Managerial Attention and Worker Engagement*

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Abstract

We study a dynamic agency problem with two-sided moral hazard: the worker chooses whether to exert effort or shirk; the manager chooses whether to invest in an attention technology to recognize worker performance. In equilibrium the worker uses past recognition to infer managerial attention. An engagement trap arises: absent recent recognition, both worker effort and managerial investment decrease, making a return to high productivity less likely as time passes. In a sample of ex-ante identical firms, firm performance, managerial quality, and worker engagement display heterogeneity across firms, positive correlation, and persistence over time.

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Performance differences across firms are sizable and persistent. A growing number of studies point to managerial practices as a main driver.[1] Bloom and Van Reenen (2007) find that higher-quality management practices are associated with higher productivity, profitability, and survival rates. Different components of management are stressed in Hansen and Wernerfelt (1989), Ichniowski, Shaw, and Prennushi (1997), Bloom and Van Reenen (2010), Kaplan, Klebanov, and Sorensen (2012), and Bloom et al. (2013); see Gibbons and Henderson (2013) for a survey. A managerial practice can be seen as a form of technology (Bloom, Sadun, and Van Reenen, 2012), and as such as an intangible asset that is subject to depreciation and in which the firm can invest.[2] The value of this asset however is difficult to observe directly, as is evident from the ongoing efforts of the economics profession to measure management quality. A large part of managerial practices relates to human resource management, and in particular to the firm's ability to define, identify, document, and reward worker performance.

At the same time, psychologists find that worker engagement is important for firm productivity. Employee engagement is positively related to individual performance (Warr, 1999; Judge et al., 2001); moreover, at the organizational and unit-business level, there is a significant link between aggregate measures of engagement and outcomes such as employee turnover, customer satisfaction, accidents, productivity, and profits (Ostroff, 1992; Ryan, Schmit, and Johnson, 1996; Harter, Hayes, and Schmidt, 2002).[3] Notably, some of the items comprising engagement measures concern workers' perceptions of human resource management practices. For example, whether workers believe that they are given "recognition or praise for doing good work" affects their engagement (Harter, Hayes, and Schmidt, 2002, p. 269).

This paper studies firms' ability to raise productivity by using better human resource management practices and increasing worker engagement. What drives investment in the managerial practice — a costly, intangible, and imperfectly observable asset? How is worker engagement, and hence effort and productivity,

[1]See Syverson (2011) for a discussion of the different determinants of firm productivity.

[2]Investment in the management technology is not limited to financial resources. Corporate leaders may need to devote more time to internal management (Bandiera et al., 2011).

[3]Recent work suggests that this link is indeed a causal one from worker engagement to firm performance; see for example Bockerman and Ilmakunnas (2012).

affected by the management technology? We provide a model where these two variables are interlinked and explore the dynamic interaction between them. At the center of our model is a two-sided moral hazard problem: the worker chooses whether to exert effort or shirk; the manager chooses whether to invest in an *attention technology* that provides information on worker performance. The worker's engagement depends on his belief about the manager's attention. Our analysis offers an explanation for why performance differences across firms are highly persistent over time: while a high attention technology increases worker engagement and effort, firms may fall into an "engagement trap" where low managerial investment and low worker effort reinforce each other.

We present a continuous-time model. At each moment, a myopic agent privately chooses effort which generates unobservable output for a principal.[4] The principal's attention technology can be either low or high. This intangible asset evolves according to a stochastic process that, in a stylized form, is similar to those used in the industrial organization literature to describe the dynamics of productive assets (see, e.g., Besanko and Doraszelski, 2004). Specifically, the principal can invest at some cost to transform a low attention technology into a high one, and a high technology can "break" at any point with a certain probability and become low. If the attention technology is high and the agent is exerting effort, the technology produces a verifiable signal with positive probability, according to a Poisson process. When a signal arrives, we say that "recognition" occurs. The agent receives a constant bonus payment each time he is recognized; the principal does not bear the cost of this bonus directly.[5] Naturally, the agent's incentive to exert effort depends on the bonus but also on his engagement, which we define as the agent's belief that the attention technology is high.

There are three main features of management that our model tries to capture. First, many forms of management practices feature positive complementarities with worker effort. These are practices that tend to identify and reward

[4]We think of output as some good or service whose quality is difficult to measure, or for which the agent's contribution to the final product is difficult to determine. As it is often the case in the real world, the agent is thus not directly compensated as a function of output.

[5]For example, assume that the principal makes payments to a bonus pool or a fund at each time, and the fund then pays the bonus to the agent when recognition occurs.

virtuous behavior, and which can thus be thought of as "good news technologies". A canonical example is continuous process improvement, pioneered by Toyota and imitated with a varying degree of success by scores of manufacturing firms (Gibbons and Henderson, 2013): workers exert effort to identify performance-enhancing changes to production, and these incremental innovations can be recognized only if a management system is in place to monitor how workers engage with the productive process.[6] Second, the signals produced by the management technology are (at least in part) verifiable, typically because they describe the details of positive contributions made by workers in a specific context familiar to them. Those details are not known to the manager unless the technology is in place; hence, she cannot increase worker engagement by simply "faking" recognition at random times: the worker must have made a positive contribution and the manager must be able to document it. Third, workers cannot perfectly observe the quality of the management technology.[7] A main reason is that managerial practices display synergies with other practices and attributes of the firm,[8] and these synergies are non-obvious.[9] A management technology can thus become ineffective if for exogenous reasons one of its elements ceases to work, or it can start working again if this element is replaced, and neither of these changes may be evident. In fact, even the manager may be unable to perfectly observe the effectiveness of her technology; we consider this possibility in an extension and show that our results are essentially unchanged.

We characterize a continuous equilibrium with positive investment. The agent's engagement is a function of recognition (or its absence) and the agent's belief about the principal's investment. Because recognition fully reveals that the principal's attention technology is high, engagement jumps up to one when

[6]As we describe subsequently, we also analyze "bad news technologies" and hybrid forms.

[7]As we note below, our qualitative results are unchanged if additional signals of the management technology are available to the worker, so long as these signals are not perfect.

[8]See for example Milgrom and Roberts (1990), Milgrom and Roberts (1995), Ichniowski, Shaw, and Prennushi (1997), Bartling, Fehr, and Schmidt (2012), and Brynjolfsson and Milgrom (2013). Toyota's management system is a case in point: its effectiveness depends on putting in place a complex set of practices.

[9]Indeed, a main goal of the empirical analysis of organizations is to identify these synergies. For instance, Griffith and Neely (2009) study a multi-unit organization and find that the individual experience of the unit manager determines whether the introduction of a Balanced Scorecard system increases value added in a particular unit.

the agent is recognized. We show that engagement then decreases continuously over time until recognition again occurs. While engagement is high, the agent exerts high effort and the principal does not invest; when engagement becomes low, the agent's effort begins to decline and the principal then starts investing, but the principal's investment also declines over time. Thus, the relationship can fall into an engagement trap, where effort and investment decrease so long as recognition does not occur, and as a consequence the probability of recognition also goes down. This engagement trap provides a partial solution to the principal's moral hazard problem: a principal who does not invest in attention technology is punished with decreasing agent engagement and firm performance.

The equilibrium predicts that in a sample of ex-ante identical firms, firm performance, managerial quality, and worker engagement will display heterogeneity across firms, positive correlation, and persistence over time. We show that this equilibrium is the unique continuous equilibrium where the agent's effort does not go to zero in the long run, and it is the unique continuous equilibrium if the cost of managerial investment is low enough. The characterization also yields testable comparative statics. Engagement and effort are higher if managerial attention depreciates less — for example because the nature of work changes less frequently — or the cost of investment is lower — for example because the manager is more experienced. The response to an increase in the conditional probability of recognition, on the other hand, is non-monotonic.

Can firms escape the engagement trap? We study the role of costly signaling. Suppose that the principal can, at any time, purchase a non-productive public signal at some fixed cost. We think of this signal as a public announcement or hiring a consulting firm. We show that there exists a continuous equilibrium that implements the first-best outcome: the principal invests in attention technology and purchases the public signal continuously when the agent's engagement becomes low, and the agent exerts effort so long as the principal purchases the signal as prescribed. If the signal is money burning, however, total welfare can be smaller than that in the absence of costly signaling.

We contrast the dynamics of our model with those that arise in a setting where the principal monitors not only good performance but also *bad* performance. Suppose that bad signals arrive with positive probability if the attention technology is high and the agent does *not* exert effort. The agent is punished when a bad signal arrives. We show that if monitoring is primarily of bad performance, the model is essentially static, with a high constant effort level when engagement is high and a low constant effort level when engagement is low. Furthermore, an engagement trap does not arise, as lower-performing firms are more likely to detect shirking and jump back to high engagement and performance. The implications are immediate: we predict more persistent performance differences among ex-ante identical firms in settings where monitoring is based on reward — e.g., continuous process innovation — than in settings where monitoring is based on punishment — e.g., random quality control.

Finally, we perform a battery of robustness checks. We consider variants of our model where: the attention technology is unobservable to the principal; the value of the bonus is endogenous and contingent; equilibria can be discontinuous; there are multiple agents; the agent is forward-looking; and the agent receives other signals about the management technology. Our qualitative results change only if the agent is able to observe or infer the principal's attention technology *perfectly*. If attention is only imperfectly observable, the principal faces a moral hazard problem, and low attention, low engagement, and low productivity arise as equilibrium phenomena.

Related literature

The paper is related to a large literature on reputation; see Cripps (2006) and Bar-Isaac and Tadelis (2008) for surveys. The standard approach, pioneered by Kreps and Wilson (1982) and Milgrom and Roberts (1982) and generalized by Fudenberg and Levine (1989, 1992), considers a firm that is either a normal type or a behavioral type committed to a strategy. The continuous-time analog of Fudenberg and Levine (1992) is studied in Faingold and Sannikov (2011), where consumers observe Brownian signals of the firm's actions, but these signals do not depend on their own actions. Closer to our setting are Board and Meyer-ter-Vehn (2012, 2014), where a firm can invest in product quality and consumers learn about quality through Poisson signals. Here however moral hazard is one-sided; moreover, signals are again independent of consumers' actions.[10] We depart by focusing on the dynamics generated by the complementarity between the principal's investment and the agent's effort. Related to our analysis of rewards and punishments, Board and Meyer-ter-Vehn (2012) compare good news and bad news learning about firm quality.[11] We note though that learning is always good news about the principal's type in our model; we distinguish between good news and bad news about the (uninformed) agent's performance.

There is also an extensive literature on monitoring. In particular, a series of early papers study the problem of monitoring or auditing when the monitor cannot commit to a monitoring strategy.[12] Graetz, Reinganum, and Wilde (1986) analyze a simple tax compliance game. Khalil (1997) shows that a principal induces overproduction to increase her incentives to audit an agent's private cost of production ex post. Strausz (1997) considers a moral hazard setting and shows that delegating monitoring to a third party can help a principal to provide incentives for effort and monitoring simultaneously. While our focus is on recognition of good behavior, monitoring in this literature is of bad behavior, as in the case that we study in Section 4.

Various models generate persistent performance differences among ex-ante identical firms. Chassang (2010) considers a party who cannot observe her partner's cost of cooperating but can learn to predict this cost over time. Because learning is costly, the parties may stop learning before it is complete, and the efficiency of the relationship can depend on the history. In Li and Matouschek (2012), a principal's cost of making contingent payments to an agent depends on a privately observed shock. As bad shocks accumulate, the agent's effort goes down and a recovery can become more difficult. Callander and Matouschek (2013) propose a theory where managers learn about the quality of managerial practices by trial and error. If managerial actions are complementary, differences in the quality of practices across firms are persistent.

Finally, by studying managerial attention, our paper relates to Geanakoplos

[10] In other models such as Ely and Välimäki (2003), information depends on both players' actions, but there is moral hazard only on the side of the informed player.

[11] See also Abreu, Milgrom, and Pearce (1991).

[12] Sappington (1986) and Melumad and Mookherjee (1989) consider partial commitment.

and Milgrom (1991) and other work on organizations under cognitive limits, although we address quite different issues.[13] Specifically, this literature is concerned with the coordination of agents without conflicting interests, while we consider how an attention technology interacts with the provision of effort incentives.[14] The role of attention is also stressed in recent empirical work on the time use of managers and firm productivity, including Bandiera et al. (2011) and Bandiera, Prat, and Sadun (2012).

1 The model

1.1 Setup

Consider a principal and an agent. Time $t \in [0, \infty)$ is continuous and infinite and the discount rate is $r > 0$. At each time t, the agent privately chooses whether to work or shirk, which we model as the agent choosing continuous effort $a_t \in [0, 1]$ at cost ca_t, for $c > 0$. The agent's effort generates a flow output for the principal whose value we normalize to be equal to a. This output is unobservable to the parties (that is, either the quality of output or the agent's contribution to final output is difficult to measure, so the agent's compensation cannot directly depend on output).

The principal's attention technology can be either low or high, $\theta_t \in \{\theta^L, \theta^H\}$. An attention level θ^H becomes θ^L, i.e. "the technology breaks", with instantaneous transition probability $\gamma > 0$. The principal can transform θ^L into θ^H, i.e. "fix the technology", by investing at cost $F > 0$.[15] The principal's attention and her investment decisions are unobservable by the agent. The agent's belief

[13] See Garicano and Prat (2013) for a survey of this literature.

[14] Dur (2009) and Dur, Non, and Roelfsema (2010) study optimal incentives when workers reciprocate managerial attention with effort. Gil and Mondria (2011) consider a multitasking setting where allocating more attention to a task increases the precision of performance measures on that task.

[15] We model the attention technology as a capital asset in dynamic industrial organization models. In Besanko and Doraszelski (2004), for example, an asset can take a number of finite values. In each period the asset is subject to two forces: endogenous investment that tends to raise its value and exogenous depreciation that tends to lower it. Our setup contains a continuous-time two-value version of this process.

that the attention technology is high at time t is $x_t = \Pr(\theta_t = \theta^H) \in [0,1]$. We refer to x as the agent's engagement.

