatrix} \mathbf{n} - \mathbf{R} \\ \mathbf{z} \end{bmatrix}' \begin{bmatrix} \bar{\mathbf{p}} \\ \bar{\mathbf{q}} \end{bmatrix} \\ &= 2\gamma \frac{\beta}{(1-\beta)^2 (1-\beta^S)} S \frac{1}{Q} \begin{bmatrix} \mathbf{n} - \mathbf{R} \\ \mathbf{z} \end{bmatrix}' \Sigma \begin{bmatrix} \mathbf{n} - \mathbf{R} \\ \mathbf{z} \end{bmatrix} = 2\gamma \frac{\beta}{(1-\beta)^2 (1-\beta^S)} SQVar(M). \end{aligned}$$

Similarly

$$\begin{aligned} \bar{p}^l &= 2\gamma \frac{\beta}{(1-\beta)^2 (1-\beta^S)} S \left(\sum_{m=1}^L (n^m - R^m) Cov(\tau^l, \tau^m) + \sum_{k=1}^K z^k Cov(\tau^l, \nu^k) \right) \\ &= 2\gamma \frac{\beta}{(1-\beta)^2 (1-\beta^S)} SQCov(l, M) \end{aligned}$$

Hence, we can write

$$\frac{\bar{p}^l}{\bar{p}^M} = \frac{Cov(l, M)}{Var(M)}.$$

The proof for k follows similar lines and is omitted. ■

Proof of Proposition 4

It is immediate to see that a solution to $\boldsymbol{\nu}(\hat{\mathbf{r}}) = \mathbf{n}$ constitutes a linear stationary equilibrium; no agent wants to change location, by definition $r_t^l = y_t^j + \bar{r}^l$, and the conditions for Propositions 1 and 2 are satisfied.

To prove existence, note that $\nu^l(\hat{\mathbf{r}})$ is continuous in $\hat{\mathbf{r}}$ and that $\lim_{\hat{r}^l \rightarrow -\infty} \nu^l(\hat{\mathbf{r}}) = 1$ and $\lim_{\hat{r}^l \rightarrow \infty} \nu^l(\hat{\mathbf{r}}) = 0$.

To prove that prices and rents are uniquely determined *given an allocation of agents to cities*, suppose that the system $\boldsymbol{\nu}(\hat{\mathbf{r}}) = \mathbf{n}$ has two distinct solutions $\bar{\mathbf{r}}$ and $\bar{\mathbf{r}}'$. Assume without loss of generality that there are a non-empty set of cities \tilde{L} for which $(\bar{r}')^l < \bar{r}^l$. The set of agents who locate in a city in \tilde{L} is given by

$$\left\{ (\boldsymbol{\varepsilon}, \boldsymbol{\rho}) : \max_{l \in \tilde{L}} (\bar{u}^l(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) - \bar{r}^l) \geq \max_{j \notin \tilde{L}} (\bar{u}^j(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) - \bar{r}^j) \right\}$$

Note, however, that this set must become strictly larger when $\bar{\mathbf{r}}$ is replaced by $\bar{\mathbf{r}}'$, because all elements $\bar{u}^l(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) - \bar{r}^l$ on one side become strictly larger and all elements $\bar{u}^j(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) - \bar{r}^j$ on the other side do not become larger. Hence, more agents will want to locate in cities in \tilde{L} , but this is impossible as the mass of agents who locate in \tilde{L} must sum up to $\sum_{l \in \tilde{L}} n^l$ in both solutions. ■

Proof of Proposition 5

Note that

$$\bar{u}^l = \varepsilon^l + \frac{1-\beta}{1-\beta^S} \left(1 - (S+1)\beta^S + S\beta^{S+1}\right) \bar{p}^l \bar{\rho}_{s+1}^l.$$

As $\bar{\rho}_{s+1}^l$ are the same for all agents and the $\tilde{\varepsilon}$ are uniformly distributed, we write

$$\nu^l(\hat{r}) = \int_{\tilde{\varepsilon}: \tilde{\varepsilon}^l - \hat{r}^l = \max_m (\tilde{\varepsilon}^m - \hat{r}^m)} d\tilde{\varepsilon}$$

