Modes of Connection

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Abstract. In recent years there has been a proliferation of theories for representing space and time in a qualitative way based on a primitive notion of topological connection. In previous work [14], we have commenced the construction of a unified framework. Independently of any foundational or applicative concerns, we attempt to delineate the space of mereotopological theories based on an account of their intended models and to place some existing theories into this framework. This paper extends this work by considering a second, orthogonal dimension along which varieties of topological connection can be classified: the strength of the connection.

1 Introduction

This paper is part of a longer project aimed at a systematic comparative study of spatial representation theories, specifically theories based on one or more primitive relations of mereotopological connection. In recent years such theories have become increasingly popular, especially since the publication of Clarke's calculus [7, 8]. However, this proliferation corresponds to a variety of approaches that are not always in agreement on the basic terms, either because of genuine philosophical dissension or because of a difference in the intended application [32]. The very notion of connection is open to a variety of understandings, typically because the ordinary set-theoretic account – based on the distinction between open and closed entities – gives rise to a number of puzzles when it comes to applications in the spatial domain [34]. This makes it difficult to chart the territory. And although the resulting variety of theories has a corresponding variety of merits, how to assess them and where to look for further progress calls for a clear, unified framework within which each theory can be interpreted and possibly integrated with the others.

In [14] we took a first step in this direction by studying three distinct families of theories, corresponding to the different ways of interpreting the connection relation vis-a-vis the options made available by the open-closed distiction. In each case, we followed the familiar course of explaining connection in terms of boundary sharing, irrespective of the type (dimension) of the relevant boundary. This paper extends that work by taking a closer look at the options that are available in this respect: two regions may share a single boundary point, an extended boundary segment, or an entire, maximal boundary. This gives us a second, orthogonal dimension along which varieties of topological connection can be classified: the strength of the connection. Further dimensions may then be obtained by taking into consideration the topological structure of the regions allowed in the domain of quantification – for instance, the distinction between one-piece and multi-piece regions.

2 Three Varieties of Connection

The three basic notions of connection introduced in our previous paper [14] are defined as follows:

$$C_1(x,y) \Leftrightarrow x \cap y \neq \emptyset$$
$$C_2(x,y) \Leftrightarrow x \cap c(y) \neq \emptyset \text{ or } c(x) \cap y \neq \emptyset$$
$$C_3(x,y) \Leftrightarrow c(x) \cap c(y) \neq \emptyset$$

(Here the double arrow is to be understood as a semantic relation specifying the truth conditions of each connection predicate 'C_k' and '$c(x)$' denotes the topological closure of a region x.) The first of these relations corresponds to Clarke's connection [7] and the second to ordinary topological connection, while the third is the relation known as "RCC" connection [27, 24]. It is easy to show that these three relations are ordered, in the sense that C_1 implies C_2 and C_2, C_3. The limit cases are illustrated in figure 1.

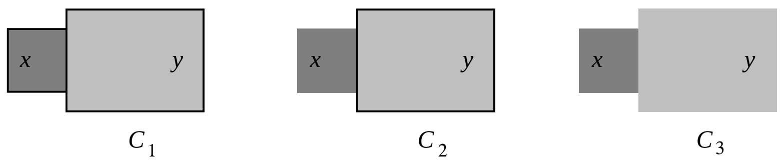

Fig. 1. The three basic connection relations (limit cases); a solid line indicates closure.

From these three connection relations we defined three corresponding notions of parthood, denoted by a predicate 'P_k' ($1 \leq k \leq 3$), and three notions of fusion, 'σ_k' ($1 \leq k \leq 3$). We then defined the notion of a *type*, characterised by a tuple $\langle i, j, k \rangle$ where i indicates a particular connection relation C_i, j a parthood relation C_j, and k a fusion operator σ_k. Many other mereotopological notions (such as tangency, boundary, closure, regularity, and so on) can be defined from these notions of connection, parthood, and fusion. Each type defines a possible mereotopological theory, giving nine possible theories in all[1]. Theories characterised by types of the form $\langle i, i, i \rangle$ we denoted *uniform*. We started to explore

[1] In fact we went on to define a higher order notion of type, giving an infinite number of types and thus many more syntactic theories.

the space of these theories and noted that certain types gave rise to theories with more or less desirable properties. For instance, assuming a domain with both regions and boundaries (i.e., lower dimensional entities), there are no reasonable uniform theories: $i = 1$ yields implausible topologies in which the boundary of a region is never connected to the region's interior (since they never share any points); $i = 2$ yields implausible mereologies in which every boundary is part of its own complement (since anything connected to the former is connected to the latter); $i = 3$ yields implausible mereotopologies in which the interior of a region is always connected to its exterior (so that boundaries make no difference) and in which the closure of a region is always part of the region's interior. In fact, we conjectured that the only reasonable type for theories which include boundaries is $\langle 2, 1, 1 \rangle$.