At any time t, if $\theta_t = \theta^H$, a verifiable signal is realized — "recognition" occurs — with instantaneous probability μa_t, where $\mu > 0$. If $\theta_t = \theta^L$, recognition cannot occur at t. The agent receives a bonus $b > 0$ each time he is recognized. To study the problem of managerial attention separately from that of enforcement of payments, we assume that the principal does not bear the cost of the bonus directly. One interpretation is that the principal makes continuous payments to a fund, which then pays the bonus to the agent when recognition occurs. Alternatively, the bonus may represent the agent's intrinsic value for being recognized, where this recognition does not entail a direct cost to the principal. The bonus is exogenous and the agent's wage is normalized to 0.[16]

Both the principal and the agent are risk neutral. Let $q_t \geq 0$ denote the instantaneous probability with which the principal invests at time t (where $q_t = 0$ if $\theta_t = \theta^H$). The principal's flow payoff is $a_t - (1 - e^{-q_t})F$. The agent's flow payoff is $b - ca_t$ if recognition occurs at t and $-ca_t$ otherwise. The agent is completely myopic and his outside option is not to work.[17]

1.2 Observable attention benchmark

Consider a benchmark setting where the principal's attention technology is observable by the agent.[18]

Proposition 1. *Suppose that θ_t is observable. An equilibrium where the agent exerts effort and the principal invests in attention technology exists if and only if $\mu b \geq c$ and $(\gamma + r)F \leq 1$.*

Proof. The agent exerts effort at time t if and only if $\theta_t = \theta^H$ and $\mu b \geq c$. Given $a_t = 1$ if $\theta_t = \theta^H$ and $a_t = 0$ if $\theta_t = \theta^L$, if the principal's attention technology

[16] Subsection 5.2 shows that allowing the principal to choose the bonus at each point and incorporating the cost of bonus payments into her payoff does not fully solve the problem.

[17] See Subsection 5.5 for a discussion of the case of a forward-looking agent. We assume throughout that the parties cannot correlate their strategies over time; see fn. 21.

[18] Another benchmark one can consider is a first-best setting where the principal can commit to a strategy. In this case, the principal commits to the minimum investment at each time that ensures an agent's belief high enough that the agent always exerts effort $a_t = 1$. This is the investment path in the equilibrium with costly signaling in Section 3.

breaks at time t, the principal prefers to fix the technology at t rather than fixing it at time $t+\delta$, for any $\delta > 0$, if and only if

$$\int_0^\delta e^{-(\gamma+r)\tau} d\tau - (1 - e^{-(\gamma+r)\delta})F \geq 0,$$

which is equivalent to the second condition in the proposition. *Q.E.D.*

Two conditions are required for trade when managerial attention is observable. First, given high attention, the agent's expected reward for performance must be large enough to compensate him for the cost of effort. Second, the principal must have incentives to invest in attention technology: the increase in output when she invests must be larger than the instantaneous rental cost of capital, given by the risk of breakdown plus the interest rate.

It is immediate that these two conditions are also necessary for trade when the principal's attention technology is unobservable. Throughout our analysis, we assume that parameters are such that these conditions are satisfied:

Assumption 1. *$\mu b \geq c$ and $(\gamma + r)F \leq 1$.*

2 Equilibrium characterization

Consider now the case where the principal's attention technology is unobservable by the agent. Unlike in the observable attention benchmark, an equilibrium where the principal fixes the technology each time it breaks does not exist: if the principal always invests, the agent's engagement is $x_t = 1$ for all t, but then the agent always exerts effort $a_t = 1$ and the principal has no incentives to invest at cost F. Of course, as long as x_t is positive and the agent exerts positive effort, recognition is possible and engagement can jump back to $x_t = 1$. We are thus looking for equilibria with ups and downs in engagement.

Subsection 2.1 constructs a continuous equilibrium with positive investment. We show that this equilibrium exists if and only if the principal's cost of investing in attention technology is not too high. Subsection 2.2 gives conditions for uniqueness and Subsection 2.3 analyzes comparative statics.

2.1 The solution

Let s be the amount of time that has passed since recognition last occurred. We construct an equilibrium where the agent's engagement as a function of s, x_s, is continuous.[19] The principal does not invest if $s < \overline{s}$, for a time $\overline{s} \geq 0$, and she invests with instantaneous probability $q_s \in (0, \infty)$ if $s \geq \overline{s}$. The agent exerts effort $a_s = 1$ if $s < \overline{s}$ and $a_s \in (0, 1)$ if $s \geq \overline{s}$. Note that the agent has incentives to exert effort at time s if and only if $\mu b x_s \geq c$. Thus, the threshold $\overline{s}$ is the time at which the agent's engagement x_s reaches $\overline{x} \equiv c/(\mu b)$ where the agent is indifferent between exerting effort and shirking.

Engagement. To solve for the threshold time $\overline{s}$, consider the law of motion for the agent's belief, x_s. At the time of recognition, $s = 0$, the belief is $x_0 = 1$, since recognition fully reveals that the principal's attention technology is high. Then, if no recognition occurs, the change in x_s over $[s, s + \delta]$ is given by three sources: (i) the possibility that a high attention technology breaks, with instantaneous probability γ; (ii) learning about the attention technology in the absence of recognition, according to Bayes' rule; and (iii) the agent's belief about the principal's investment. For δ sufficiently small and a_s and q_s continuous in s over $[s, s + \delta]$, the change in x_s over $[s, s + \delta]$ absent recognition is

$$x_{s+\delta} - x_s = -\gamma \delta x_s - \frac{x_s(1 - x_s)\mu a_s \delta}{x_s(1 - \mu a_s \delta) + (1 - x_s)} + (1 - x_s) q_s \delta + o(\delta).$$

In the limit as $\delta \to 0$, x_s in the absence of recognition is then governed by the following differential equation:

$$\dot{x}_s = -\gamma x_s - x_s(1 - x_s)\mu a_s + (1 - x_s) q_s. \tag{1}$$

This law of motion is similar to that in Board and Meyer-ter-Vehn (2012), with an important difference: our Bayesian learning term, $x_s(1 - x_s)\mu a_s$, depends on the agent's action, while it is simply $x_s(1 - x_s)\mu$ in their paper (see their

[19] That is, the agent's belief as a function of time is continuous in the absence of publicly observable events. This restriction is similar to the one used by Board and Meyer-ter-Vehn (2012) in their model of reputation. See Subsection 5.3 for a discussion.

equation 2.2). In our setting, the learning process is endogenous and depends on the agent's behavior.

For $s < \overline{s}$, effort and investment are $a_s = 1$ and $q_s = 0$ respectively, so the law of motion is

$$\dot{x}_s = -\gamma x_s - x_s(1 - x_s)\mu. \tag{2}$$

Solving this differential equation with initial condition $x_0 = 1$ and setting $x_{\overline{s}} = \overline{x} = c/(\mu b)$, we obtain

$$\overline{s} = \frac{log\left(\frac{(\gamma+\mu)\mu b-\mu c}{\gamma c}\right)}{\gamma + \mu}. \tag{3}$$

Note that $\mu b \geq c$ by Assumption 1 and thus $\overline{s} \geq 0$.

Investment and effort. Consider next the principal's incentives to invest. The principal's payoff at any point s depends on her attention technology or *type*. Let π_s^H be the principal's expected payoff when her type is $\theta_s = \theta^H$ and π_s^L when her type is $\theta_s = \theta^L$. The principal is willing to invest at s only if

$$\pi_s^L \leq \pi_s^H - F.$$

Since at any point in the equilibrium the principal either does not want to invest or is indifferent between investing and not investing, $\pi_s^L \geq \pi_s^H - F$ for all s and we can compute the principal's payoff as if she never invested:

$$\pi_s^L = \int_s^\infty e^{-r(\tau-s)} a_\tau d\tau, \tag{4}$$

$$\pi_s^H = \int_s^\infty e^{-(\gamma+r)(\tau-s)-\int_s^\tau \mu a_{\tilde{\tau}} d\tilde{\tau}} \left(a_\tau + \gamma\pi_\tau^L + \mu a_\tau \pi_0^H\right) d\tau. \tag{5}$$

Let $\Lambda_s \equiv \pi_s^H - \pi_s^L$ denote the value of investing and $\Psi_s \equiv \pi_0^H - \pi_s^H$ the value of recognition. Using (4) and (5), the principal's value of investing in attention technology is equal to the probability of obtaining recognition before the technology breaks, times the value of recognition:

$$\Lambda_s = \int_s^\infty e^{-(\gamma+r)(\tau-s)} \mu a_\tau \Psi_\tau d\tau. \tag{6}$$

Consider $s \geq \overline{s}$ where the principal invests with instantaneous probability $q_s \in (0, \infty)$. The principal must be indifferent between investing and not investing:

$$\Lambda_s = F. \tag{7}$$

Moreover, since the principal must be indifferent at each time $s \geq \overline{s}$, we must have $\dot{\Lambda}_s = 0$. Differentiating (6) and substituting with (7), this condition yields

$$\Psi_s \mu a_s = (\gamma + r)F, \tag{8}$$

which has a standard interpretation of equalizing the instantaneous benefit of investment with the instantaneous rental cost of capital.

We use the principal's indifference conditions (7) and (8) to solve for the agent's effort a_s and the principal's investment q_s for $s \geq \overline{s}$. Using (7), $\Psi_s = \pi_0^H - \pi_s^L - F$, and thus

$$\dot{\Psi}_s = -\dot{\pi}_s^L = a_s - r\pi_s^L,$$

where the second equality follows from (4). Using (4), (7), and (8), we obtain a system of two differential equations for $s \geq \overline{s}$:

$$\begin{aligned} \dot{\Psi}_s &= \frac{(\gamma + r)F}{\mu \Psi_s} - r\pi_s^L, & (9)\\ \dot{\pi}_s^L &= -\dot{\Psi}_s, & (10) \end{aligned}$$

with initial conditions $\Psi_{\overline{s}} = \overline{\Psi}$ and $\pi_{\overline{s}}^L = \overline{\pi}^L$. The values of $\overline{\Psi}$ and $\overline{\pi}^L$ are obtained from the solution for $s < \overline{s}$, as we show subsequently. Let Ψ_s^* and π_s^{L*} denote the solution to (9)-(10) given these initial conditions. Then using (8) the agent's effort for $s \geq \overline{s}$ is

$$a_s^* = \frac{(\gamma + r)F}{\Psi_s^* \mu}. \tag{11}$$

To solve for the principal's investment, note that the agent's belief must be constant at $x_s = \overline{x}$ at any time $s \geq \overline{s}$. Setting $\dot{x}_s = 0$, $x_s = \overline{x}$, and $a_s = a_s^*$ in the law of motion for the agent's belief given in (1) and solving for q_s yields that the principal's instantaneous probability of investment for $s \geq \overline{s}$ is

$$q_s^* = \gamma \frac{\overline{x}}{1-\overline{x}} + \overline{x}\mu a_s^*. \tag{12}$$

The last step is to characterize the solution for $s < \overline{s}$. Since here the agent's effort is $a_s = 1$, we have the following system of differential equations:

$$\begin{aligned} \dot{\Lambda}_s &= (\gamma + r)\Lambda_s - \mu\Psi_s, & (13)\\ \dot{\Psi}_s &= 1 - (\gamma + r)\Lambda_s + \mu\Psi_s - r\pi_s^L, & (14)\\ \dot{\pi}_s^L &= -1 + r\pi_s^L, & (15) \end{aligned}$$

with boundary conditions $\Psi_0 = 0$, $\Lambda_{\overline{s}} = F$, and $\Psi_{\overline{s}} = \frac{(\gamma+r)F}{\mu}$. The first two boundary conditions are straightforward from the definitions and discussion above. To understand the third condition, note that we must have $\Lambda_s < F$ for $s < \overline{s}$, so that the principal has no incentives to invest before time $\overline{s}$. Given $\Lambda_{\overline{s}} = F$, this requires $\dot{\Lambda}_s \geq 0$ for $s < \overline{s}$, s close to $\overline{s}$. Now suppose this inequality is strict for s arbitrarily close to $\overline{s}$. Since Λ_s is continuous, it approaches F as s approaches $\overline{s}$. Hence, in the limit as s goes to $\overline{s}$, $\dot{\Lambda}_s > 0$ implies $\mu\Psi_s < (\gamma+r)F$. But then condition (8) requires that a_s jump up above one at $\overline{s}$. Thus, we must have $\lim_{\varepsilon\to 0} \dot{\Lambda}_{\overline{s}-\varepsilon} = 0$.

The system (13)-(15) admits a closed-form solution that we derive in Appendix A. We show that in this solution, Λ_s and Ψ_s are increasing (and thus, using the boundary condition, $\Lambda_s < F$ for $s < \overline{s}$) and π_s^L is decreasing. The solution gives the values of $\overline{\Psi}$ and $\overline{\pi}^L$, which are then used to solve the system (9)-(10). By Peano's existence theorem, (9)-(10) also has a solution. Moreover, if $\overline{\pi}^L \in (0, \frac{1}{r})$, the solution to (9)-(10) has Ψ_s increasing, π_s^L decreasing, and $\pi_s^L \geq 0$ for all $s \geq \overline{s}$, implying (together with the other boundary conditions) that the agent's effort decreases continuously for $s \geq \overline{s}$ towards a value $\underline{a} \in (0, 1)$. We show in Appendix A that $\overline{\pi}^L < \frac{1}{r}$ for any set of parameters, while $\overline{\pi}^L > 0$ if and only if

$$F < \frac{\mu\left[(\gamma+r)(\gamma+\mu)e^{\overline{s}(\gamma+2r+\mu)} - \gamma(\gamma+r+\mu)e^{\overline{s}(\gamma+r+\mu)} - r\mu e^{r\overline{s}}\right]}{r(\gamma+r)(\gamma+u)(\gamma+r+\mu)e^{\overline{s}(\gamma+2r+\mu)}} \equiv \overline{F},$$

where $\overline{F} > 0$ and $\overline{s}$ is independent of F and given by (3).

