RECURSION THEOREM

Suppose that the subset C of U is freely generated from B by two functions f and g such that

$f : \mathrm{U} \times \mathrm{U} \to \mathrm{U}$
$g : \mathrm{U} \to \mathrm{U}$

Further, let V be any set, and let F and G be corresponding functions such that

$F : V \times V \to V$
$G : V \to V$

Then, given any map

$h : B \to V$

there exists a unique map

$\bar{h} : C \to V$

such that, for all $x,y \quad C$:

(i) if $x \quad B$, then $\bar{h}(x) = h(x)$
(ii) $\bar{h}(g(x)) = G(\bar{h}(x))$
(iii) $\bar{h}(f(x,y)) = F(\bar{h}(x), \bar{h}(y))$

[In other words, any “basic” map h of B into V can be extended to a unique homomorphism $\bar{h}$ from C (with operations f and g) into V (with operations F and G)]

PROOF

Call a function *acceptable* if it maps a subset of C into V in agreement with the desired conditions imposed on $\bar{h}$. That is, a function h_i is acceptable iff there exists a subset D_i of C such that

$h_i : D_i \to V$

and the following conditions hold for all $x,y \quad C$

(i') If $x \quad B$ and also $x \quad D_i$, then $h_i(x) = h(x)$
(ii') If $g(x) \quad D_i$, then also $x \quad D_i$ and $h_i(g(x)) = G(h_i(x))$
(iii') If $f(x,y) \quad D_i$, then also $x,y \quad D_i$, and $h_i(f(x,y)) = F(h_i(x), h_i(y))$.

Now let

$\bar{h} = \quad \{h_i : h_i \text{ is an acceptable function}\}$.

That is, for all $x \quad C$ and all $v \quad V$, we set

$x,v) \quad \bar{h}$ iff $h_i(x)=v$ for some acceptable h_i

(recall that every function is a set of ordered pairs). We now show that $\bar{h}$ thus defined meets our requirements. This will involve four main steps:

(a) *Show that* $\overline{h}$ *is a function on* C
(b) *Show that* $\overline{h}$ *is acceptable*
(c) *Show that* $\overline{h}$ *is defined throughout* C
(d) *Show that* $\overline{h}$ *is unique*

Step (a): *Show that* $\overline{h}$ *is a function on* C

This amounts to establishing three facts:

(a.1) $\overline{h}$ *is a set of ordered pairs*. This follows immediately from the definition of $\overline{h}$ as the union of a set of functions: by set theory, the union of any set of sets ordered pairs is in turn a set of ordered pairs.

(a.2) *The domain of* $\overline{h}$ *is included in* C*, i.e.,* x,y $\overline{h}$ *only if* x C*.* This also follows immediately from the definition using set theory: the domain of $\overline{h}$ is the union of the domains of the acceptable functions, and these domains are all subsets of C. By set theory, the union of any set of subsets of a given set C is always sure to be a subset of C.

(a.3) $\overline{h}$ *is functional, i.e., for every* x *dom(*$\overline{h}$*) we have at most one* y *so that* x,y $\overline{h}$. This can be proved by induction. Let

$$S=\{x \quad C\text{: for at most one } y,\ x,y \quad \overline{h}\}.$$

We now show that S is inductive. As always, this involves two steps:

(a.3.1) *Base step: Show that* B *is included in* S. To see this, pick any x B and note that $h_i(x) = h(x)$ for every acceptable function h_i (by definition of “acceptable”). Thus, we have

$$x,y \quad \overline{h} \text{ iff } y=h(x)$$

which entails that x S. Since x was arbitrarily chosen, this holds for *every* element of B, i.e., B is included in S.

(a.3.2) *Inductive step: Show that* S *is closed under* g *and* f.

(a.3.2.1) Suppose that x S — Show that $g(x)$ S. We distinguish two cases:

(a.3.2.1.1) If x dom($\overline{h}$), then x dom(h_i) for all acceptable h_i’s

So $g(x)$ dom(h_i) for all such h_i (by clause (ii') in def.)
hence $g(x)$ dom($\overline{h}$) (since dom($\overline{h}$) is the union of the dom(h_i)’s)
hence $g(x)$ S (vacuously)

(a.3.2.1.2) If x dom($\overline{h}$), then suppose $g(x),y$ and $g(x),z$ are both in $\overline{h}$. Then there exist acceptable functions h_i and h_j such that

$$h_i(g(x)) = y = G(h_i(x))$$
$$h_j(g(x)) = z = G(h_j(x))$$

But since x S, we must have

$$h_i(x) = h_j(x) = \overline{h}(x)$$

Hence we have

$$y = G(h_i(x)) = G(h_j(x)) = z$$

Thus, the pairs $g(x),y$ and $g(x),z$ must be identical, i.e., $\overline{h}$ assigns exactly one value to $g(x)$. Hence $g(x)$ S

(a.3.2.2) Suppose x,y S — Show $f(x,y)$ S. This case is perfectly similar to case (a.3.2.1) above.