For boundary-free theories (where boundaries do not form part of the domain) we investigated the properties of each of the three uniform theories (all of which exist in the literature) and noted the advantages and disadvantages of each. For example, only theories of type $\langle 3, 3, 3 \rangle$ do not violate the suplementation principle of Simons [28], which demands that no region can have a single proper part.

Our purpose here is not to replicate our previous work of course, but simply to note that by varying the interpretation of a connection predicate by considering whether regions actually share a point or whether it is sufficient that one or both of the closures do, one can obtain a rich variety of mereotopological theories. In the remainder of this paper we will consider an entirely separate dimension along which mereotopological theories may vary.

3 An orthogonal dimension of variety

In our study of C_1–C_3, we were only concerned with whether two regions (or perhaps their closures) shared a point. However, in the limit this gives rise to a notion of connection where two entities are connected if they (or their closures) share just a single point (cf figure 2). For many applications, this may be considered too weak a form of connection. For example, a worm cannot pass from the interior of one apple to another, which touch just at a point, without becoming visible to the exterior; from the worm's point of view, we might as well say that the two apples are not "sufficiently" connected.

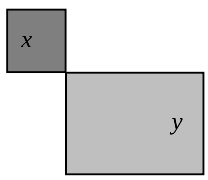

Fig. 2. A very weak kind of connection

One can find mereotopologies (e.g. [22, 5]) in which a predicate is defined to distinguish simple point connection from a stronger form of connection (where a worm could travel from one body to another without becoming visible to the exterior). Indeed whole mereotopological theories have been built taking this notion of connection as primitive [3]. The general question then arises as to what varieties of strength of connection there might be. We are not interested here in metrical notions but in topological distinctions. We want to explore this second way of varying the interpretation of the connection relation.

We will consider four cases of strong connection by distinguishing the different relations by subscripts: a, b, c, d. Thus, in general, a connection relation is now characterised by a pair of subscripts, $C_{\kappa,\lambda}$, where the subscript $\lambda \in \{1, 2, 3\}$ denotes a connection relation amongst those described in section 2, and the subscript $\kappa \in \{a, b, c, d\}$ indicates the strength of the connection. Where there is no confusion, we will omit either subscript.

Examining the limiting cases of connection, it would appear that one way of characterising the different cases of strong connection is by relating portions of the boundaries of a pair of regions – is their common boundary a single point, an extended portion of boundary, or a maximal boundary? Thus we will make the following definitions:

Definition 1 (Minimal, extended, maximal, and perfect boundaries).

- *z is a minimal boundary of x iff it is a boundary of x and contains no proper parts which are boundaries of x.*
- *z is an extended boundary of x iff z is a one-piece non-minimal boundary of x of dimension $n - 1$, where x is of dimension n.*[2]
- *z is a maximal boundary of x iff it is a one-piece boundary of x and it is not a proper part of any one-piece boundary of x.*
- *z is a perfect boundary between x and y iff z is a shared maximal boundary and z is not connected to the complement of x+y.*

We can now define the four new kinds of connection relation. In each case our definition will represent the minimum condition of that relation to hold. It is easy to see that these relations are ordered in terms of strength, starting from the weakest.

Definition 2 (Minimal, extended, maximal, and perfect connection).

a) *$C_{a,\lambda}(x, y)$ (minimal connection): x and y are C_λ connected (they share at least a minimal boundary).*
b) *$C_{b,\lambda}(x, y)$ (extended connection): x and y are C_λ connected and they share an extended boundary.*
c) *$C_{c,\lambda}(x, y)$ (maximal connection): x and y are C_λ connected and a maximal boundary of x is an extended boundary of y (or vice versa).*
d) *$C_{d,\lambda}(x, y)$ (perfect connection): x and y are C_λ connected and there is a perfect boundary between x and y, unless $x = y$.*

[2] Below we will give an alternative account with no reference to dimension.