Proposition 2. *If the cost of managerial investment is $F < \overline{F}$, there exists an equilibrium characterized by equations (1), (2), (3), (11), and (12). In this equilibrium, the agent's engagement decreases continuously with the time that has passed since recognition, s. If $s < \overline{s}$, the agent exerts effort $a_s = 1$ and the principal does not invest. If $s \geq \overline{s}$, the agent exerts effort $a_s^* \in (0, 1)$ and the principal invests with instantaneous probability $q_s^* \in (0, \infty)$. Effort a_s^* decreases continuously for $s \geq \overline{s}$. Investment q_s^* jumps at $\overline{s}$ and decreases continuously for $s > \overline{s}$. The probability of recognition is continuously decreasing in s.*

Proof. See Appendix A. *Q.E.D.*

Figure 1 illustrates the equilibrium.[20] When the agent is highly engaged, the principal's benefit from being revealed to be a high-attention type is small. Following recognition, there is thus a period of time during which the agent exerts high effort and the principal does not invest. Engagement declines during this period because the probability that the principal's attention technology has broken increases as time passes without recognition. Eventually, engagement becomes low enough that the principal must start investing to prevent the agent from shirking. So long as no recognition occurs, however, the principal's investment and the agent's effort decrease continuously, and so does the probability of obtaining recognition. Hence, low worker engagement and poor managerial practices reinforce each other, and the relationship's chances of returning to high engagement and productivity decline as productivity continues to go down.

Our results have direct implications for firm performance. In a sample of ex-ante identical firms, worker engagement, managerial quality, and firm performance display heterogeneity across firms and positive correlation. Additionally, as performance becomes lower and lower, firms are less likely to "snap out" of the engagement trap, implying that low levels of engagement, managerial quality, and performance are persistent over time.

[20] We consider $r = 0.01$, $\gamma = 0.4$, $F = 1.5$, $\mu = 0.95$, $b = 0.1$, and $c = 0.03$.

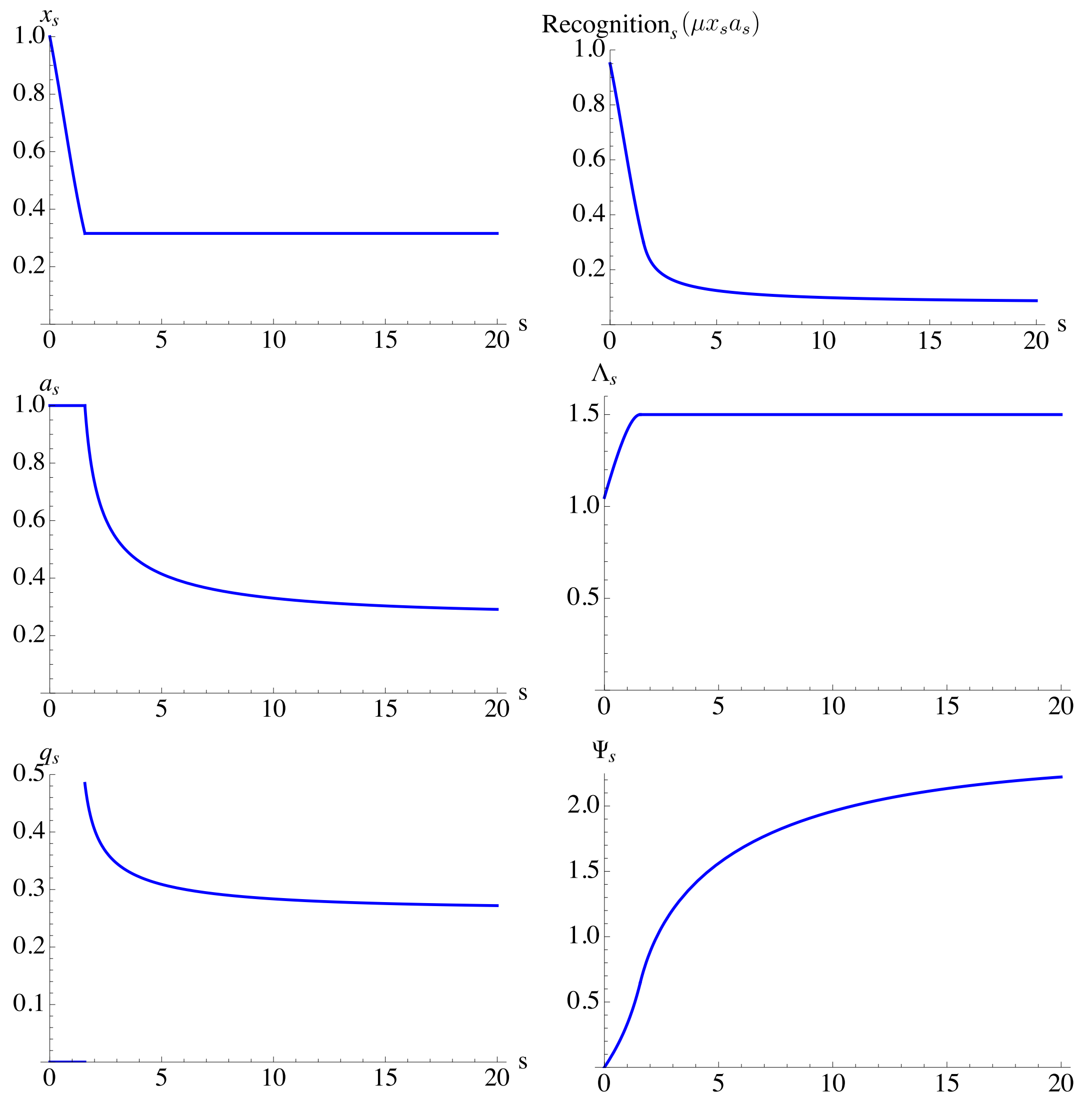

Figure 1: Equilibrium dynamics.

Corollary 1. *In the equilibrium of Proposition 2, firm performance is positively correlated with engagement and managerial quality. The equilibrium gives rise to persistent performance differences among ex-ante identical firms.*

2.2 Uniqueness

There may be equilibria different from the one described in Proposition 2.[21] We show, however, that the equilibrium of Proposition 2 is the unique continuous equilibrium where the agent's effort does not go to zero in the long run. The argument is based on two claims. First, the equilibrium of Proposition 2 is the unique continuous equilibrium where the agent's belief never falls strictly below $\overline{x}$. This follows from the fact that the principal cannot be indifferent between investing and not investing over a period of time where the agent's effort is constant at one, and she cannot have strict incentives to invest unless there is a continuation where the agent's belief falls strictly below $\overline{x}$. Second, we show that if the belief falls strictly below $\overline{x}$ at some point s in a continuous equilibrium, it never increases back. Since the probability of reaching that point s is strictly positive, the agent's effort goes to zero in the long run.

If the principal's cost of investing in attention technology, F, is low enough, we can further show that the equilibrium of Proposition 2 is the unique equilibrium within the class of continuous equilibria. Intuitively, we show that if F is sufficiently low, a continuous equilibrium where the agent's belief falls strictly below $\overline{x}$ at some point does not exist: since the agent stops exerting effort forever when the belief falls strictly below $\overline{x}$, the principal has strict incentives to invest before, and thus the belief stays above $\overline{x}$. Therefore, using the claims above, the equilibrium that we characterize is the unique continuous equilibrium.

[21] We restrict attention to uncorrelated strategies. Allowing the parties to condition on their past decisions (i.e., mix over paths), however, has no effects in the context of the equilibrium of Proposition 2. In particular, we cannot improve by allowing the principal to autocorrelate her strategy, as what matters is the principal's type rather than her investment. As for the agent, he is not willing to follow a different effort path: if the agent exerts more effort, his belief that the principal is a high type following no recognition is lower, so he wants to lower his effort; if the agent exerts less effort, his belief is higher, so he wants to increase his effort.

Let

$$\overline{\overline{F}} \equiv \max_{s' \in [0,\bar{s}]} \frac{\mu \left\{ \begin{array}{c} (\gamma + r)e^{rs'} \left(e^{\bar{s}(\gamma+\mu)} - e^{s'(\gamma+\mu)} \right) \\ -e^{-r\bar{s}} \left[\gamma \left(e^{\bar{s}(\gamma+r+\mu)} - e^{s'(\gamma+r+\mu)} \right) + \mu \left(e^{r\bar{s}} - e^{rs'} \right) \right] \end{array} \right\}}{r(\gamma+\mu)\left[(\gamma+r)e^{\bar{s}(\gamma+r+\mu)} + \mu \right]},$$

where $\overline{\overline{F}} > 0$ and $\bar{s}$ is independent of F and given by (3). We obtain:

Proposition 3. *The equilibrium of Proposition 2 is the unique continuous equilibrium where the agent's effort does not go to zero in the long run. Moreover, if $F < \overline{\overline{F}}$, this equilibrium is the unique continuous equilibrium.*

Proof. See Appendix B. *Q.E.D.*

2.3 Comparative statics

We study how the equilibrium described above varies with parameters. Consider the threshold time $\bar{s}$ at which the agent's effort begins to decline.

Proposition 4. *In the equilibrium of Proposition 2, the threshold time $\bar{s}$ is increasing in b, decreasing in c and γ, non-monotonic with respect to μ, and independent of F.*

Proof. See Appendix B. *Q.E.D.*

These comparative statics are illustrated in Figure 2 and Figure 3.[22] The intuition for the comparative static with respect to the bonus is simple: if b increases, the agent is willing to exert effort for lower levels of engagement; i.e., the threshold belief at which the agent is indifferent, $\bar{x} = c/(\mu b)$, decreases. The principal can then enjoy high effort for a longer period of time without investing, so $\bar{s}$ increases. The intuition for the cost of effort c is similar. As for the probability that the principal's attention technology breaks, γ, the result is due to the fact that, in the absence of recognition, the agent updates his belief down faster if the attention technology is less persistent. Hence, if γ increases, $\bar{s}$ decreases because the threshold belief $\bar{x}$ is reached more quickly.

[22] We do not show the comparative static with respect to the cost of effort c as it follows a similar pattern, though with the opposite sign, as that for the bonus b.

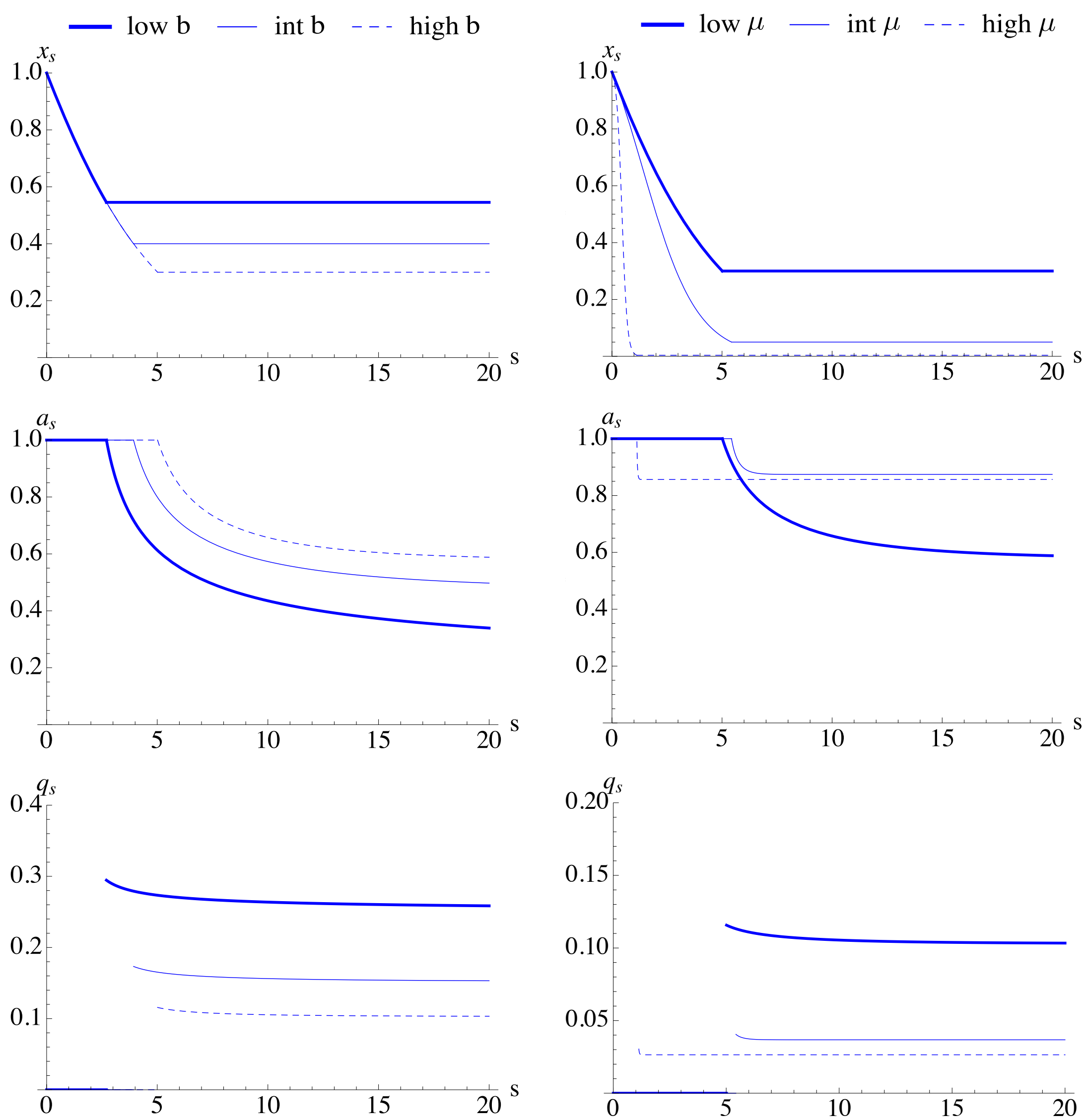

Figure 2: Comparative statics with respect to b and μ.