This problem is symmetric in l . Hence, the unique solution to $\nu^l(\hat{r}) = \frac{1}{L}N$ for $l = 1, \dots, L$ must be symmetric in l , namely $\bar{r}^l = \bar{r}$. This means that in every period, the mass of young agent who locates in the countryside is $\nu^0(\hat{r}) = \bar{r}^L$. This implies $\bar{r} = (1-N)^{\frac{1}{L}}$. According to equation (4), the equilibrium rent is given by

$$r^l = \bar{r} + \frac{(1-\beta)^2}{1-\beta^S} \bar{p}^l \sum_{s=1}^S \left(1 - \beta^{S-s-1}\right) \bar{\rho}_s^l.$$

■

Proof of Proposition 6

Consider any allocation of agents to cities. Suppose an agent with $(\varepsilon^l, \varepsilon^m)$ is allocated to city l and another agent with $((\varepsilon^l)', (\varepsilon^m)')$ is allocated to m . Swapping agents does not increase total expected production if and only if

$$\varepsilon^l - (\varepsilon^l)' \geq \varepsilon^m - (\varepsilon^m)'$$

If this holds true for every agent, one can find a unique vector $\bar{\varepsilon}$ such that the condition in the statement is satisfied. ■

Proof of Proposition 7

According to Proposition 4, in a linear stationary equilibrium agents are assigned to cities according to

$$\bar{u}^l(\varepsilon, \rho) = \varepsilon^l + \frac{(1-\beta)^2}{1-\beta^S} \bar{p}^l \sum_{s=1}^S \left(1 - \beta^{S-s-1}\right) \rho_s^l.$$

Suppose that an agent with a certain (ε, ρ) locates in city l . The next preferred city is m , and the utility difference between the two cities is given by

$$D = \bar{u}^l(\varepsilon, \rho) - \bar{u}^m(\varepsilon, \rho),$$

where D is sufficiently low. Consider another agent with (ε', ρ') that is identical to (ε, ρ) except that $(\varepsilon')^l = \varepsilon^l + \delta$ and $\sum_{s=1}^S \left(1 - \beta^{S-s-1}\right) (\rho')_s^l = \sum_{s=1}^S \left(1 - \beta^{S-s-1}\right) \rho_s^l - \alpha$. Given a positive \bar{p}^l , it is always possible to find α and δ such that $\bar{u}^l(\varepsilon', \rho') < \bar{u}^m(\varepsilon', \rho')$. By the assumption that ϕ has full support, agents with (ε, ρ) and (ε', ρ') exist. The sum of expected outputs of the two agents would be higher if the agents switch cities.

The only time this cannot be done is when \bar{p}^l is the same for all cities. In that case, it is easy to see that agents choose location in order to maximize ε^l . ■

Proof of Proposition 9

The first part is immediate. If \mathbf{E} is an allocation and prices are linear, then every agent is using $\bar{u}^l(\varepsilon, \rho, \mathbf{E}^l) - \bar{r}^l$ as a criterion to locate and rents must equate demand and supply. The argument for the uniqueness of \bar{r}^l (given \mathbf{E}) is unchanged from the proof of Proposition 4. \blacksquare

Proof of Proposition 10

Consider an agent born in period t with parameters (ε, ρ) who locates in city l . His or her wealth at the end of life is

$$w_t = \sum_{s=0}^{S-1} \beta^{s-S} \left(y_{t-1}^l + \varepsilon^l + \sum_{m=0}^s (1 - \rho_m^l) \tau_{t+m}^l + \sum_k b_{t,t+s}^k \left(\frac{\beta}{1-\beta} \nu_{t+s+1}^j + (1-\beta) \bar{q}^k \right) \right) - \frac{1}{\beta^S} p_t^l + p_{t+S}^l.$$