Figure 3 illustrates[3] the various possibilities – there are 12 cases since we had three relations previously defined and each of these can now be split into four specialisations. Note that the depictions of maximal and perfect connection perhaps suggest that the relationship between x and y is asymmetric – it appears as though y surrounds x. However it is not our intention to model this closely related concept. Clearly if y does surround x then there is a perfect boundary between them, but this need not be the only way that perfect connection can be achieved – consider, for example, 2D Euclidean space split into two half planes, x and y: neither surrounds the other, but they are perfectly connected. Another interesting example (in 2D) is three nested rings, the innermost and outermost bands summing to a region y with x being the inner ring: clearly x has two perfect boundaries in common with y.

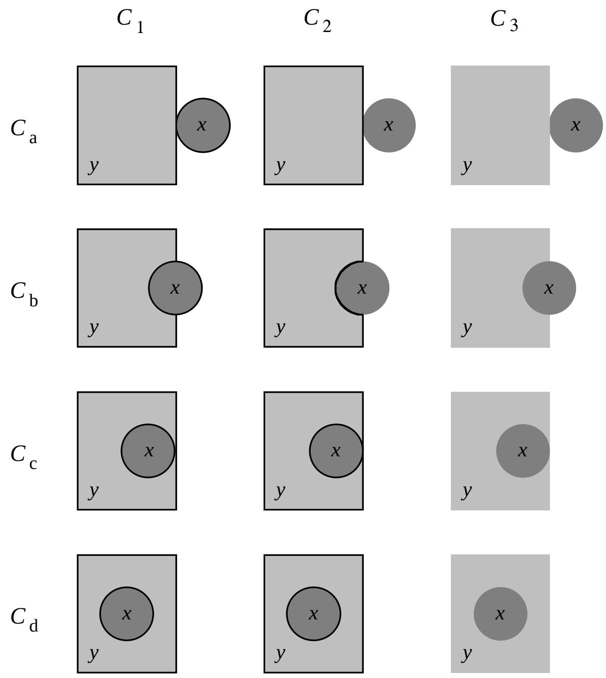

Fig. 3. An illustration of all 12 connection relations.

A reason we chose not to define asymmetric surround relationships is that in every mereotopological theory we know of the connection relation is axiomatised to be reflexive and symmetric, which would clearly be at odds with a surround

[3] These illustrations, and the others in this paper are 2D, but this is simply for ease of drawing; the defined concepts are applicable to higher dimensions as well. The analysis does not really make sense in 1D since the distinctions collapse – all connections are pointlike.

predicate. We now show that the new connection relations we have defined do indeed have these properties, at least in the domain of extended regions. (If either x or y is a boundary, some obvious exceptions arise for $\kappa = b$, c, d.)

Proposition 1 (Reflexivity and symmetry).

a) $C_{a,\lambda}(x, y)$*: the symmetry and reflexivity follow immediately from the symmetry and reflexivity of* C_λ.
b) $C_{b,\lambda}(x, y)$*: clearly* x *shares an extended boundary with itself (reflexivity) and sharing is a symmetric relationship.*
c) $C_{c,\lambda}(x, y)$*: note that the condition does not state that* x *and* y *share a maximal boundary as this would not lead to the configurations in the 3rd line of figure 3 satisfying* $C_{c,\lambda}(x, y)$*. Rather, the definition requires that a maximal boundary of one is an extended boundary of the other; this condition is clearly both reflexive and symmetric.*
d) $C_{d,\lambda}(x, y)$*: if* $x \neq y$ *then there is a shared maximal boundary between* x *and* y *(and sharing is a symmetric relationship) and since* $x \neq y$*, the condition that it is not connected to the complement of* $x + y$ *can be satisfied. Reflexivity follows immediately from the "unless" clause.*

It is worth pointing out that maximal imperfect connection need not have just a single point of "non-perfection" where the exterior touches both boundaries; for instance consider the "flower" shape depicted in figure 4(a). Equally, notice that there may be multiple connections between two regions – in such cases the most specific connection relation still holds (as do all the less specific ones of course); for example, consider the configurations in figure 4(b,c). Finally notice that the configuration in figure 4(d) is a case of C_b rather than C_c.