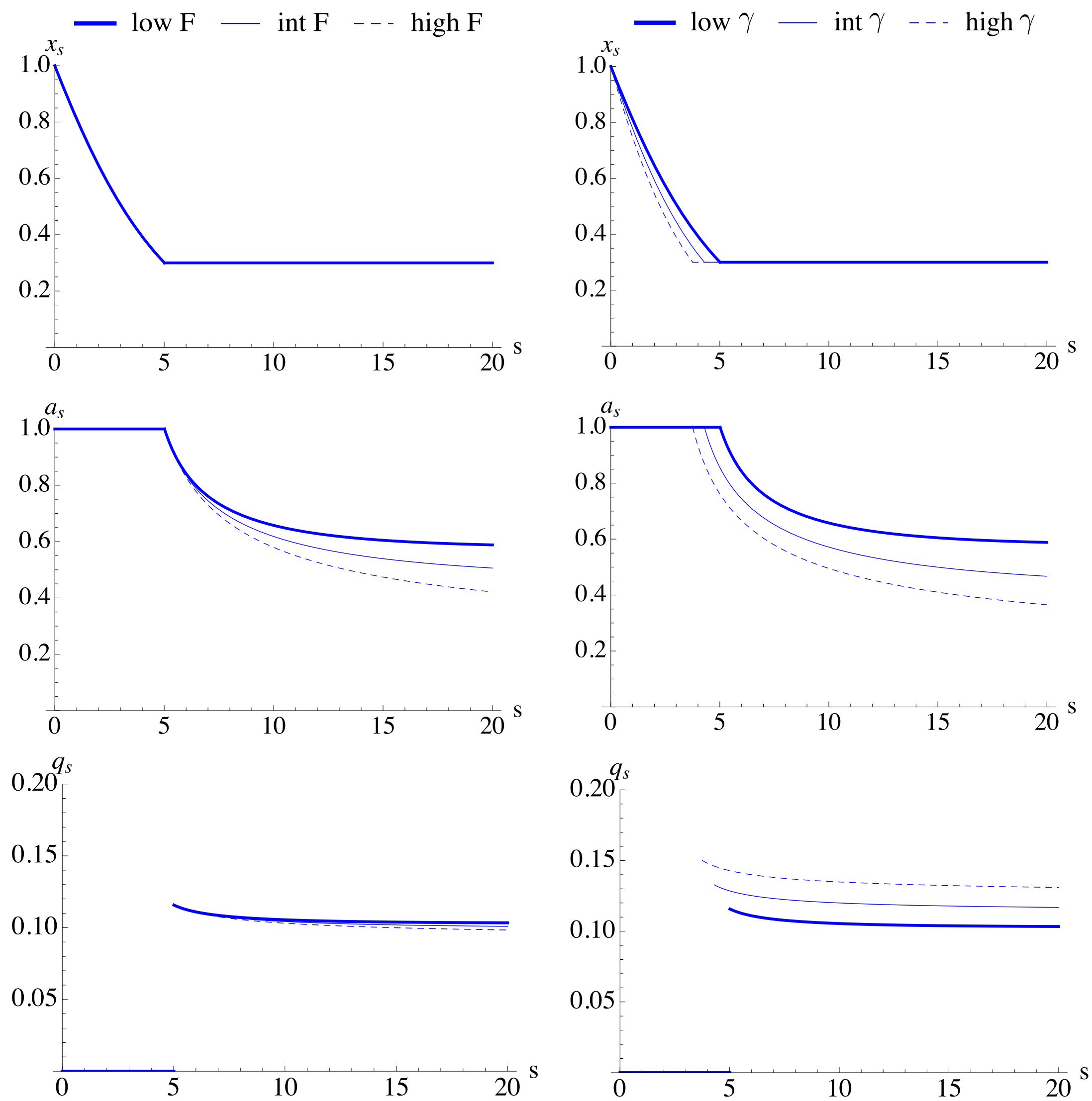

Figure 3: Comparative statics with respect to F and γ.

The comparative static with respect to the conditional probability of recognition, μ, is more interesting. As shown in Figure 2, the relationship between $\overline{s}$ and μ is non-monotonic. On the one hand, if μ increases, the agent's incentive to exert effort for a given belief increases, so the threshold belief $\overline{x}$ decreases. On the other hand, if μ increases, the agent updates his belief down faster absent recognition. Consequently, as μ increases, $\overline{s}$ first increases as the incentive effect pushes $\overline{x}$ down, but it then decreases as the updating effect makes the belief decline and reach $\overline{x}$ more quickly.

The figures also show how the paths of effort and investment for $s \geq \overline{s}$ depend on parameters. We can formally characterize the limit level of effort as s increases, $\underline{a}$, in the limit of no discounting. As the discount rate r goes to zero, $r\pi_s^L$ becomes equal to the effort level $\underline{a}$. Using this, we obtain that in the limit of no discounting,

$$\underline{a} = \frac{\mu[\mu + \gamma(\gamma+\mu)\overline{s}] - F\gamma(\gamma+\mu)^2 - \mu^2 e^{-(\gamma+\mu)\overline{s}}}{\mu[\mu + \gamma(\gamma+\mu)\overline{s}] - \mu^2 e^{-(\gamma+\mu)\overline{s}}}. \tag{16}$$

Substituting $\overline{s}$ from (3), we compute the comparative statics:

Proposition 5. *Consider the equilibrium of Proposition 2 and a discount rate r arbitrarily close to zero. The effort level in the limit as s increases, $\underline{a}$, is decreasing in F, γ and c, increasing in b, and non-monotonic with respect to μ.*

Proof. See Appendix B. *Q.E.D.*

Similar to the threshold time $\overline{s}$, the relationship between the limit effort level $\underline{a}$ and the conditional probability of recognition μ is ambiguous. This is illustrated in Figure 2. Of course, a higher μ also implies that effort is more likely to remain high immediately following recognition, as the probability of again obtaining recognition, $\mu x_s a_s$, is higher. Yet, as s increases, we find that both the agent's effort and the probability of recognition can decrease with μ.

3 Costly signaling

The equilibrium characterized above gives rise to an engagement trap, where both worker effort and managerial investment decrease and a return to high productivity becomes less likely as time passes. Can firms escape this trap? We explore the role of costly signaling. Suppose that the principal can, at any time, purchase a non-productive public signal at some fixed cost. We think of this signal as a public announcement or hiring a consulting firm. We show that if the equilibrium of Proposition 2 exists, there exists a continuous equilibrium with costly signaling that implements the first-best outcome.

Consider the following equilibrium. As above, let s be the time since recognition and $\overline{s}$ the time at which the agent's engagement reaches $\overline{x} = c/(\mu b)$. The principal does not invest in attention technology if $s < \overline{s}$. If $s \geq \overline{s}$, the principal continuously invests with instantaneous probability $q_s \in (0, \infty)$ and pays a cost $m > 0$. The agent exerts effort $a_s = 1$ so long as the principal has paid m as prescribed. If the principal fails to pay m at some time $s \geq \overline{s}$, the agent believes that the principal does not invest and he shirks forever (and so the principal indeed prefers not to invest).

The agent's belief x_s follows the law of motion (2) for $s < \overline{s}$ and is constant at $\overline{x}$ for $s \geq \overline{s}$, where $\overline{s}$ is given by (3). At each time $s \geq \overline{s}$, the principal must be indifferent between investing and not investing, given that she pays m continuously. Since $a_s = 1$ for all s, the principal's indifference conditions are

$$\Lambda_s = F, \tag{17}$$

$$\Psi_s \mu = (\gamma + r)F. \tag{18}$$

Given indifference, the principal invests with instantaneous probability q_s such that the agent's belief is constant at $\overline{x}$. Using (1), we thus have that for $s \geq \overline{s}$,

$$q_s = \gamma \frac{\overline{x}}{1 - \overline{x}} + \overline{x}\mu.$$

The principal must also be willing to pay the cost m at each time $s \geq \overline{s}$. Given that the agent stops exerting effort forever if m is not paid, the low type of principal is willing to pay m at each $s \geq \overline{s}$ if and only if $\pi_s^L \geq 0$, where

$$\pi_s^L = \frac{1-m}{r}$$

for $s \geq \overline{s}$ and thus $m = 1 - r\pi_{\overline{s}}^L$. As for the high principal type, effort is more valuable to her than to the low type because she can obtain recognition when the agent exerts effort; hence, the high type prefers to pay m whenever the low type does.

The last step is to obtain the solution for $s < \overline{s}$. This solution is given by the same system of differential equations, (13)-(15), in Subsection 2.1, using the boundary conditions described in that section. Since here the value of investment Λ_s and the value of recognition Ψ_s are constant for $s \geq \overline{s}$, the boundary conditions ensure that (17) and (18) are satisfied. The system yields $\overline{\pi}^L$, and the equilibrium requires $\pi_{\overline{s}}^L = \overline{\pi}^L$ and thus $m = 1 - r\overline{\pi}^L$. If the equilibrium of Proposition 2 exists, then $\overline{\pi}^L \in \left(0, \frac{1}{r}\right)$, and hence we show that an equilibrium with costly signaling and constant effort exists.

Proposition 6. *Consider parameters such that the equilibrium of Proposition 2 exists. Suppose that at any time the principal can purchase a public signal. There exists a continuous equilibrium where the cost of the public signal is $m = 1 - r\overline{\pi}^L$ and the agent exerts effort $a_t = 1$ at all times. If m is money burning and the principal bears the cost of bonus payments to the agent, total welfare is lower than in the equilibrium of Proposition 2 without m. Otherwise welfare is higher.*

Proof. See Appendix C. *Q.E.D.*

The logic behind the construction is simple. In the equilibrium of Proposition 2, the principal has incentives to invest in attention technology to obtain recognition and avoid decreasing effort by the agent. Here, instead, the agent exerts constant effort at all times in equilibrium, and the principal has incentives to invest to obtain recognition and avoid having to make the payments m. Since these payments are public, the principal can be incentivized to make them by the threat of breakup.

Note that there is a unique cost m that sustains effort $a_s = 1$ for all $s \geq 0$. If the public signal is more expensive (i.e. $m > 1 - r\overline{\pi}^L$), the principal has strict incentives to invest in attention technology before time $\overline{s}$ so she can avoid making the payments m. However, this cannot occur in equilibrium: if the principal has strict incentives to invest before $\overline{s}$, the agent's belief does not reach $\overline{x}$, but then the agent exerts effort $a_s = 1$ while the principal never pays m and as a result the principal has no incentives to invest. On the other hand, if the public signal is cheaper (i.e. $m < 1 - r\overline{\pi}^L$), the principal has no incentives to invest after time $\overline{s}$, as the value of recognition (namely, saving the cost of the public signal) is too low to compensate for the cost of investment. But then of course the agent shirks after $\overline{s}$. More generally, if the cost of the signal is $m < 1 - r\overline{\pi}^L$, an equilibrium with positive investment requires that the agent exert less than full effort after time $\overline{s}$, so that the principal is incentivized to invest by a combination of public payments and decreasing effort.

Is the equilibrium with costly signaling characterized in Proposition 6 more efficient than the one without (characterized in Proposition 2)? One can show that the principal's expected payoff excluding the bonus b is the same in the two equilibria. The reason is that the principal's payoff is determined by the solution for $s < \overline{s}$, which is described by the same system of differential equations (13)-(15) with the same boundary conditions. Intuitively, the payment m exactly offsets the benefit that the principal receives from the agent's higher effort.

The effect on total welfare depends on how we handle the payment m. If welfare is the sum of the payoff to the principal including b and the payoff to the agent (so m is money burning), then the equilibrium with costly signaling results in lower welfare. This is because the payment b washes out, the principal's payoff (excluding b) is unchanged, and the agent's payoff (excluding b) is lower as the agent works harder. Instead, if we assume that m is not a waste of resources but a payment to a third party (whose payoff enters social welfare), then efficiency always increases, since introducing the payment m allows to implement the first-best outcome.

4 Good news and bad news

We have considered a principal who can recognize and reward *good* performance by the agent. What happens if the principal can also monitor and punish *bad* performance?

Consider the setup of Section 1 but assume now that there are two types of signals: signals of good performance — "good news" — and signals of bad performance — "bad news". No signal arrives if the principal's attention technology is low. If the attention technology is high and the agent exerts effort $a_t \in [0, 1]$ at time t, good news arrives with instantaneous probability $\mu_G a_t$ and bad news arrives with instantaneous probability $\mu_B(1 - a_t)$, where $\mu_G, \mu_B \geq 0$. The agent receives a bonus $\overline{b} > 0$ if good news arrives and a punishment $-\underline{b} < 0$ if bad news arrives. Given his belief x_t that the attention technology is high, the agent then has incentives to exert effort at time t if and only if $(\mu_G \overline{b} + \mu_B \underline{b}) x_t \geq c$. Consistent with Assumption 1, assume $(\mu_G \overline{b} + \mu_B \underline{b}) \geq c$.

It is straightforward that the benchmark case where managerial attention is observable is qualitatively the same as above. Now suppose that the principal's attention technology is unobservable by the agent. Let s be the amount of time since a signal — either good or bad — arrived. As in Section 2, we construct an equilibrium where the agent's belief as a function of s, x_s, is continuous. If $s < \widehat{s}$, the agent exerts effort $a_s = 1$ and the principal does not invest in attention technology. If $s \geq \widehat{s}$, the agent exerts effort $a_s \in (0, 1)$ and the principal invests with instantaneous probability $q_s \in (0, \infty)$. The threshold time $\widehat{s}$ is the time at which the agent's belief x_s reaches $\widehat{x} \equiv c/(\mu_G \overline{b} + \mu_B \underline{b})$ where the agent is indifferent between exerting effort and shirking.

The agent's belief jumps to one when a signal arrives, i.e. $x_0 = 1$. Then if no signal arrives, the law of motion for the belief is

$$\dot{x}_s = -\gamma x_s - x_s(1 - x_s)[\mu_G a_s + \mu_B(1 - a_s)] + (1 - x_s) q_s. \tag{19}$$

For $s < \widehat{s}$, effort and investment are $a_s = 1$ and $q_s = 0$ respectively, so the law of motion is

$$\dot{x}_s = -\gamma x_s - x_s(1 - x_s)\mu_G. \tag{20}$$

Solving this differential equation with initial condition $x_0 = 1$ and setting $x_{\widehat{s}} = \widehat{x} = c/(\mu_G \overline{b} + \mu_B \underline{b})$ yields the value of the threshold time $\widehat{s}$.