Conjecture $p_t^l = \frac{1}{1-\beta} y_t^l - \bar{p}^l$. This implies $p_{t+S}^l = \frac{1}{1-\beta} y_{t+S}^l - \bar{p}^l = \frac{1}{1-\beta} (y_t^l + \tau_{t+1} + \dots + \tau_{t+S-1} + \tau_{t+S}) - \bar{p}^l = p_t^l + \frac{1}{1-\beta} \left(\sum_{s=0}^{S-1} \tau_{t+s+1} \right)$. Replacing in the above equation yields

$$\begin{aligned} w_t &= \sum_{s=0}^{S-1} \beta^{s-S} \left(y_{t-1}^l + \varepsilon^l + \sum_{m=0}^s (1 - \rho_m^l) \tau_{t+m}^l + \sum_k b_{t,t+s}^k \left(\frac{\beta}{1-\beta} \nu_{t+s+1}^j + (1-\beta) \bar{q}^k \right) \right) \\ &\quad + \left(1 - \frac{1}{\beta^S} \right) p_t^l + \sum_{s=1}^S \frac{1}{1-\beta} \tau_{t+s}^l \\ w_t &= \sum_{s=0}^{S-1} \beta^{s-S} \left(y_{t-1}^l + \varepsilon^l + \sum_k b_{t,t+s}^k \left(\frac{\beta}{1-\beta} \nu_{t+s+1}^j + (1-\beta) \bar{q}^k \right) \right) + \sum_{s=0}^{S-1} \beta^{s-S} \sum_{m=0}^s (1 - \rho_m^l) \tau_{t+m}^l \\ &\quad + \left(1 - \frac{1}{\beta^S} \right) p_t^l + \sum_{s=0}^{S-1} \frac{1}{1-\beta} \tau_{t+s+1}^l. \end{aligned}$$

Note

$$\begin{aligned}
\sum_{s=0}^{S-1} \beta^{s-S} \sum_{m=0}^s (1 - \rho_m^l) \tau_{t+m}^l &= \frac{1}{\beta^S} \sum_{s=0}^{S-1} (1 - \rho_s^l) \tau_{t+s}^l \sum_{m=s}^{S-1} \beta^m \\
&= \frac{1}{\beta^S} \sum_{s=0}^{S-1} (1 - \rho_s^l) \tau_{t+s}^l \beta^s \frac{1 - \beta^{S-s}}{1 - \beta} \\
&= \sum_{s=0}^{S-1} (1 - \rho_s^l) \frac{\beta^{s-S} - 1}{1 - \beta} \tau_{t+s}^l \\
&= \sum_{s=1}^S (1 - \rho_s^l) \frac{\beta^{s-S} - 1}{1 - \beta} \tau_{t+s}^l + (1 - \rho_0^l) \frac{\beta^{0-S} - 1}{1 - \beta} \tau_t^l \\
&\quad - (1 - \rho_S^l) \frac{\beta^{S-S} - 1}{1 - \beta} \tau_{t+S}^l \\
&= \sum_{s=0}^{S-1} (1 - \rho_{s+1}^l) \frac{\beta^{s-S+1} - 1}{1 - \beta} \tau_{t+s+1}^l + (1 - \rho_0^l) \frac{\beta^{-S} - 1}{1 - \beta} \tau_t^l \\
&\quad - (1 - \rho_S^l) \frac{\beta^{S-S} - 1}{1 - \beta} \tau_{t+S}^l \\
&= \sum_{s=0}^{S-1} \beta^{s-S} \frac{\beta}{1 - \beta} (1 - \beta^{S-s-1}) (1 - \rho_{s+1}^l) \tau_{t+s+1}^l + \frac{\beta^{-S} - 1}{1 - \beta} \tau_t^l.
\end{aligned}$$