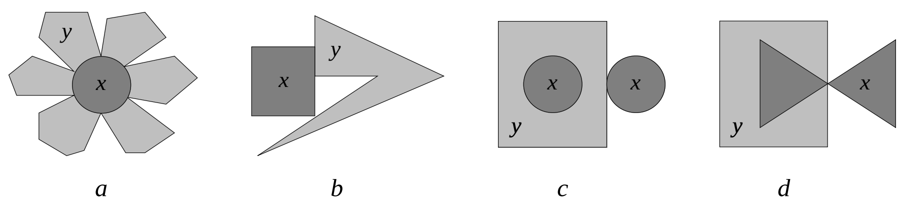

Fig. 4. (a) There may be multiple points of "imperfection" in between two maximally connected regions; (b) in the case of multiple connections, the most specific relationship still holds – x and y are extendedly connected; (c) similarly x and y are perfectly connected; (d) is a case of C_b rather than C_c.

There are however problems with the boundary based definitions we have just given. One problem is the need to refer to the dimension of the entities (in the definition of an extended boundary). This condition was needed because we do not want a cube balanced edge onto another cube to count as a case of extended connection. Another problem is that these definitions concentrate on the limit case of connection – external connection – whereas partial overlap, parthood and equality are also cases of connection, and the definition of $C_{b,\lambda}$ fails to take account of this. Imagine two partially overlapping circles: their boundaries share but two isolated points, but surely we would want to regard this as a case of $C_{b,\lambda}$; similarly if one region was a non-tangential part of another, it would not even be connected according to these definitions. Of course we could fix this problem by disjoining an appropriate condition, but here we shall pursue another approach, which we now present.

We will give an alternative set of definitions which do not focus on boundaries but rather rely on the intuition given by the example of the worm and the apple above. These new definitions will try to capture this notion of a "path" from one body to another which does not touch the exterior. Rather than use the term "path", which already has a specific meaning in topology, we shall speak of "conduits":

Definition 3 (Conduits, direct conduits, ideal conduits, components).

Let $i(x)$, $b(x)$ and $c(x)$ be the interior, boundary, and closure of x respectively; then:

- *A conduit between x and y is a one-piece region z which overlaps $i(x)$ and overlaps $i(y)$.*
- *A direct conduit between x and y is a conduit z such that $c(z) \cap b(x)$ is one-piece and $c(z) \cap b(y)$ is one-piece.*
- *A direct conduit z is ideal iff for every component w which is part of z, $z - w$ is not a direct conduit between x and y.*
- *A component is a region w whose interior $i(w)$ is one-piece.*

Intuitively: a conduit is just a one-piece region, possibly "pinched" to a point; a direct conduit is a conduit which only crosses the boundary of x and of y once. Since a conduit overlaps the interior of both regions, this effectively forces a direct conduit to "start" in x, exit x, and enter into the interior of y. An ideal direct conduit is a direct conduit which is minimal with respect to the number of "pinchings". Note that there is always a direct conduit between *any* two regions, though not necessarily an ideal direct conduit.

We can now use these concepts to redefine our four new kinds of connection relations:

Definition 4 (Conduit-based connection relations).

a) $C_{a,\lambda}(x,y)$ (minimal connection): Some direct conduit between x and y does not overlap the exterior of $x + y$ (i.e. they just C_λ).
b) $C_{b,\lambda}(x,y)$ (extended connection): Some direct conduit between x and y does not connect to the exterior.
c) $C_{c,\lambda}(x,y)$ (maximal connection): No ideal direct conduit between x and y overlaps the exterior.
d) $C_{d,\lambda}(x,y)$ (perfect connection): No ideal direct conduit between x and y connects to the exterior.