At each time $s \geq \widehat{s}$, the principal must be indifferent between investing and not investing. The principal's indifference conditions are

$$\begin{aligned} \Lambda_s &= F, \\ \Psi_s[\mu_G a_s + \mu_B(1-a_s)] &= (\gamma + r)F. \end{aligned} \tag{21}$$

From condition (21), we see that the solution depends on the sign of $\mu_G - \mu_B$. If $\mu_G - \mu_B > 0$, the solution is qualitatively the same as that in Section 2. On the other hand, if $\mu_G - \mu_B < 0$, (21) shows that the agent's effort a_s and the principal's value of a signal Ψ_s must move in the same direction for the principal's value of investing to be constant over time. Now note that effort cannot be decreasing, as that would imply $\dot{\Psi}_s = a_s - r\pi_s^L > 0$ and hence the value of investment increases. Similarly, effort cannot be increasing, as then $\dot{\Psi}_s = a_s - r\pi_s^L < 0$ and the value of investment decreases. Hence, when $\mu_G - \mu_B < 0$, a_s and Ψ_s must be constant for $s \geq \widehat{s}$. Denote these constant values by $\widehat{a}$ and $\widehat{\Psi}$ respectively. Note that

$$\pi_0^H = \int_0^{\widehat{s}} e^{-(r+\gamma+\mu_G)\tau}(1 + \gamma\pi_\tau^L + \mu_G \pi_0^H)d\tau + e^{-(r+\gamma+\mu_G)\widehat{s}}\pi_{\widehat{s}}^H, \tag{22}$$

where $\pi_{\widehat{s}}^H = F + \pi_{\widehat{s}}^L$ and for $s \leq \widehat{s}$,

$$\pi_s^L = \frac{1 - e^{-r(\widehat{s}-s)}(1-\widehat{a})}{r}.$$

Therefore, using these expressions, $\widehat{a}$ and $\widehat{\Psi}$ are the solution to

$$\widehat{\Psi} = \frac{(\gamma + r)F}{[\mu_G \widehat{a} + \mu_B(1-\widehat{a})]}, \tag{23}$$

$$\widehat{\Psi} = \pi_0^H - \left(F + \frac{\widehat{a}}{r}\right). \tag{24}$$

Finally, setting $\dot{x}_s = 0$, $x_s = \widehat{x}$, and $a_s = \widehat{a}$ in (19), the principal's instantaneous probability of investment at $s \geq \widehat{s}$ is constant and equal to

$$\widehat{q} = \gamma \frac{\widehat{x}}{1-\widehat{x}} + \widehat{x}[\mu_G \widehat{a} + \mu_B(1-\widehat{a})]. \tag{25}$$

We show in Appendix C that the equilibrium with positive investment is unique when $\mu_G - \mu_B < 0$.

Proposition 7. *Consider a setting with good signals and bad signals about the agent's performance, where the arrival rates given attention θ^H and effort a are $\mu_G a$ and $\mu_B(1-a)$ respectively. If $\mu_G - \mu_B > 0$, equilibria are as characterized in Proposition 2. If $\mu_G - \mu_B < 0$, the equilibrium with positive investment is generically unique. This equilibrium has effort $a_s = 1$ and no investment if the time that has passed since a signal is $s < \widehat{s}$, and constant effort $\widehat{a} \in (0,1)$ and investment $\widehat{q} \in (0,\infty)$ if $s \geq \widehat{s}$. The probability of a signal conditional on a high attention technology is increasing in s.*

Proof. See Appendix C. *Q.E.D.*

Figure 4 illustrates the equilibrium for the same parameter values used in Figure 1, setting $\mu_G = 0 < \mu_B = \mu$. We stress two important differences. First, the equilibrium is essentially static in the bad news case — i.e., when $\mu_G - \mu_B < 0$ so monitoring is predominantly of bad performance — while it is dynamic in the good news case — i.e., when $\mu_G - \mu_B > 0$ so monitoring is predominantly of good performance. Second, in the bad news case, the probability of a signal when performance is low is constant and higher than for higher levels of performance.[23] Hence, in the bad news case, one cannot speak of an engagement trap: the probability of a signal is independent of s for $s \geq \widehat{s}$. If we define an engagement trap as a situation where effort is $a_s < 1$ and the probability of a signal $(x_s[\mu_G a_s + \mu_B(1-a_s)])$ is decreasing, we can state:

Corollary 2. *An engagement trap is present only in industries with a good news technology (i.e. where $\mu_G - \mu_B > 0$).*

[23]The figure has $\mu_G = 0$. If $\mu_G > 0$, the probability of a signal may first decrease with s (as x_s decreases with s), but it must become increasing in s at some $s' \in (0, \widehat{s})$ if $\mu_G - \mu_B < 0$.

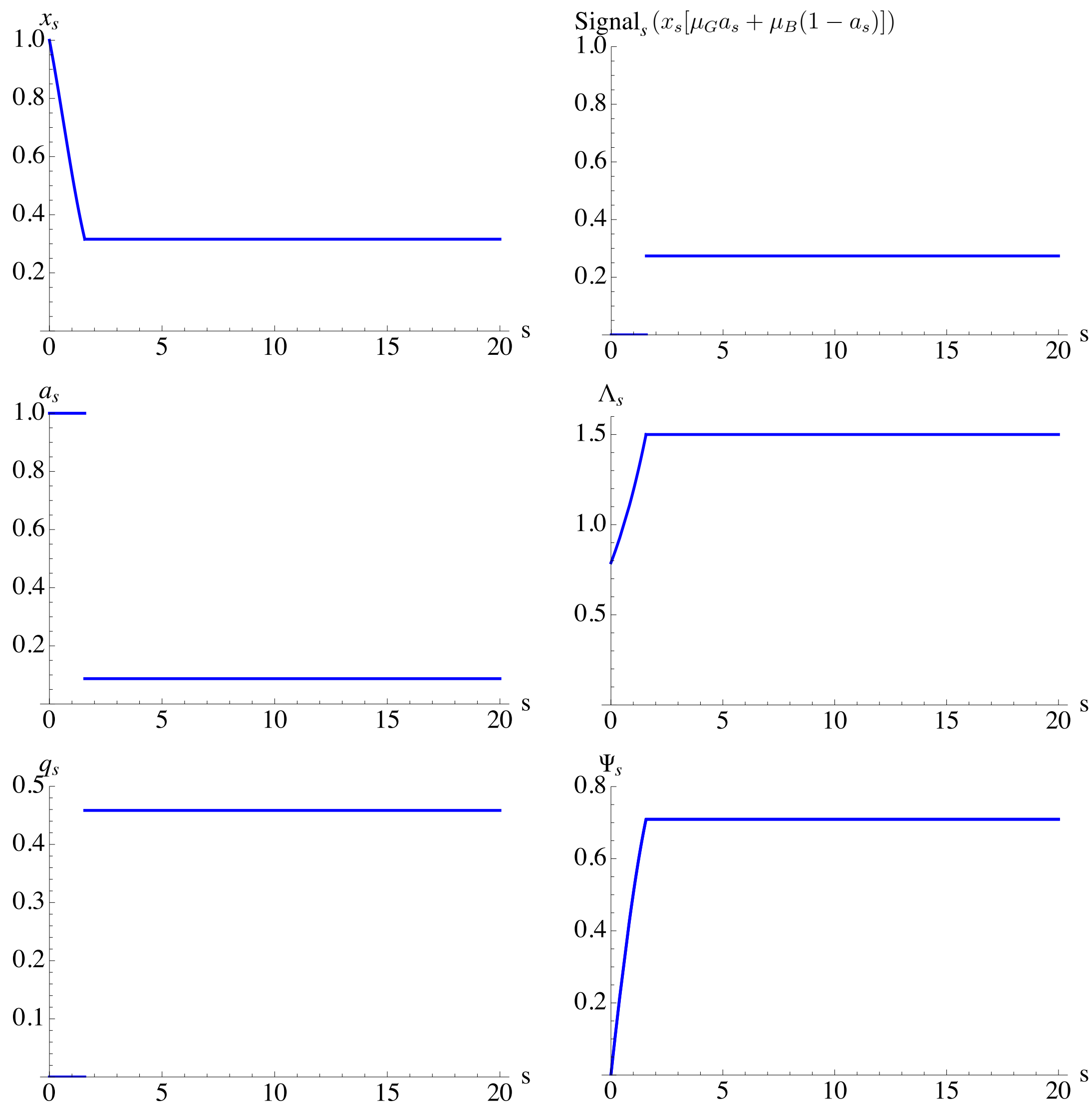

Figure 4: Equilibrium with a bad news technology.

As mentioned in the Introduction, an example of a good news attention technology is continuous process innovation. In general, any innovation-driven company requires good news monitoring because the verifiable event is the presence of something positive — an innovation. Here no news is bad news. Instead, a bad news attention technology is more likely to be found in companies where employees are required to perform well-defined tasks — like maintenance — and a verifiable event is the presence of something negative — like a fault. Then no news is good news. Corollary 2 predicts that an industry based on a bad news technology is less likely to end in an engagement trap. The logic is intuitive: while a good news technology becomes less useful when engagement and effort go down, the manager's incentive to invest in a bad news technology increases, as it is then when this technology is most effective to detect the agent's shirking.

5 Discussion

We consider a number of extensions and variants of our model. We find that our main qualitative results remain unchanged unless the agent is able to observe or infer the principal's attention technology perfectly.

5.1 Unknown attention technology

As discussed in the Introduction, a manager may be unable to perfectly assess the quality of her management practices. Suppose that in our model the principal does not observe whether her attention technology is high or low. We show that the continuous equilibrium that we characterized is an equilibrium of this modified model for an adjusted cost of investment:

Proposition 8. *Consider the setup of Section 1 but where θ_t is unobservable by both the principal and the agent. If all parameters are unchanged except that the cost of managerial investment is now $F' \equiv (1 - \overline{x})F$, the equilibrium characterized in Proposition 2 is also an equilibrium of this modified game.*

Proof. See Appendix D. *Q.E.D.*

The intuition is straightforward: the only difference when the principal is unable to observe θ_t is that when she invests, she may be investing in an unbroken attention technology. This occurs with probability x_s and implies that fixing the technology is now more expensive — the principal wastes $x_s F$ in expectation when she invests. Hence, by reducing the cost of investment to F', the principal's incentives to invest in the equilibrium are kept unchanged: she has no incentives to invest at any time $s < \overline{s}$ (since $x_s > \overline{x}$ and thus $F' > (1 - x_s)F$ at all such times), and she is indifferent between investing and not investing at all times $s \geq \overline{s}$ (where $x_s = \overline{x}$ and thus $F' = (1 - x_s)F$).

5.2 Endogenous bonus

Our model made two assumptions about the recognition bonus b: it is paid not by the principal but by some external, unmodeled party and its value is set exogenously. The first assumption was made to focus on the moral hazard due to the principal's cost of investment and abstract from another source of moral hazard: if the principal pays for b, she has an additional reason not to invest in attention technology, as she can save on the expected bonus payment. The second assumption was a logical consequence of the first: the value of the bonus cannot be endogenous unless we model the preferences of the party who sets it.

Removing the first assumption while keeping the second one would make the model less tractable without significantly changing the analysis. Instead, removing both assumptions may lead to different dynamics as the principal could use the size of the bonus b as a way to boost engagement. While we do not provide a full solution to this case here, we can show a negative result: endogenizing the bonus does not fully eliminate low engagement and inefficiency.

Proposition 9. *Consider the setup of Section 1 but where at each time t, b_t is chosen by the principal and subtracted from the principal's payoff if recognition occurs at t. An equilibrium with efficient effort $a_t = 1$ at all t does not exist.*

Proof. See Appendix D. *Q.E.D.*

To see the intuition, let s be the amount of time since recognition. Suppose for contradiction that the agent's effort is $a_s = 1$ at all s. Then the principal does not invest, as she receives the largest possible payoff, $1/r$, when her attention technology is low and she bears no investment nor bonus costs. Moreover, note that the principal cannot signal her type through the bonus offer: the low type can always mimic the high type at no cost because she does not pay the bonuses. This implies that for any bonus sequence, given no investment, the agent's engagement goes down as time passes without recognition, approaching $x_s = 0$ in the limit. For the agent to exert effort, the bonus b_s must increase fast enough so that $\mu x_s b_s \geq c$ is still satisfied. However, as s increases, this condition requires that the bonus become arbitrarily large, and a high principal type is not willing to make such an offer: the gain from offering b_s is no larger than $1/r$, while the cost is proportional to b_s as the high type has to pay the bonuses if recognition occurs before her attention technology breaks. Therefore, the agent shirks for x_s low enough, and an efficient equilibrium does not exist.

5.3 Discontinuous equilibria

Our analysis restricted attention to continuous equilibria, where the agent's belief cannot jump in the absence of publicly observable events. We can show that in any stationary discontinuous equilibrium where effort does not go to zero in the long run, the agent alternates between shirking and working.

To see this, let s be the time since recognition. We know that effort cannot be $a_s = 1$ at all s, as the principal would not invest. We can further show that a discontinuous equilibrium with $a_s > 0$ at all s does not exist. Intuitively, this equilibrium would have a time s' such that the agent's belief jumps to some level $x' > \overline{x}$ at s' without recognition, it eventually declines and reaches $\overline{x}$ at some time $s' + \Delta$, $\Delta > 0$, and it stays at $\overline{x}$ for some period of time $[s' + \Delta, s'']$ (during which $a_s < 1$). But since the principal cannot be indifferent between investing and not investing while the agent's effort is constant at one, this requires that for some $\delta \in (0, \Delta)$ the value of investing satisfies $\Lambda_{s'+\delta} = F$, $\Lambda_s < F$ for $s \in (s' + \delta, s' + \Delta)$, and $\Lambda_{s'+\Delta} = F$, which cannot occur given $a_s = 1$ for $s \in [s' + \delta, s' + \Delta]$.[24] It thus follows that any discontinuous equilibrium has zero

[24]The reasoning is similar to the one used in Claim 1 of the proof of Proposition 3. The principal must have strict incentives not to invest over a period $(s' + \delta, s' + \Delta)$ during which

effort for some period of time, and unless shirking is an absorbing state, it must be followed by a jump in the agent's belief so that a new working period starts.

Therefore, the path for effort in discontinuous equilibria is such that the relationship completely shuts down for some period of time and then suddenly re-starts work at a specific, coordinated date. Without an observable event at the time of re-start, this path appears unrealistic. Future work may explore the role of cathartic corporate events such as a change of leadership in triggering these dynamics.

5.4 Many agents

Our model considered a principal and a single agent. Suppose instead that there are multiple agents. Does the engagement trap still arise?