We therefore conclude that S is inductive, and by the induction principle we infer that $S=C$. Since dom($\overline{h}$) is included in C, the desired result follows: $\overline{h}$ is indeed a function.

Step (b): *Show that $\overline{h}$ is acceptable*

This follows immediately from the definition of $\overline{h}$, given that it is a function on C. For, corresponding to the three defining clauses (i')–(iii'), we can reason as follows:

(b.1) If $x \quad B$, then surely $x \quad \mathrm{dom}(h_i)$ for some acceptable function h_i, hence $x \quad \mathrm{dom}(\overline{h})$. Moreover, every such function h_i agrees on the value $h_i(x)=h(x)$; hence we also have $\overline{h}(x) = h(x)$, as desired.

(b.2) If $g(x) \quad \mathrm{dom}(\overline{h})$, then $g(x) \quad \mathrm{dom}(h_i)$ for some acceptable function h_i. By (ii') it follows that $x \quad \mathrm{dom}(h_i)$ and $h_i(g(x)) = G(h_i(x))$. Since $h_i \quad \overline{h}$, it follows that $x \quad \mathrm{dom}(\overline{h})$ and $\overline{h}(g(x)) = G(h_i(x)) = G(\overline{h}(x))$, as desired.

(b.3) The case for $f(x,y)$ is perfectly similar.

Thus, $\overline{h}$ is a function on C satisfying clauses (i')–(iii'), i.e., $\overline{h}$ is acceptable.

Step (c): *Show that $\overline{h}$ is defined throughout C.*

To this end it will suffice to show that the domain of $\overline{h}$ is inductive: the induction principle will then ensure that it coincides with C.

(c.1) *Base step: Show that B is included in $\mathrm{dom}(\overline{h})$.* This is obvious: for each $x \quad B$ there is sure to be an acceptable function defined (at least) on x. Since $\overline{h}$ is the union of *all* such functions, it will be defined for *every* $x \quad$ B.

(c.2) *Inductive step: Show that $\mathrm{dom}(\overline{h})$ is closed under g and f.*

(c.2.1) Suppose that $x \quad \mathrm{dom}(\overline{h})$ — Show that $g(x) \quad \mathrm{dom}(\overline{h})$. To this end, define

$$h^* = \overline{h} \qquad \{ \; g(x),\ G(\overline{h}(x)) \; \}$$

i.e., h^* is the obtained by expanding $\overline{h}$ (if necessary) so as to assign to $g(x)$ the value $G(\overline{h}(x))$. (If $g(x)$ is already in $\mathrm{dom}(\overline{h})$, then h^* coincides with $\overline{h}$ itself.) We shall show that h^* is an acceptable function. From this, and from the definition of $\overline{h}$ as the union of all acceptable functions, it will follow that $\mathrm{dom}(h^*) \quad \mathrm{dom}(\overline{h})$. Hence, in particular, since $g(x) \quad \mathrm{dom}(h^*)$ by definition, it will follow that $g(x) \quad \mathrm{dom}(\overline{h})$.

The proof amounts to establishing three facts:

(c.2.1.1) *h^* is a function.* To see this, recall that $\overline{h}$ is a function, and consider the two cases:

(c.2.1.1.1) If $g(x) \quad \mathrm{dom}(\overline{h})$, then adding the pair $\; g(x),\ G(\overline{h}(x)) \;$ won't affect the property of being a function: h^* will only assign one value to $g(x)$, namely $G(\overline{h}(x))$.

(c.2.1.1.2) If $g(x) \quad \mathrm{dom}(\overline{h})$, then the pair $\; g(x),\ G(\overline{h}(x)) \;$ is already in $\overline{h}$ (because $\overline{h}$ is acceptable). Thus, $h^*=\overline{h}$: a function.

(c.2.1.2) *The domain of h^* is included in C.* This is obvious, since $\mathrm{dom}(h^*) = \mathrm{dom}(\overline{h}) \quad \{g(x)\}$, and both of these are included in C.

(c.2.1.3) *h^* is closed under the rules of acceptability (i')–(iii').* It is here that the assumption of freeness finally becomes relevant.

(c.2.1.3.1) First of all, $h^*(x) = \overline{h}(x) = h(x)$ for all $x \quad B$, since $\overline{h}$ is acceptable, and h^* agrees with $\overline{h}$ on everything except at most $g(x)$. So clause (i') is satisfied.