These definitions seem somewhat more intuitive. For $C_{a,\lambda}(x,y)$ we just require that there be some direct conduit between x and y, which does not overlap their exterior; since a direct conduit is one-piece, this forces x and y to be C_λ connected – see figure 5(a). For $C_{b,\lambda}(x,y)$, since the direct conduit cannot connect with the exterior, it forces the connection to be "wider" – see figure 5(b). For $C_{c,\lambda}(x,y)$, the intuition is that rather than just requiring some direct conduit not to overlap the exterior, this must be true for every direct conduit; thus every "path" which starts in x, crosses x's boundary, and enters y must do so directly without overlapping the exterior, which effectively forces a complete boundary of one of them to be contained in a boundary of the other; the ideality restriction is necessary because otherwise there could be an additional component of the direct conduit which overlapped the exterior – see 5(c): one direct conduit is ideal, and illustrates a direct conduit that is not connected to the exterior; the other direct conduit connects with the exterior, but it is not ideal, since a component could be removed whilst it remained a direct conduit between x and y. Notice that this latter direct conduit with the outer component removed is still connected to the exterior – the definition of $C_{d,\lambda}(x,y)$ ensures that no such direct conduit exists, thus ensuring "perfect" connection.

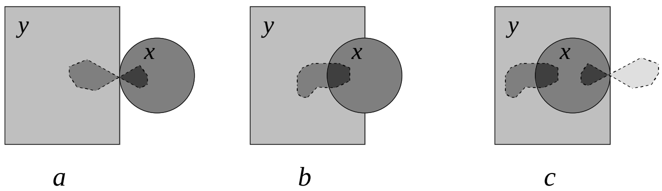

Fig. 5. An illustration of how "conduits" induce different connection relations

The revised definition overcomes the limits of Definition 2 without, of course, altering the intuitive understanding of the four connection relations. In particular, it is easy to verify that the new definitions (like the old ones) yield relations which are reflexive and symmetric in the domain of extended regions, and which are ordered in terms of increasing strength (starting from $C_{a,\lambda}$).

4 Domain examples

Some of the twelve relations that we have defined can be found in the literature. For example:

- $C_{a,1}(x,y)$ is that of Clarke [7, 8].
- $C_{a,2}(x,y)$ is standard topological connection [29, 33].
- $C_{a,3}(x,y)$ is "RCC" connection [27, 24, 12]
- $C_{b,3}(x,y)$ is that of Borgo *et al.* [3] who eschew C_a connection. $C_{b,3}$ is also essentially Bennett's [2] firm connection (though his notion is actually external connection rather connection more generally).

- $C_{c,1}(x,y)$ is closely related to the notion of tangential surround defined in [26], except that this notion is asymmetric.
- $C_{c,2}(x,y)$ is "niche" connection as described by Smith and Varzi [31].
- $C_{d,1}(x,y)$ is closely related to the notion of non-tangential surround defined in [26], except that this notion is asymmetric.

The significance of the distinctions we have been making can also be illustrated with reference to some domain examples. We have already mentioned the case of a worm travelling from one body to another and many other three dimensional examples exist. For another example, consider a chunk of Swiss cheese. A hole hidden in the interior (an internal cavity) is C_d connected to the piece of cheese: a worm cannot leave the hole without going through the cheese. If the worm starts digging, eventually the hole will "open up" – then it will be merely C_b connected to the cheese. And at the "magical moment" when the worm sees the light for the first time – when the worm breaks through the last layer of cheese – the hole is C_c connected to the cheese, though only for an instant. (Cf. [4, chapter 6]).

Further interesting examples arise in the geographic domain. For instance, consider the portion of the USA illustrated in figure 6. Utah and New Mexico are only C_a connected (as are Colorado and Arizona). Utah and Colorado are C_b connected (as are many other states, e.g. Arizona and Nevada). Utah is C_c connected with the sum of all the states except New Mexico and Utah. Finally, Utah is C_d connected with the sum of all the other states. The reader might question what motivates taking the sum of a set of States and treating it as a uniform region, but one can easily imagine that there might be some property (e.g. social, economic, meteorological, political) which some set of States might share but others not.

Another, essentially identical, example exists in France, where four departments all meet at a point – see figure 6. There are many other examples of these relationships, for example Vatican City and Italy are C_d connected since they are separate countries and Italy surrounds Vatican City. The same relationship holds between San Marino and Italy[4], and between the former countries of East and West Germany (because West Berlin was completely surrounded by East Germany).