We have therefore

$$\begin{aligned}
w_t &= \sum_{s=0}^{S-1} \beta^{s-S} \left(y_{t-1}^l + \varepsilon^l + \sum_k b_{t,t+s}^k \left(\frac{\beta}{1 - \beta} \nu_{t+s+1}^j + (1 - \beta) \bar{q}^k \right) \right) \\
&\quad + \sum_{s=0}^{S-1} \beta^{s-S} \frac{\beta}{1 - \beta} (1 - \beta^{S-s-1}) (1 - \rho_{s+1}^l) \tau_{t+s+1}^l \\
&\quad + \frac{\beta^{-S} - 1}{1 - \beta} \tau_t^l + \left(1 - \frac{1}{\beta^S} \right) p_t^l + \sum_{s=0}^{S-1} \frac{1}{1 - \beta} \tau_{t+s+1}^l \\
w_t &= \sum_{s=0}^{S-1} \beta^{s-S} \left(y_{t-1}^l + \varepsilon^l + \sum_k b_{t,t+s}^k \left(\frac{\beta}{1 - \beta} \nu_{t+s+1}^j + (1 - \beta) \bar{q}^k \right) \right) \\
&\quad + \sum_{s=0}^{S-1} \beta^{s-S} \left(\frac{\beta}{1 - \beta} (1 - \beta^{S-s-1}) (1 - \rho_{s+1}^l) + \frac{\beta}{1 - \beta} \beta^{S-s-1} \right) \tau_{t+s+1}^l \\
&\quad + \frac{\beta^{-S} - 1}{1 - \beta} \tau_t^l + \left(1 - \frac{1}{\beta^S} \right) p_t^l \\
w_t &= \sum_{s=0}^{S-1} \beta^{s-S} \left(y_{t-1}^l + \varepsilon^l + \left((1 - \rho_{s+1}^l) (1 - \beta^{S-s-1}) + \beta^{S-s-1} \right) \frac{\beta}{1 - \beta} \tau_{t+s+1}^l \right) \\
&\quad + \sum_k b_{t,t+s}^k \left(\frac{\beta}{1 - \beta} \nu_{t+s+1}^j + (1 - \beta) \bar{q}^k \right) \\
&\quad + \frac{\beta^{-S} - 1}{1 - \beta} \tau_t^l + \left(1 - \frac{1}{\beta^S} \right) p_t^l.
\end{aligned}$$

Note $\frac{\beta^{-S}-1}{1-\beta}\tau_t^l + \left(1 - \frac{1}{\beta^S}\right)p_t^l = \frac{1}{\beta^S}\frac{1-\beta^S}{1-\beta}\tau_t^l - \frac{1}{\beta^S}\frac{1-\beta^S}{1-\beta}(1-\beta)p_t^l = \sum_{s=0}^{S-1}\beta^{s-S}(\tau_t^l - (1-\beta)p_t^l)$. Using $p_t^l = \frac{1}{1-\beta}(y_{t-1} + \tau_t) - \bar{p}^l$ this yields

$$\begin{aligned} \frac{\beta^{-S}-1}{1-\beta}\tau_t^l + \left(1 - \frac{1}{\beta^S}\right)p_t^l &= \sum_{s=0}^{S-1}\beta^{s-S}(\tau_t^l - (1-\beta)p_t^l) \\ &= \sum_{s=0}^{S-1}\beta^{s-S}\left(\tau_t^l - (1-\beta)\left(\frac{1}{1-\beta}(y_{t-1} + \tau_t^l) - \bar{p}^l\right)\right) \\ &= \sum_{s=0}^{S-1}\beta^{s-S}(-y_{t-1} + (1-\beta)\bar{p}^l). \end{aligned}$$

So we now have

$$\begin{aligned} w_t &= \sum_{s=0}^{S-1}\beta^{s-S}\left(y_{t-1}^l + \varepsilon^l - y_{t-1} + (1-\beta)\bar{p}^l + \left((1-\rho_{s+1}^l)(1-\beta^{S-s-1}) + \beta^{S-s-1}\right)\frac{\beta}{1-\beta}\tau_{t+s+1}^l\right. \\ &\quad \left. + \sum_k b_{t,t+s}^k\left(\frac{\beta}{1-\beta}\nu_{t+s+1}^j + (1-\beta)\bar{q}^k\right)\right) \\ w_t &= \sum_{s=0}^{S-1}\beta^{s-S}\left(\varepsilon^l + (1-\beta)\bar{p}^l + \left((1-\rho_{s+1}^l)(1-\beta^{S-s-1}) + \beta^{S-s-1}\right)\frac{\beta}{1-\beta}\tau_{t+s+1}^l\right. \\ &\quad \left. + \sum_k b_{t,t+s}^k\left(\frac{\beta}{1-\beta}\nu_{t+s+1}^j + (1-\beta)\bar{q}^k\right)\right). \end{aligned}$$