Suppose that there are n identical agents who observe recognition of each other perfectly. We can show that the one-agent problem with parameters $\{\mu, b, c, \gamma, F\}$ is equivalent to this n-agent problem with parameters $\{\tilde{\mu}, \tilde{b}, c, \gamma, \tilde{F}\}$, where

$$\tilde{\mu} = \frac{\mu}{n}, \ \tilde{b} = nb, \ \text{and } \tilde{F} = nF. \tag{26}$$

To see the equivalence, let s be the time that has passed since recognition of any one agent occurred. When recognition occurs, all agents learn that the principal's attention technology is high and their beliefs jump to one. The instantaneous probability that at least one agent is recognized at s is simply $n\tilde{\mu}a_s$. Thus, the law of motion for the agents' beliefs is

$$\dot{x}_s = -\gamma x_s - x_s\left(1 - x_s\right) n\tilde{\mu}a_s + \left(1 - x_s\right) q_s,$$

which is the same as that in the one-agent case. Note also that each agent's effort decision at any time s is determined by $\tilde{\mu}x_s\tilde{b} \geq c$, which is equivalent to the one-agent condition $\mu x_s b \geq c$. Finally, the principal's benefits and costs are simply n times those in the one-agent case, so her investment decision is also the same.

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the belief declines towards $\overline{x}$. But given $a_s = 1$ for $s \in [s' + \delta, s' + \Delta]$, the principal's incentives to invest at $s' + \delta$ and $s' + \Delta$ imply that she wants to invest at $s \in (s' + \delta, s' + \Delta)$.

This equivalence shows that, given a set of parameters, the analysis with n agents who observe recognition perfectly is analogous to that with one agent but where the arrival rate of recognition, μ, is larger. That is, agents' inferences of the principal's attention technology become more precise as the number of agents increases. We obtain that in the limit, as n and thus μ approaches infinity, the principal's attention technology becomes effectively observable, and consequently the principal can be induced to always maintain high attention.

However, in order for full knowledge to obtain, in practice we need four conditions to be satisfied: (a) the attention technology is the same for all agents; (b) the number of agents is arbitrarily large; (c) each agent observes recognition of an unbounded number of agents; (d) each agent observes the behavior of an unbounded number of those agents. As we argued in the Introduction, the reason why the principal cannot "fake" recognition is that she must be able to provide details about the agent's positive behavior which only the agent knows. With multiple agents, this requires that an agent observe the behavior of the other agents. We believe (d) is unlikely to hold in practice as it is based on direct interaction.

5.5 Forward-looking agent

We assumed throughout that the agent is myopic. The presence of a forward-looking player and a myopic one makes the analysis tractable and is in line with the literature on reputation.

However, it is also interesting to consider what would happen if the agent is forward-looking. A forward-looking agent would benefit from experimentation. The agent benefits from knowing in the future whether the principal's attention technology is working, and thus he is willing to exert effort in the present even when this yields a negative myopic payoff. Let s be the time since recognition, so that $x_s = 1$ for $s = 0$, and denote by U_s the forward-looking agent's expected payoff at s. Note that

$$U_0 = \int_0^\infty e^{-\int_0^s (x_\tau \gamma + r) d\tau} a_s \left[\mu x_s (b + U_0 - U_s) - c\right] ds.$$

Hence, an optimal strategy for the agent is to exert effort if and only if

$$\mu x_s(b + U_0 - U_s) \geq c.$$

This contrasts with the myopic-agent case, where the agent exerts effort if and only if $\mu x_s b \geq c$. Clearly, since $U_0 > U_s$, the level of engagement x_s that makes the agent indifferent between working and shirking is now lower. Note that $U_0 - U_s$ is bounded, and hence there exists a value $x > 0$ such that if engagement falls below x, the agent will shirk for sure.[25]

5.6 Other signals of managerial attention

Besides recognition, an agent may be able to observe other signals of the principal's attention technology. If the arrival rate of these other signals depends on the agent's effort but the signals do not carry a payment to the agent, we can just treat them as recognition where the expected bonus for recognition is now smaller. If the arrival rate of the signals is independent of the agent's effort, on the other hand, then our engagement trap result is weakened, as the existence of these signals gives firms the possibility of jumping up to high engagement and performance regardless of their current performance level. However, insofar as the principal's attention technology is not perfectly observable, our mechanism still plays a role and implies persistence in performance differences across firms.

6 Concluding remarks

This paper studied firms' ability to raise productivity by improving managerial practices and increasing worker engagement. We considered a dynamic two-sided moral hazard problem where the worker chooses how much effort to exert and the manager chooses how much to invest in an attention technology that can recognize good worker performance. We showed that persistent performance differences among ex-ante identical firms arise in equilibrium: the relationship falls

[25] One can combine the agent's strategy above with the principal's problem and obtain a system of differential equations. The problem is more complicated than with a myopic agent because it involves an additional state variable U_s.

into an engagement trap where both worker effort and managerial investment decrease and a return to high productivity becomes less likely as time passes. This engagement trap provides a partial solution to the manager's moral hazard problem, as it implies that a manager who does not invest faces decreasing engagement and productivity. Signaling devices such as hiring a consulting firm can allow managers to maintain high productivity, but they may reduce total welfare. Finally, by contrasting monitoring systems based on reward and punishment, we offered predictions on the types of industries or occupations where performance differences across firms are likely to be more persistent.

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Appendix

A Proof of Proposition 2

We first derive the solution to (13)-(15). Let $\Pi_s \equiv -1 + r\pi_s^L$, so $\dot{\Pi}_s = r\dot{\pi}_s^L$. We rewrite (13)-(15) as a system of homogeneous first-order linear differential equations:

$$\begin{aligned}
\dot{\Lambda}_s &= (\gamma + r)\Lambda_s - \mu\Psi_s, \\
\dot{\Psi}_s &= -(\gamma + r)\Lambda_s + \mu\Psi_s - \Pi_s,
\end{aligned}$$

$$\dot{\Pi}_s = r\Pi_s.$$

In matrix form,

$$\begin{bmatrix} \dot{\Lambda}_s \\ \dot{\Psi}_s \\ \dot{\Pi}_s \end{bmatrix} = \begin{bmatrix} \gamma + r & -\mu & 0 \\ -\gamma - r & \mu & -1 \\ 0 & 0 & r \end{bmatrix} \begin{bmatrix} \Lambda_s \\ \Psi_s \\ \Pi_s \end{bmatrix}.$$

The system has three distinct real eigenvalues, $r+\gamma+\mu$, 0, and r, with respective associated eigenvectors so that the solution is

$$\begin{bmatrix} \Lambda_s \\ \Psi_s \\ \Pi_s \end{bmatrix} = C_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{(r+\gamma+\mu)s} + C_2 \begin{bmatrix} \frac{\mu}{r+\gamma} \\ 1 \\ 0 \end{bmatrix} + C_3 \begin{bmatrix} -\frac{\mu}{r\gamma+r\mu} \\ -\frac{\gamma}{r\gamma+r\mu} \\ 1 \end{bmatrix} e^{rs}, \quad (27)$$

where C_1, C_2, and C_3 are constants for which we solve using the boundary conditions: $\Psi_0 = 0$; $\Lambda_{\overline{s}} = F$; and $\Psi_{\overline{s}} = \frac{(\gamma+r)F}{\mu}$. These conditions imply

$$\begin{aligned} C_1 + C_2 - C_3 \frac{\gamma}{r\gamma + r\mu} &= 0, \\ -C_1 e^{(r+\gamma+\mu)\overline{s}} + C_2 \frac{\mu}{r+\gamma} - C_3 \frac{\mu}{r\gamma + r\mu} e^{r\overline{s}} &= F, \\ C_1 e^{(r+\gamma+\mu)\overline{s}} + C_2 - C_3 \frac{\gamma}{r\gamma + r\mu} e^{r\overline{s}} &= \frac{(\gamma+r)F}{\mu}. \end{aligned}$$

Solving for C_1, C_2, and C_3 and recalling that $\Pi_s = -1 + r\pi_s^L$, we obtain

$$\begin{aligned} \Lambda_s &= \frac{F\left[(\gamma+r)(\gamma+r+\mu)e^{\overline{s}(\gamma+r+\mu)+rs} - r(\gamma+r)e^{s(\gamma+r+\mu)+r\overline{s}} - \gamma(\gamma+r+\mu)e^{\overline{s}(\gamma+r+\mu)} - r\mu e^{r\overline{s}}\right]}{(\gamma+r)(\gamma+\mu)e^{\overline{s}(\gamma+2r+\mu)} - \gamma(\gamma+r+\mu)e^{\overline{s}(\gamma+r+\mu)} - r\mu e^{r\overline{s}}}, \\ \Psi_s &= \frac{F(\gamma+r)\left[r\mu e^{s(\gamma+r+\mu)+r\overline{s}} + \gamma(\gamma+r+\mu)e^{\overline{s}(\gamma+r+\mu)+rs} - \gamma(\gamma+r+\mu)e^{\overline{s}(\gamma+r+\mu)} - r\mu e^{r\overline{s}}\right]}{\mu\left[(\gamma+r)(\gamma+\mu)e^{\overline{s}(\gamma+2r+\mu)} - \gamma(\gamma+r+\mu)e^{\overline{s}(\gamma+r+\mu)} - r\mu e^{r\overline{s}}\right]}, \\ r\pi_s^L &= 1 - \frac{Fr(\gamma+r)(\gamma+\mu)(\gamma+r+\mu)e^{\overline{s}(\gamma+r+\mu)+rs}}{\mu\left[(\gamma+r)(\gamma+\mu)e^{\overline{s}(\gamma+2r+\mu)} - \gamma(\gamma+r+\mu)e^{\overline{s}(\gamma+r+\mu)} - r\mu e^{r\overline{s}}\right]}. \end{aligned}$$

Note that $\dot{\Lambda}_s > 0$ and $\dot{\Psi}_s > 0$ for all $s < \overline{s}$. Given the boundary condition

$\Lambda_{\overline{s}} = F$, this implies $\Lambda_s < F$ for $s < \overline{s}$. Note also that $r\pi_s^L < 1$ and $\dot{\pi}_s^L < 0$ for all $s \leq \overline{s}$. To ensure $r\pi_s^L > 0$ for all $s \leq \overline{s}$, since π_s^L is decreasing, it suffices to ensure that $r\pi_{\overline{s}}^L > 0$:

$$r\pi_{\overline{s}}^L > 0 \iff F < \frac{\mu\left[(\gamma+r)(\gamma+\mu)e^{\overline{s}(\gamma+2r+\mu)} - \gamma(\gamma+r+\mu)e^{\overline{s}(\gamma+r+\mu)} - r\mu e^{r\overline{s}}\right]}{r(\gamma+r)(\gamma+u)(\gamma+r+\mu)e^{\overline{s}(\gamma+2r+\mu)}} \equiv \overline{F},$$

where $\overline{F} > 0$. Substituting $\overline{s}$ from (3),

$$\overline{F} = \frac{\mu\left[c\left((\gamma+r)\left(\frac{\mu(b(\gamma+\mu)-c)}{c\gamma}\right)^{\frac{r}{\gamma+\mu}+1} + \mu\right) - b\mu(\gamma+r+\mu)\right]}{cr(\gamma+r)(\gamma+r+\mu)\left(\frac{\mu(b(\gamma+\mu)-c)}{c\gamma}\right)^{\frac{r}{\gamma+\mu}+1}}.$$

Next, consider the system (9)-(10). By Peano's existence theorem, this system always has a solution. Note that given $\pi_{\overline{s}}^L \in \left(0, \frac{1}{r}\right)$ and $\Psi_{\overline{s}} = \frac{(\gamma+r)F}{\mu}$, $\dot{\Psi}_{\overline{s}} > 0$ and hence $\dot{\pi}_{\overline{s}}^L < 0$. Furthermore, it follows that $\dot{\Psi}_s \geq 0$ and $\dot{\pi}_s^L \leq 0$ for all $s > \overline{s}$. The reason is that if Ψ_s decreases at some point $s' > \overline{s}$, then $\dot{\Psi}_{s'} = 0$, but this implies $\dot{\pi}_{s'}^L = 0$ and hence Ψ_s and π_s^L remain constant from then on, leading to a contradiction. Lastly, integrating (9) and (10), we verify that $\pi_s^L \geq 0$ for all $s > \overline{s}$.

We thus conclude that the equilibrium of Proposition 2 exists if and only if $F < \overline{F}$ as shown above. We end by showing that the equilibrium dynamics are as described in the proposition. The path for the agent's belief x_s follows from the construction and the solution to (2) given $x_0 = 1$. Also by construction, $a_s = 1$ and $q_s = 0$ for $s < \overline{s}$. Consider now a_s for $s \geq \overline{s}$. Since Ψ_s is continuous, (11) implies that a_s is continuous for $s > \overline{s}$. As shown in the text, the boundary condition $\Psi_{\overline{s}} = \frac{(\gamma+r)F}{\mu}$ must hold and implies that a_s is also continuous at $\overline{s}$. Because a_s cannot be equal to one at all s (as the principal would not invest), we obtain that a_s decreases continuously from one at $\overline{s}$. Note further that a_s must be decreasing for all $s > \overline{s}$: if a_s increases at some point $\tilde{s} > \overline{s}$, then by (11) Ψ_s must decrease at $\tilde{s}$, but this contradicts the fact that Ψ_s is increasing for all $s \geq \overline{s}$ as shown above. Lastly, the path for investment, q_s, follows immediately from (12) and the path for the probability of recognition, $\mu x_s a_s$, follows from the paths for the agent's belief and effort.

Supplementary Appendix for Online Publication

This Online Appendix contains the formal proofs that are not included in the paper.

B Proofs for Section 2

B.1 Proof of Proposition 3

We proceed by proving three claims.

Claim 1: *The equilibrium of Proposition 2 is the unique continuous equilibrium where the agent's belief never falls strictly below* $\overline{x} = c/(\mu b)$.

First, it is straightforward that an equilibrium where x_s is always strictly above $\overline{x}$ does not exist: if $x_s > \overline{x}$ for all s, the agent always exerts full effort and thus the principal has no incentives to invest, but then x_s must fall below $\overline{x}$.

Second, we show that the principal cannot mix between investing and not investing continuously over a period $[s', s' + \delta]$ where $x_{s'} > \overline{x}$. Suppose by contradiction that she did. Then there is $\delta' \in (0, \delta]$ such that conditions (7) and (8) must hold for $s \in [s', s' + \delta']$ with $a_s = 1$. This implies $\Psi_s = (\gamma + r)F/\mu$ for $s \in [s', s' + \delta']$ and thus $\dot{\pi}^L_s = -\dot{\Psi}_s = 0$ for $s \in [s', s' + \delta']$. Now note that since $a_s = 1$ and without loss (given indifference) the principal does not invest at $s \in [s', s' + \delta']$, the principal's payoff π^L_s cannot be constant for $s \in [s', s' + \delta']$ if $a_s < 1$ at some $s > s' + \delta'$. But then a_s does not fall below one, which leads to a contradiction.