5 Further Analysis

It is worth taking a second look at figure 3. The intended interpretation is that the differently shaded areas represent distinct, non-overlapping regions (x and y) – thus although y starts as a rectangle on the top row, it changes shape in each successive row, ending up with a hole in it. Notice that if we reinterpreted the figure, so that y remained a rectangle, then the two regions would overlap (mereologically) in the bottom three rows, and x would, respectively, partially

[4] Italy is perhaps unique in being doubly holed in this way?

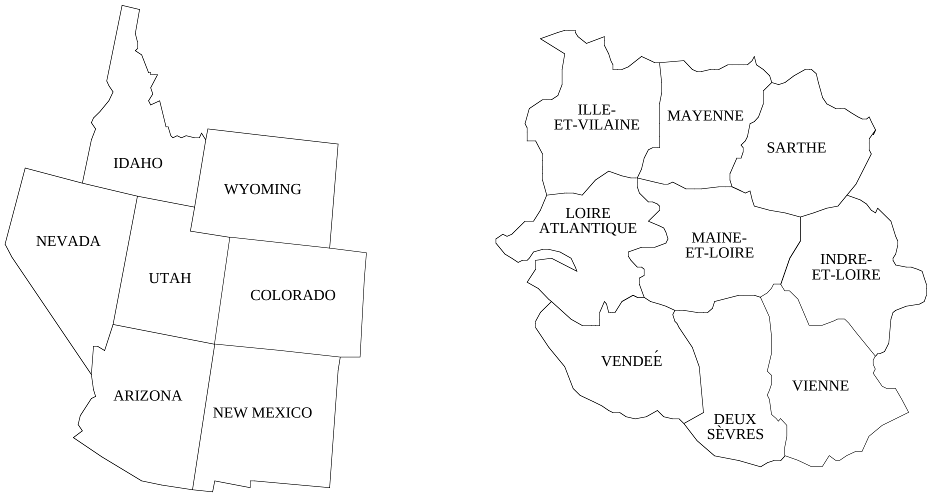

Fig. 6. Left: the meeting of four States at a single point illustrates all four new connection relations; right: similarly for the meeting of four Departments at a single point.

overlap, be a tangential part, and be a non- tangential part of y.[5] Thus, in the terminology of RCC [27], or that of [1] or [7,8][6], we would write $EC(x, y)$, $PO(x, y)$, $TPP(x, y)$ and $NTPP(x, y)$ respectively, to describe the situation in each of the four rows. Notice that these four relations, together with the inverses of TPP and $NTPP$ are six of the eight pairwise disjoint and jointly exhaustive relations to be found in RCC and similar systems. The remaining two relations are disconnection $DC(x, y)$ and equality. If one thinks of the region x moving from right to left as one moves down the columns, then $DC(x, y)$ would be represented by a missing row above all the others. We are treating equality as a special case of perfect connection, as mentioned earlier. Thus we essentially have an homomorphism between the eight RCC relations and the connection relations presented here. (The sequence of relations as one moves from row to row is known as a conceptual neighbourhood diagram [19, 11], or as a continuity network [11].)

Although we are treating both dimensions of variety in the connection relation as yielding primitive relations, there is a sense in which the original dimension [14] is the more primitive: one can imagine defining the relations in the second dimension from the first, but not vice versa. Also note that one can move from one relation of the second dimension to another by movement of the region, whilst it may be necessary for a region to actually change topologically to move

[5] Note that this analogy strengthens the motivation for regarding the configuration in figure 4(d) as a case of C_b rather than C_c connection.

[6] Essentially the same relations, but with differing names are defined in many other papers, e.g. in [15].

between relations of the first dimension (e.g. it might need to grow or lose its boundary).

6 Strong connection with different shapes

The patterns of interaction displayed in figure 3 can become considerably more complex if other shapes are considered. For example, consider replacing the circular shape with a simple concave shape, as displayed in figure 7. If we visualise the sequence of pictures as the "boomerang" sinks into the "water", we can see that a variety of topological configuations ensue, with single and multiple connections.