Then

$$\begin{aligned} E[w_t] &= \sum_{s=0}^{S-1}\beta^{s-S}\left(\varepsilon^l + (1-\beta)\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ b_{t,t+s}^1 \\ \vdots \\ b_{t,t+s}^K \end{bmatrix}' \begin{bmatrix} \bar{p}^1 \\ \vdots \\ \bar{p}^l \\ \vdots \\ \bar{p}^L \\ \bar{q}^1 \\ \vdots \\ \bar{q}^K \end{bmatrix}\right), \\ \text{Var}[w_t] &= \frac{\beta^2}{(1-\beta)^2}\sum_{s=0}^{S-1}\beta^{2(s-S)}\begin{bmatrix} 0 \\ \vdots \\ \omega_s(\rho_{s+1}^l) \\ \vdots \\ 0 \\ b_{t,t+s}^1 \\ \vdots \\ b_{t,t+s}^K \end{bmatrix}' \mathbf{\Sigma} \begin{bmatrix} 0 \\ \vdots \\ \omega_s(\rho_{s+1}^l) \\ \vdots \\ 0 \\ b_{t,t+s}^1 \\ \vdots \\ b_{t,t+s}^K \end{bmatrix}, \end{aligned}$$

where

$$\omega_s(\rho_{s+1}^l) = (1-\rho_{s+1}^l)(1-\beta^{S-s-1}) + \beta^{S-s-1}.$$

The first-order condition for the optimal stock investment is

$$\begin{bmatrix} \bar{q}^1 \\ \vdots \\ \bar{q}^K \end{bmatrix} = H \begin{bmatrix} \Sigma_{\nu\tau} & \Sigma_{\nu\nu} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ \omega_s (\rho_{s+1}^l) \\ \vdots \\ 0 \\ b_{t,t+s}^1 \\ \vdots \\ b_{t,t+s}^K \end{bmatrix} = H \begin{bmatrix} cov(\nu^1, \tau^l) \\ \vdots \\ cov(\nu^k, \tau^l) \end{bmatrix} \omega_s (\rho_{s+1}^l) + H \Sigma_{\nu\nu} \begin{bmatrix} b_{t,t+s}^1 \\ \vdots \\ b_{t,t+s}^K \end{bmatrix},$$

where

$$H = 2\gamma \frac{\beta^{s-S+2}}{(1-\beta)^3}.$$

The vector of individual demands for stocks is

$$\begin{bmatrix} b_{t,t+s}^1 \\ \vdots \\ b_{t,t+s}^K \end{bmatrix} = \frac{\Sigma_{\nu\nu}^{-1}}{H} \begin{bmatrix} \bar{q}^1 \\ \vdots \\ \bar{q}^K \end{bmatrix} - \Sigma_{\nu\nu}^{-1} \begin{bmatrix} cov(\nu^1, \tau^l) \\ \vdots \\ cov(\nu^k, \tau^l) \end{bmatrix} \omega_s (\rho_{s+1}^l).$$

Let

$$\mathbf{h}^l = \begin{bmatrix} cov(\nu^1, \tau^l) \\ \vdots \\ cov(\nu^k, \tau^l) \end{bmatrix}.$$

The expectation and the variance of final wealth are given respectively by

$$\begin{aligned} E[w_t] &= \sum_{s=0}^{S-1} \beta^{s-S} (\varepsilon^l + (1-\beta) (\bar{p}^l + \mathbf{b}' \bar{\mathbf{q}})) \\ &= \sum_{s=0}^{S-1} \beta^{s-S} \left(\varepsilon^l + (1-\beta) \left(\bar{p}^l + \left(\frac{\Sigma_{\nu\nu}^{-1}}{H} \bar{\mathbf{q}} - \Sigma_{\nu\nu}^{-1} \begin{bmatrix} cov(\nu^1, \tau^l) \\ \vdots \\ cov(\nu^k, \tau^l) \end{bmatrix} \omega_s \right)' \bar{\mathbf{q}} \right) \right) \\ &= \sum_{s=0}^{S-1} \beta^{s-S} \left(\varepsilon^l + (1-\beta) \left(\bar{p}^l + \bar{\mathbf{q}}' \frac{\Sigma_{\nu\nu}^{-1}}{H} \bar{\mathbf{q}} - \bar{\mathbf{q}}' \Sigma_{\nu\nu}^{-1} \mathbf{h}^l \omega_s \right) \right), \end{aligned}$$