Finally, we show that the principal cannot have strict incentives to invest at a point s' unless the agent's belief eventually falls strictly below $\overline{x}$. Suppose by contradiction that the principal has strict incentives to invest at s' and the belief never falls strictly below $\overline{x}$. Strict incentives at s' implies that the belief is $x_{s'} = 1$. Since the belief cannot stay strictly above $\overline{x}$ forever and the principal cannot be indifferent for a period of time while the belief is strictly above $\overline{x}$, at some point the principal must have strict incentives not to invest and the belief must decrease towards $\overline{x}$. Moreover, if the belief never falls strictly below $\overline{x}$, it must stay at $\overline{x}$ when it reaches that level (the belief cannot jump up and it cannot increase continuously as that would require the principal to mix while the belief is strictly above $\overline{x}$). But then there must be s'', s''' such that the principal has strict incentives to invest for $s \in (s', s'')$, strict incentives not to invest for $s \in (s'', s''')$, and is indifferent for $s \geq s'''$. We show that this cannot occur in a continuous equilibrium.

To see this, note that for $s \geq s''$ we can use the expression for Λ_s in (6). Note that $\Lambda_{s''} = \Lambda_{s'''} = F$. As the principal has strict incentives not to invest for $s \in (s'', s''')$, this means that there exists an interval within (s'', s''') where $\dot{\Lambda}_s < 0$. As $a_s = 1$ for $s \in [s'', s''']$, the change in Λ_s over the interval is given by (13):

$$\dot{\Lambda}_s = (\gamma + r)\Lambda_s - \mu \Psi_s.$$

Furthermore, note that Ψ_s is increasing for $s \geq s''$. This means that if $\dot{\Lambda}_s < 0$ at some point within (s'', s'''), then $\dot{\Lambda}_s$ must continue to be negative in the whole interval. But that contradicts $\Lambda_{s''} = \Lambda_{s'''} = F$.

It follows from the arguments above that in any continuous equilibrium where the agent's belief never falls strictly below $\overline{x}$, the principal does not invest while the belief is strictly above $\overline{x}$ and the belief must stay at $\overline{x}$ forever when it reaches that level. The equilibrium must thus coincide with the equilibrium of Proposition 2.

Claim 2: *In any continuous equilibrium where the agent's belief falls strictly below $\overline{x}$ at some point, effort goes to zero in the long run.*

Suppose that the belief falls below $\overline{x}$ at a time s'. The principal cannot invest continuously over a period $[s', s' + \delta]$ where $x_{s'} < \overline{x}$: since $a_{s'} = 0$ and thus recognition cannot occur, the principal has strict incentives to not invest and wait until the belief reaches $\overline{x}$. This implies that once x_s falls below $\overline{x}$, x_s cannot increase continuously above $\overline{x}$, so there is a "breakup": the relationship goes into an absorbing state where the principal never invests and the agent never exerts effort. Since the relationship reaches point s' with strictly positive probability and this is an absorbing state, effort goes to zero in the long run.

Claim 3: *If $F < \overline{\overline{F}}$, the equilibrium of Proposition 2 is the unique continuous equilibrium.*

Step 1: Consider an equilibrium with no investment, where the belief falls from one governed by equation (2). The agent's effort is $a_s = 1$ for $s \in [0, \overline{s}]$ and $a_s = 0$ for $s > \overline{s}$, where $\overline{s}$ is given by (3). Call this path of effort $\tilde{a}$. Given $\tilde{a}$, the values of recognition and investment are $\Psi_s(\tilde{a}) \equiv \pi_0^H(\tilde{a}) - \pi_s^H(\tilde{a})$ and $\Lambda_s(\tilde{a}) \equiv \pi_s^H(\tilde{a}) - \pi_s^L(\tilde{a})$ respectively. This equilibrium with no investment exists if and only if $\Lambda_s(\tilde{a}) \leq F$ for all $s \in [0, \overline{s}]$, where

$$\Lambda_s(\tilde{a}) = \int_s^{\overline{s}} e^{-(r+\gamma)(\tau - s)} \mu \Psi_s(\tilde{a}) d\tau.$$

Since neither $\Lambda_s(\tilde{a})$ nor $\overline{s}$ are functions of F, it is immediate that this condition fails to hold if F is small enough. Indeed, we show that this condition is violated if $F < \overline{\overline{F}}$. For this, note that we can solve for $\Lambda_s(\tilde{a})$ by using the system of differential equations (13)-(15), where now the boundary conditions are $\Psi_0 = 0$, $\Lambda_{\overline{s}} = 0$, and $\pi_{\overline{s}}^L = 0$. The solution to the system is given by (27) (in the proof of Proposition 2), and using the new boundary conditions we obtain that for $s \leq \overline{s}$,

$$\Lambda_s(\tilde{a}) = \frac{\mu \Big[(\gamma + r) e^{rs} \left(e^{\overline{s}(\gamma+\mu)} - e^{s(\gamma+\mu)} \right) - e^{-r\overline{s}} \left[\gamma \left(e^{\overline{s}(\gamma+r+\mu)} - e^{s(\gamma+r+\mu)} \right) + \mu \left(e^{r\overline{s}} - e^{rs} \right) \right] \Big]}{r(\gamma+\mu) \left[(\gamma + r) e^{\overline{s}(\gamma+r+\mu)} + \mu \right]}. \tag{28}$$

For the principal to have no incentives to invest, we must have $F \geq \max_{s' \in [0, \overline{s}]} \Lambda_{s'}(\tilde{a})$, which is equivalent to $F \geq \overline{\overline{F}}$.

Step 2: As shown in Claims 1-2 above, in any continuous equilibrium that does not coincide with the equilibrium of Proposition 2, there is a time s', with $x_{s'} = \overline{x}$, at which the agent's belief falls strictly below $\overline{x}$ and the agent stops exerting effort forever. We show that there must be a time $s'' \equiv s' - \overline{s}$ such that $\overline{s}$ is given by (3), $x_{s''} = 1$, and the belief is governed by equation (2) between s'' and s' (i.e., the principal does not invest from s'' on). Suppose not. Then since the principal cannot be indifferent between investing and not investing while the belief is strictly above $\overline{x}$, the only other possibility is that the belief is constant at $x_s = \overline{x}$ for $s \in [s' - \delta, s']$ and some $\delta > 0$. Consider time $s' - \varepsilon$

for $\varepsilon \in (0, \delta)$. The principal's value of investing is

$$\Lambda_{s'-\varepsilon} = \int_{s'-\varepsilon}^{s'} e^{-(r+\gamma)(\tau-(s'-\varepsilon))} \mu a_\tau \Psi_\tau d\tau.$$

Since $\mu a_s \Psi_s$ is finite for any s, there exists $\varepsilon \in (0, \delta)$ small enough such that $\Lambda_s < F$ for $s \in [s' - \varepsilon, s']$. This means that the principal stops investing at $s' - \varepsilon$. But then the agent's belief falls below $\overline{x}$ at $s' - \varepsilon$, and using the same logic we can show that there exists $\varepsilon' \in (0, \delta - \varepsilon)$ small enough such that the principal wants to stop investing at $s' - \varepsilon - \varepsilon'$. Continuing with this reasoning gives that the principal cannot invest continuously over a period of time after which the agent's belief falls strictly below $\overline{x}$.

Step 3: From Step 2, in any continuous equilibrium that does not coincide with the equilibrium of Proposition 2, there is some time $\hat{s}$ such that the continuation equilibrium from $\hat{s}$ on coincides with the no investment equilibrium, i.e. the principal never again invests and the belief falls from one governed by equation (2). Note further that if $\hat{s} > 0$, then $x_s = 1$ for $s < \hat{s}$, since $x_{\hat{s}} = 1$ and the agent's belief cannot increase continuously above $\overline{x}$. This means that the principal has strict incentives to invest at $s < \hat{s}$, and the value of investment at $s' \in [\hat{s}, \hat{s} + \overline{s}]$ cannot be smaller than $\Lambda_{s'-\hat{s}}(\tilde{a})$. But then it follows from Step 1 that the continuation equilibrium from $\hat{s}$ on cannot be sustained if $F < \overline{\overline{F}}$. The claim follows.

B.2 Proof of Proposition 4

The comparative statics with respect to b, c, and γ are computed as follows:

$$\begin{aligned}
\frac{\partial \overline{s}(\gamma, \mu, b, c)}{\partial b} &= \frac{1}{b(\gamma+\mu) - c} \geq 0, \\
\frac{\partial \overline{s}(\gamma, \mu, b, c)}{\partial c} &= \frac{b}{c(c - b(\gamma+\mu))} \leq 0, \\
\frac{\partial \overline{s}(\gamma, \mu, b, c)}{\partial \gamma} &= -\frac{\frac{\mu}{\gamma} - \frac{c}{b(\gamma+\mu)-c} + log\left(\frac{\mu(b(\gamma+\mu)-c)}{c\gamma}\right)}{(\gamma+\mu)^2} \leq 0,
\end{aligned}$$

where we have used the fact that, by Assumption 1, $\mu b \geq c$.

The comparative static with respect to μ is given by

$$\frac{\partial\overline{s}(\gamma,\mu,b,c)}{\partial\mu} = \frac{2+\frac{\gamma}{\mu}+\frac{c}{b(\gamma+\mu)-c}-log\left(\frac{\mu(b(\gamma+\mu)-c)}{c\gamma}\right)}{(\gamma+\mu)^2}.$$

This expression is positive for low enough μ and negative for μ sufficiently high.

B.3 Proof of Proposition 5

As the discount rate r approaches zero, $r\pi_s^L$ becomes equal to the limit level of effort as s increases, $\underline{a}$. Using this, we simplify the system characterizing the solution for $s \geq \overline{s}$. This system becomes:

$$\begin{aligned}\dot{\Lambda}_s &= \gamma\Lambda_s - \mu\Psi_s,\\ \dot{\Psi}_s &= 1-\gamma\Lambda_s+\mu\Psi_s-\underline{a},\end{aligned}$$

with boundary conditions $\Psi_0 = 0$, $\Lambda_{\overline{s}} = F$. The solution is

$$\Lambda_s = \frac{\left\{\begin{array}{c}(1-\underline{a})e^{(\gamma+\mu)\overline{s}}\mu(1+\gamma s)+\mu[F(\gamma+\mu)+(1-\underline{a})\mu(s-\overline{s})]\\ +e^{(\gamma+\mu)s}[F\gamma(\gamma+\mu)-(1-\underline{a})\mu(1+\gamma\overline{s})]\end{array}\right\}}{(\gamma+\mu)(e^{(\gamma+\mu)\overline{s}}\gamma+\mu)},$$

$$\Psi_s = \frac{(1-e^{(\gamma+\mu)s})F\gamma(\gamma+\mu)+(1-\underline{a})\left\{e^{(\gamma+\mu)\overline{s}}\gamma^2 s-\mu[1-\gamma(s-\overline{s})-e^{(\gamma+\mu)s}(1+\gamma\overline{s})]\right\}}{(\gamma+\mu)(e^{(\gamma+\mu)\overline{s}}\gamma+\mu)}. \quad (29)$$

At $\overline{s}$, $\Psi_{\overline{s}} = F\gamma/\mu$. Substituting in (29) and solving for $\underline{a}$ gives

$$\underline{a} = \frac{\mu[\mu+\gamma(\gamma+\mu)\overline{s}]-F\gamma(\gamma+\mu)^2-\mu^2e^{-(\gamma+\mu)\overline{s}}}{\mu[\mu+\gamma(\gamma+\mu)\overline{s}]-\mu^2e^{-(\gamma+\mu)\overline{s}}}.$$

Since $\overline{s}$ is independent of F, one can immediately verify that $\underline{a}$ is decreasing in F, with $\underline{a}$ approaching one as F approaches zero and becoming negative for F sufficiently high. Substituting $\overline{s}$ from equation (3) into the expression above, we obtain:

$$\underline{a} = \frac{(\gamma+\mu)\left\{c[F\gamma(\gamma+\mu)-\mu]-b[F\gamma(\gamma+\mu)^2-\mu^2]\right\}+\gamma\mu[b(\gamma+\mu)-c]log\left(\frac{\mu(b(\gamma+\mu)-c)}{c\gamma}\right)}{\mu\left\{(\gamma+\mu)(b\mu-c)+\gamma[b(\gamma+\mu)-c]log\left(\frac{\mu(b(\gamma+\mu)-c)}{c\gamma}\right)\right\}}.$$

Differentiating $\underline{a}$ with respect to γ, b, and c yields

$$\begin{aligned}
\frac{\partial\underline{a}(F,\gamma,\mu,b,c)}{\partial\gamma} &= -\frac{F(\gamma+\mu)\left\{\begin{array}{c}(\gamma+\mu)(b\mu-c)[b(\gamma+\mu)(4\gamma+\mu)-c(3\gamma+\mu)]\\ +2\gamma^2[c-b(\gamma+\mu)]^2log\left(\frac{\mu(b(\gamma+\mu)-c)}{c\gamma}\right)\end{array}\right\}}{\mu\left\{(\gamma+\mu)(b\mu-c)+\gamma[b(\gamma+\mu)-c]log\left(\frac{\mu(b(\gamma+\mu)-c)}{c\gamma}\right)\right\}^2}\le 0,\\
\frac{\partial\underline{a}(F,\gamma,\mu,b,c)}{\partial b} &= \frac{bF\gamma^2(\gamma+\mu)^4}{\mu\left\{(\gamma+\mu)(b\mu-c)+\gamma[b(\gamma+\mu)-c]log\left(\frac{\mu(b(\gamma+\mu)-c)}{c\gamma}\right)\right\}^2}\ge 0,\\
\frac{\partial\underline{a}(F,\gamma,\mu,b,c)}{\partial c} &= -\frac{b^2F\gamma^2(\gamma+\mu)^4}{c\mu\left\{(\gamma+\mu)(b\mu-c)+\gamma[b(\gamma+\mu)-c]log\left(\frac{\mu(b(\gamma+\mu)-c)}{c\gamma}\right)\right\}^2}\le 0,
\end{aligned}$$

where we have used the fact that, by Assumption 1, $\mu b \ge c$.