(c.2.1.3.2) Suppose $g(y) \quad \mathrm{dom}(h^*)$ — Show that $y \quad \mathrm{dom}(h^*)$ and $h^*(g(y)) = G(h^*(y))$. [*Note*: We are here using a variable 'y' rather than 'x' because 'x' is already in use by our assumption that $x \quad \mathrm{dom}(\overline{h})$: see the beginning of (c.2.1)].
We distinguish here two cases:

(c.2.1.3.2.1) Suppose that $y \neq x$.
Then $g(y) \neq g(x)$ (since g is one-one, by the assumption that C is *freely* generated).
Hence $g(y) \in \mathrm{dom}(\bar{h})$ and $h^*(g(y)) = \bar{h}(g(y))$ since h^* and $\bar{h}$ agree on everything except at most $g(x)$).
So $y \in \mathrm{dom}(\bar{h})$ and $\bar{h}(g(y)) = G(\bar{h}(y))$ (since $\bar{h}$ is acceptable, by clause (ii'))
Hence $y \in \mathrm{dom}(h^*)$ (since $\bar{h} \subseteq h^*$ by definition) and $h^*(g(y)) = \bar{h}(g(y)) = G(\bar{h}(y)) = G(h^*(y))$ (again, since h^* and $\bar{h}$ agree on y)

(c.2.1.3.2.2) Suppose that $y = x$.
Then $y = x \in \mathrm{dom}(\bar{h}) \subseteq \mathrm{dom}(h^*)$ (by assumption)
and $h^*(g(y)) = h^*(g(x)) = G(\bar{h}(x))$ (by definition of h^*)
So we only need to check that $\bar{h}(x) = h^*(x)$, and this also follows from the fact that $\bar{h} \subseteq h^*$ by definition of h^*.

(c.2.1.3.3) Suppose that $f(y,z) \in \mathrm{dom}(h^*)$ — Show that $y,z \in \mathrm{dom}(h^*)$ and $h^*(f(y,z)) = F(h^*(y), h^*(z))$.

Here we note that $f(y,z) \neq g(x)$ (since f and g have disjoint ranges, by the assumption that C is *freely* generated). Then we reason as in (c.2.1.3.2.1):
We must have $f(y,z) \in \mathrm{dom}(\bar{h})$ and $h^*(f(y,z)) = \bar{h}(f(y,z))$ (since h^* and $\bar{h}$ agree on everything except at most $g(x)$).
So $y,z \in \mathrm{dom}(\bar{h})$ and $\bar{h}(f(y,z)) = F(\bar{h}(y), \bar{h}(z))$ (since $\bar{h}$ is acceptable, by (iii'))
Hence $y,z \in \mathrm{dom}(h^*)$ (since $\bar{h} \subseteq h^*$ by definition) and $h^*(f(y,z)) = \bar{h}(f(y,z)) = F(\bar{h}(y), \bar{h}(z)) = F(h^*(y), h^*(z))$ (again, since h^* and $\bar{h}$ agree on y and z)

By (c.2.1.1)–(c.2.1.3), h^* is a function on C satisfying clauses (i')–(iii'), i.e., h^* is acceptable. From this, as already anticipated, it follows that $h^* \subseteq \bar{h}$, and therefore that $g(x) \in \mathrm{dom}(\bar{h})$, as desired.

(c.2.2) Suppose $x,y \in \mathrm{dom}(\bar{h})$ — Show $f(x,y) \in \mathrm{dom}(\bar{h})$. This case is analogous to case (c.2.1), setting

$$h^* = \bar{h} \cup \{ \langle f(x,y), F(\bar{h}(x), \bar{h}(y)) \rangle \}.$$

By (c.1) and (c.2), we see that $\mathrm{dom}(\bar{h})$ includes B and is closed under g and f, i.e., it is inductive. By the induction principle, we can therefore conclude that $\mathrm{dom}(\bar{h}) = C$, i.e., *$\bar{h}$ is indeed defined throughout C.*

Step (d): *Show that $\bar{h}$ is unique*

Suppose there are two such functions, $\bar{h}_1$ and $\bar{h}_2$. Let S be the set on which they agree. We show that S is inductive.

(d.1) *Base step: Show that B is included in S.* This is immediate, since for each $x \in B$ we have $\bar{h}_1(x) = \bar{h}_2(x) = h(x)$ (because $\bar{h}_1$ and $\bar{h}_2$ are acceptable)

(d.2) *Inductive step: Show that S is closed under g and f.*

(d.2.1) Suppose that $x \in S$ — Show that $g(x) \in S$. To this end, just note that

$$\begin{aligned} \bar{h}_1(g(x)) &= G(\bar{h}_1(x)) && \text{(since } \bar{h}_1 \text{ is acceptable)} \\ &= G(\bar{h}_2(x)) && \text{(since } x \in S\text{)} \\ &= \bar{h}_2(g(x)) && \text{(since } \bar{h}_2 \text{ is also acceptable)} \end{aligned}$$

(d.2.2) The case for $f(x,y)$ is perfectly similar.

So S is inductive. By the induction principle, this implies that $S = C$, i.e., $\bar{h}_1$ and $\bar{h}_2$ agree throughout C. Since C is the domain of these functions by (a) and (c), it follows that $\bar{h}_1$ and $\bar{h}_2$ agree on everything — i.e., they are the same.