and

$$\begin{aligned} Var[w_t] &= \frac{\beta^2}{(1-\beta)^2} \sum_{s=0}^{S-1} \beta^{2(s-S)} \left(Var(\tau^l) \omega_s^2 + \mathbf{b}' \begin{bmatrix} cov(\nu^1, \tau^l) \\ \vdots \\ cov(\nu^k, \tau^l) \end{bmatrix} \omega_s + \mathbf{b}' \Sigma_{\nu\nu} \mathbf{b} \right) \\ &= \frac{\beta^2}{(1-\beta)^2} \sum_{s=0}^{S-1} \beta^{2(s-S)} \left(Var(\tau^l) \omega_s^2 + \left(\frac{\Sigma_{\nu\nu}^{-1}}{H} \bar{\mathbf{q}} - \Sigma_{\nu\nu}^{-1} \mathbf{h}^l \omega_s \right)' \mathbf{h}^l \omega_s \right. \\ &\quad \left. + \left(\frac{\Sigma_{\nu\nu}^{-1}}{H} \bar{\mathbf{q}} - \Sigma_{\nu\nu}^{-1} \mathbf{h}^l \omega_s \right)' \Sigma_{\nu\nu} \left(\frac{\Sigma_{\nu\nu}^{-1}}{H} \bar{\mathbf{q}} - \Sigma_{\nu\nu}^{-1} \mathbf{h}^l \omega_s \right) \right) \\ &= \frac{\beta^2}{(1-\beta)^2} \sum_{s=0}^{S-1} \beta^{2(s-S)} \left(Var(\tau^l) \omega_s^2 + \frac{1}{H} (\mathbf{h}^l)' \Sigma_{\nu\nu}^{-1} \bar{\mathbf{q}} \omega_s - (\mathbf{h}^l)' \Sigma_{\nu\nu}^{-1} \mathbf{h}^l \omega_s^2 \right. \\ &\quad \left. + \frac{1}{H^2} \bar{\mathbf{q}}' \Sigma_{\nu\nu}^{-1} \bar{\mathbf{q}} - 2 \bar{\mathbf{q}}' \Sigma_{\nu\nu}^{-1} \mathbf{h}^l \omega_s + (\mathbf{h}^l)' \Sigma_{\nu\nu}^{-1} \mathbf{h}^l \omega_s^2 \right). \end{aligned}$$

Note that

$$\begin{aligned}\kappa_1 &= \sum_{s=0}^{S-1} \beta^{s-S} (1 - \beta) = \frac{1 - \beta^S}{\beta^S} > 0 \\ \kappa_2 &= \sum_{s=0}^{S-1} \beta^{s-S} = \frac{1 - \beta^S}{\beta^S (1 - \beta)} > 0\end{aligned}$$