Finally, the comparative static with respect to μ is given by

$$\frac{\partial\underline{a}(F,\gamma,\mu,b,c)}{\partial\mu} = \frac{F\gamma^2(\gamma+\mu)\left\{\begin{array}{c}(\gamma+\mu)\left[2c^2+b^2(\gamma+\mu)(\gamma+4\mu)-bc(3\gamma+5\mu)\right]\\ +(\gamma-\mu)[c-b(\gamma+\mu)]^2log\left(\frac{\mu(b(\gamma+\mu)-c)}{c\gamma}\right)\end{array}\right\}}{\mu^2\left\{(\gamma+\mu)(b\mu-c)+\gamma[b(\gamma+\mu)-c]log\left(\frac{\mu(b(\gamma+\mu)-c)}{c\gamma}\right)\right\}^2}.$$

This expression can be positive or negative depending on parameters.

C Proofs for Section 3 and Section 4

C.1 Proof of Proposition 6

The construction is as in the text. For $s < \overline{s}$, the solution is given by the system (13)-(15) with boundary conditions $\Psi_0 = 0$, $\Lambda_{\overline{s}} = F$, and $\Psi_{\overline{s}} = \frac{(\gamma+r)F}{\mu}$. Note that given $a_s = 1$ for all s and the payments m for $s \ge \overline{s}$, Ψ_s and Λ_s

are (weakly) increasing in s. The system then yields the values $\overline{\Psi}$ and $\overline{\pi}^L$ such that for $\Psi_{\overline{s}} = \overline{\Psi}$ and $\pi^L_{\overline{s}} = \overline{\pi}^L$ the principal has no incentives to invest before time $\overline{s}$ and is indifferent between investing and not investing at $\overline{s}$. The solution to the system and the values $\overline{\Psi}$ and $\overline{\pi}^L$ are as characterized in the proof of Proposition 2.

Note that for $s \geq \overline{s}$,

$$\pi^L_s = \frac{1-m}{r}.$$

Hence, setting $\pi^L_{\overline{s}} = \overline{\pi}^L$, we obtain that the cost of the public signal must be

$$m = 1 - r\overline{\pi}^L.$$

At each time $s \geq \overline{s}$, the principal must be indifferent between investing and not investing. The indifference conditions are (17) and (18) in the text. The boundary conditions and the fact that Ψ_s and Λ_s are constant for $s \geq \overline{s}$ imply that these conditions are satisfied. Using (1) (and given $a_s = 1$ for all s), we can then set the instantaneous probability with which the principal invests $q_s \in (0, \infty)$ to be such that $\dot{x}_s = 0$ and $x_s = \overline{x}$ for $s \geq \overline{s}$:

$$q_s = \gamma \frac{\overline{x}}{1-\overline{x}} + \overline{x}\mu.$$

The agent's belief on the equilibrium path is then $x_s > \overline{x}$ for $s < \overline{s}$ and $x_s = \overline{x}$ for $s \geq \overline{s}$; hence, the agent is willing to exert effort at all times so long as the principal pays m. If the principal does not pay m at some $s \geq \overline{s}$ (off the equilibrium path), the agent believes that the principal does not invest, so his belief falls from $\overline{x}$. Hence, if the principal does not pay m, the agent prefers to shirk and the principal indeed does not invest.

The last thing to check is that the principal is willing to pay the cost m at all times $s \geq \overline{s}$. If at any point $s \geq \overline{s}$ the principal does not pay m, the agent stops exerting effort forever. Note that the high type of principal prefers to pay m if the low type does, as the high type can obtain recognition and thus the value of the agent's effort is higher than for the low type. The low type prefers to pay m if and only if $\pi^L_s \geq 0$ for $s \geq \overline{s}$. This condition is satisfied if the value of $\overline{\pi}^L$ from the solution to (13)-(15) is positive, which must be true if the equilibrium of Proposition 2 exists.

Finally, consider the claims regarding efficiency. As shown, the solution for $s < \overline{s}$ is the same as in the continuous equilibrium without costly signaling. Hence, the principal's expected payoff — which does not take into account bonus payments — is also the same. As a result, if m is money burning and b is a transfer from the principal to the agent, efficiency goes down with money burning: the principal's payoff is unchanged while the agent bears the cost of effort more often so his payoff ignoring bonuses is lower. If m is money burning and b is not a transfer but rather an exogenous benefit that the agent receives, then efficiency increases with money burning: the principal's payoff is unchanged while the agent obtains recognition with higher probability and thus has strict incentives to exert effort more often. Lastly, if m is not money burning but a transfer to a third party whose payoff enters social welfare, efficiency increases since the first-best outcome is implemented when m is introduced.

C.2 Proof of Proposition 7

Assume first that $\mu_G - \mu_B > 0$. Note that for $s \leq \widehat{s}$, $a_s = 1$, so only the good signal can arrive. Furthermore, as shown in the text, when $\mu_G - \mu_B > 0$ the system for $s > \widehat{s}$ is qualitatively the same as that analyzed in the absence of bad signals. The construction of the equilibrium thus follows the same steps as those in the proof of Proposition 2, and we omit it. The proof of Proposition 3 can also be applied directly to this case. In particular, note that once the agent's belief falls strictly below $\widehat{x}$, it cannot increase continuously above $\widehat{x}$. To see this, note that if x_s falls strictly below $\widehat{x}$ at s', then $a_s = 0$ for $s \in [s', s' + \delta]$ and some $\delta > 0$, and for x_s to increase continuously above $\widehat{x}$, the principal must be indifferent between investing and not investing at $s \in [s', s' + \delta]$. However, since a_s is constant over $[s', s' + \delta]$, the principal's indifference conditions require that Ψ_s be constant over the interval too, which requires that a_s stay at zero in all future periods. Thus, $x_s < \widehat{x}$ for $s \geq s'$, and since the probability of reaching s' and managerial attention being low or becoming low after that point is strictly positive, the agent's effort goes to zero in the long run.

Assume next that $\mu_G - \mu_B < 0$. We proceed in three steps.

Step 1: We show that in any generic equilibrium with positive investment, there is no point s' such that $a_s = 0$ for all $s \geq s'$. Consider an equilibrium with such an effort profile, where s' is the point at which effort becomes zero forever. The principal cannot have strict incentives to invest at $s \geq s'$, as x_s and a_s would jump to one. If the principal is indifferent between investing and not investing, conditions (23) and (24) must hold for $\widehat{a} = 0$. Now note that the principal cannot have incentives to invest before s', since given $\mu_G - \mu_B < 0$, both the value of recognition and the probability of recognition conditional on high attention are lower than at $s \geq s'$ where the principal is indifferent. Then $s' = \widehat{s}$ and $\widehat{a} = 0$ and $\widehat{s}$ pin down the value of π_0^H as shown in (22). But then the case where (23) and (24) hold for such π_0^H and $\widehat{a} = 0$ is non-generic. Therefore, generically, the principal has strict incentives not to invest for $s \geq s'$ in the equilibrium where $a_s = 0$ for all $s \geq s'$. The equilibrium then cannot have positive investment. Suppose it does. Then the principal invests at some $s < s'$. But given $\mu_G - \mu_B < 0$, at any $s < s'$ the principal's value of investing is lower than at $s \geq s'$. Therefore, if the principal is willing to invest at $s < s'$, she must have incentives to invest at $s \geq s'$, a contradiction.

Step 2: We show that in any generic equilibrium with positive investment, there is no point s' at which the agent's belief falls strictly below $\widehat{x}$. Suppose by contradiction that such a point s' exists. Then $a_s = 0$ for $s \in [s', s' + \delta]$ and some $\delta > 0$, and the principal either is indifferent between investing and not investing or strictly prefers not to invest at $s \in [s', s' + \delta]$. But if the principal is indifferent, then a_s must be constant for all $s \geq s'$; otherwise Ψ_s is not constant over $s \in [s', s' + \delta]$ and, as shown in the text, the principal's indifference conditions cannot be satisfied. But then $a_s = 0$ for $s \geq s'$ and by Step 1 this equilibrium with indifference at $s \geq s'$ is non-generic. Hence, the principal must have strict incentives not to invest for $s \in [s', s' + \delta]$. Now note that again a_s cannot increase above zero at $s \geq s' + \delta$. Suppose it does. This means that there is a point $s \geq s' + \delta$ where $a_s \geq 0$ and the principal invests. Let s'' be the smallest such point. Then a_s is increasing over $[s', s'']$, and given $\mu_G - \mu_B < 0$ the principal's value of investing must be decreasing.

But this means that the principal has incentives to invest at some $s \in [s', s'+\delta]$, a contradiction. It follows that if the belief falls strictly below $\widehat{x}$ at a point s', then $a_s = 0$ for all $s \geq s'$, and by Step 1 the equilibrium has no investment.

Step 3: Consider a generic equilibrium with positive investment. Since the agent cannot exert effort with certainty at all times (as the principal would have no incentives to invest) and by Step 2 the agent's belief cannot fall strictly below $\widehat{x}$, there must be a point s' and $\delta > 0$ such that $x_s = \widehat{x}$ and $a_s < 1$ for $s \in [s', s'+\delta]$. But then the principal must be indifferent over $s \in [s', s'+\delta]$ and, as shown in Step 2, a_s must stay constant at $a_s = \widehat{a} \in (0, 1)$ for $s \geq s'$. Hence, the belief must stay at $x_s = \widehat{x}$ for all $s \geq s'$. Moreover, if s' is the point at which the agent's belief reaches $\widehat{x}$, then the principal's value of investing must be lower at all $s < s'$, so the principal's indifference for $s \geq s'$ implies that the principal does not invest at $s < s'$. Therefore, the belief must reach $\widehat{x}$ at time $s' = \widehat{s}$ and $x_s > \widehat{x}$ for $s < \widehat{s}$. It follows that the generically unique equilibrium with positive investment is as characterized in Proposition 7.

D Proofs for Section 5

D.1 Proof of Proposition 8

The principal does not know whether θ_t is high or low. At any point s in the putative equilibrium, she shares the agent's posterior belief, $x_s = \Pr\left(\theta_s = \theta^H\right)$.[26] The expressions for π_s^L and π_s^H are unchanged, and hence Λ_s is unchanged too. However, the principal cannot condition her investment strategy on θ and thus the benefit of investing at s is no longer Λ_s but rather $(1 - x_s)\,\Lambda_s$. The principal is willing to invest if and only if

$$(1 - x_s)\,\Lambda_s \geq F'.$$

To see that the equilibrium of Proposition 2 is still an equilibrium given an investment cost F', we proceed by considering behavior before and after $\overline{s}$.

[26] Recall that the principal's strategy has no time correlation, so the fact that she uses a mixed strategy does not generate asymmetric information.

Suppose first that behavior before $\bar{s}$ is the same as in Proposition 2 and analyze what happens for $s \geq \bar{s}$. Taking into account that $(1-\bar{x})\,\Lambda_s = F'$ for any such s, the system of differential equations is

$$\begin{aligned}\dot{\Psi}_s &= \frac{(\gamma + r)\frac{F'}{1-\bar{x}}}{\mu \Psi_s} - r\pi_s^L, \\ \dot{\pi}_s^L &= -\dot{\Psi}_s.\end{aligned}$$

Hence, replacing $F' \equiv (1-\bar{x})F$ yields the same system used to obtain Proposition 2.

Consider next $s < \bar{s}$. Here the system of differential equations, which does not contain the cost F, is unchanged. We thus only need to check whether the condition $(1-x_s)\,\Lambda_s \leq F'$ is satisfied. Now note that we can rewrite this condition as

$$\frac{1-x_s}{1-\bar{x}}\Lambda_s \leq F. \tag{30}$$

Since $\frac{1-x_s}{1-\bar{x}} \leq 1$ and we know from the proof of Proposition 2 that $\Lambda_s \leq F$ for any $s < \bar{s}$, it follows that (30) is satisfied for any such s. The proof is complete.

D.2 Proof of Proposition 9

Assume that at each time t, the bonus b_t is chosen by the principal and subtracted from the principal's payoff if recognition occurs at t. Hence, the principal's expected flow payoff at t depends on her type as follows: it is $a_t - (1-e^{-q_t})F$ if $\theta_t = \theta^L$ and $a_t\,(1-\mu b)$ if $\theta_t = \theta^H$.

Let s be the time that has passed since recognition. Suppose for contradiction that there exists an equilibrium where $a_s = 1$ for all s. We first show that this equilibrium cannot have positive investment. To see this, note that the expected payoff to a low principal type who never invests is $\pi_s^L = 1/r$, since she never bears the cost of investment nor makes bonus payments (because she never produces recognition). But this is the largest payoff that the principal can receive, and it is thus superior to any payoff stream that involves investing at cost F with some positive probability. It follows that the principal will not invest.

Next, note that the principal cannot signal her type through her bonus offer. The reason is that the low type never pays the bonuses because she never produces recognition, and thus she can mimic the high type at no cost. Given no investment, this implies that for any bonus sequence, the agent's engagement goes down as time passes without recognition, governed by equation (2):

$$\dot{x}_s = -\gamma x_s - x_s(1 - x_s)\mu.$$

For the agent to exert effort $a_s = 1$ at all s, the bonus sequence must be such that the agent's incentive constraint, $\mu x_s b_s \geq c$, is satisfied at all s. Let $\{\hat{b}_s\}_{s \geq s'}$ be a sequence of bonuses from s' on such that the agent's incentive constraint is satisfied at all $s \geq s'$. By the law of motion above, x_s approaches zero as s becomes large enough, and so the agent's incentive constraint requires that the bonuses increase unboundedly. However, note that the high type's benefit from offering such arbitrarily large bonuses is no more than $1/r$ — this is the benefit if the agent stops exerting effort forever when the principal fails to offer the prescribed bonuses and the principal bears no bonus costs. On the other hand, the high type's cost of offering a sequence $\{\hat{b}_s\}_{s \geq s'}$ is bounded below by

$$\int_{s'}^{\infty} e^{-(\gamma+\mu+r)(s-s')} \mu \hat{b}_s ds,$$

since the high type must pay the bonus $\hat{b}_s$ if recognition occurs at $s \geq s'$ before her attention technology breaks. Because the benefit is no more than $1/r$ while the cost is proportional to the bonuses $\hat{b}_s$ which increase unboundedly as s increases, it follows that there is a finite time s' sufficiently large such that the cost is larger than the benefit. Thus, at such time s', the high type is not willing to offer bonuses large enough to keep the agent exerting effort at all times, and the agent starts shirking. This gives us the contradiction.