■

Proof of Proposition 11

The market clearing condition is

$$\begin{aligned}S \begin{bmatrix} z^1 \\ \vdots \\ z^K \end{bmatrix} &= \sum_{s=0}^{S-1} \sum_{l=0}^L \int_{\hat{l}(\boldsymbol{\varepsilon}, \boldsymbol{\rho})=l} \begin{bmatrix} b_{t,t+s}^1 \\ \vdots \\ b_{t,t+s}^K \end{bmatrix} \phi(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) d(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) \\ &= \sum_{s=0}^{S-1} \sum_{l=0}^L \int_{\hat{l}(\boldsymbol{\varepsilon}, \boldsymbol{\rho})=l} \left(\frac{\Sigma_{\nu\nu}^{-1}}{H} \begin{bmatrix} \bar{q}^1 \\ \vdots \\ \bar{q}^K \end{bmatrix} - \Sigma_{\nu\nu}^{-1} \begin{bmatrix} \text{cov}(\nu^1, \tau^l) \\ \vdots \\ \text{cov}(\nu^K, \tau^l) \end{bmatrix} \omega(\rho_{s+1}^l) \right) \phi(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) d(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) \\ &= S \frac{\Sigma_{\nu\nu}^{-1}}{H} \begin{bmatrix} \bar{q}^1 \\ \vdots \\ \bar{q}^K \end{bmatrix} - \sum_{l=0}^L \Sigma_{\nu\nu}^{-1} \begin{bmatrix} \text{cov}(\nu^1, \tau^l) \\ \vdots \\ \text{cov}(\nu^K, \tau^l) \end{bmatrix} \sum_{s=0}^{S-1} \int_{\hat{l}(\boldsymbol{\varepsilon}, \boldsymbol{\rho})=l} \omega(\rho_{s+1}^l) \phi(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) d(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) \\ &= S \frac{\Sigma_{\nu\nu}^{-1}}{H} \begin{bmatrix} \bar{q}^1 \\ \vdots \\ \bar{q}^K \end{bmatrix} - \Sigma_{\nu\nu}^{-1} \sum_{l=0}^L \begin{bmatrix} \text{cov}(\nu^1, \tau^l) \\ \vdots \\ \text{cov}(\nu^K, \tau^l) \end{bmatrix} \Omega^l\end{aligned}$$

where

$$\Omega^l = \sum_{s=0}^{S-1} \int_{\hat{l}(\boldsymbol{\varepsilon}, \boldsymbol{\rho})=l} \omega(\rho_{s+1}^l) \phi(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) d(\boldsymbol{\varepsilon}, \boldsymbol{\rho}).$$

Then,

$$\begin{aligned}\begin{bmatrix} \bar{q}^1 \\ \vdots \\ \bar{q}^K \end{bmatrix} &= H \Sigma_{\nu\nu} \begin{bmatrix} z^1 \\ \vdots \\ z^K \end{bmatrix} + \frac{H}{S} \sum_{l=0}^L \begin{bmatrix} \text{cov}(\nu^1, \tau^l) \\ \vdots \\ \text{cov}(\nu^K, \tau^l) \end{bmatrix} \Omega^l \\ &= H \Sigma_{\nu\nu} \mathbf{z} + \frac{H}{S} \Sigma_{\nu\tau} \boldsymbol{\Omega},\end{aligned}$$

where

$$\boldsymbol{\Omega} = [\Omega^1 \quad \dots \quad \Omega^L]'.$$

■

Proof of Proposition 12

Given a vector of housing premiums $\bar{\mathbf{p}}$, and agent with $(\boldsymbol{\varepsilon}, \boldsymbol{\rho})$ locates in l if

$$U^l = \max_{j=0, \dots, L} U^j$$

but this is equivalent to

$$\kappa_1 \bar{p}^l + \kappa_2 \varepsilon^l + \sum_{s=1}^S \xi_s \rho_s^l + \varsigma_s (\rho_s^l)^2 = \max_{j=0, \dots, L} \left(\kappa_1 \bar{p}^j + \kappa_2 \varepsilon^j + \sum_{s=1}^S \xi_s \rho_s^j + \varsigma_s (\rho_s^j)^2 \right)$$

or

$$\frac{1}{\kappa_1} \left(\kappa_2 \varepsilon^l + \sum_{s=1}^S \xi_s \rho_s^l + \varsigma_s (\rho_s^l)^2 \right) + \bar{p}^l = \max_{j=0, \dots, L} \left(\frac{1}{\kappa_1} \left(\kappa_2 \varepsilon^j + \sum_{s=1}^S \xi_s \rho_s^j + \varsigma_s (\rho_s^j)^2 \right) + \bar{p}^j \right).$$

The rest of the proof is similar to the proof of Proposition 4, and it is omitted. Note that in our conjecture we treat \bar{r}^l and \bar{p}^l asymmetrically, one with a positive sign, the other with a negative sign. This explains the difference in signs for each of these two terms between Proposition 4 and the Proposition 12. ■