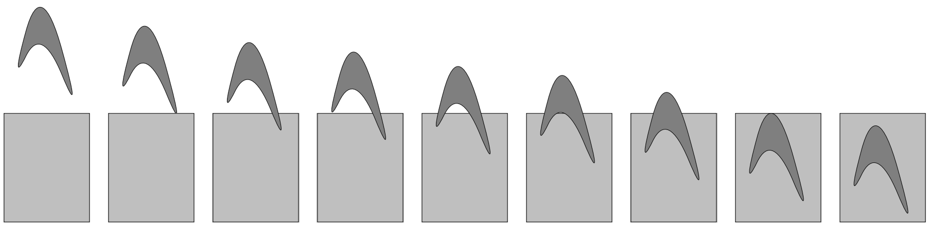

Fig. 7. Connection relations involving a concave, boomerang-shaped region.

It is tempting to consider that counting the number of connections made between two regions might also count as an indicator of the strength of their connection, but we reject this – it does not seem that any number of single points of connection are "as good as" a single extended connection (unless perhaps one is lying on a bed of nails).

The pattern of interaction between regions of different shapes and topological genus (e.g. regions with holes) deserves further investigation, but there is not space here to do it.

7 Multi-piece regions – a third dimension?

The boundary-based definitions presented in section 3 will work for one-piece or multi-piece regions, since each of the connection relations is defined in terms of whether a particular kind of boundary exists or not; adding an additional (separated) piece to a region will not affect the existence of this boundary. However, the conduit-based definitions of C_c and C_d fail for multi-piece regions because they are defined by demanding that no ideal direct conduit exists satisfying a particular condition; adding a new piece to a region will mean that many new ideal direct conduits exist (from the new piece to the other region) and some of these may fail the condition. For example, consider the configuration depicted in figure 4(c) – there is certainly an ideal direct conduit which connects to the exterior.

If we wish to handle multi-piece regions, then to remedy this is fairly straightforward: we just add an additional condition selecting a particular component from x and y for the condition to hold over:

Definition 5 (Conduit-based maximal and perfect connection, revised).

c) $C_{c,\lambda}(x,y)$ *(maximal connection): There are components x' of x and y' of y such that no ideal direct conduit between x' and y' overlaps the exterior of $x \cup y$.*
d) $C_{d,\lambda}(x,y)$ *(perfect connection): There are components x' of x and y' of y such that no ideal direct conduit between x' and y' connects to the exterior of $x \cup y$.*

In fact, considering the notion of multi-piece regions leads to the idea that the degree of connection between the various components of a multi-piece region is a third dimension of variation of the connection relation. Consider two multi-piece regions x and y. The weakest form of connection is that each has a single component[7] , x' and y' respectively, which is connected to the other (by one of our existing 12 connection relations). A stronger form of multi-piece connection is that every component of one region connects to some component of the other. By quantifying appropriately, we can come up with a new variety of connection relations.

Let us add a third subscript, chosen from the initial portion of the Greek alphabet to indicate this variety. Since connection is a binary relation, if we confine ourselves to the two basic quantifiers we naturally obtain four relations, indexed by α, β, γ, and δ.

Definition 6 (Multi-piece connection).

α) $C_{\alpha,\kappa,\lambda}(x,y)$*: some component of x is $C_{\kappa,\lambda}$ connected to some component of y.*
β) $C_{\beta,\kappa,\lambda}(x,y)$*: all components of x are $C_{\kappa,\lambda}$ connected to some component of y, and vice versa.*
γ) $C_{\gamma,\kappa,\lambda}(x,y)$*: some component of x is $C_{\kappa,\lambda}$ connected to all components of y, and vice versa.*[8]
δ) $C_{\delta,\kappa,\lambda}(x,y)$*: all components of x are $C_{\kappa,\lambda}$ connected to all components of y, and vice versa.*

These relations are illustrated in figure 8 for the case $\lambda = 1$. The first row illustrates $C_{\alpha,a,1}$, $C_{\beta,a,1}$, $C_{\gamma,a,1}$, and $C_{\delta,a,1}$ connection, in order from left to right. The next rows illustrate the corresponding patterns for $\kappa = b$, c, d.

[7] In fact the definition of what constitutes a multi-piece region and thus a component is connection dependent, depending on both the first two dimensions of variation. Here we will assume the existing definition of component.

[8] It would be possible to define another relationship, similar to this one, by replacing the "vice versa" condition with the requirement that some component of one is connected to all the components of the other. This requirement is weaker, but sufficient to ensure symmetry.

Notice that that for $C_{\gamma,c}$ and $C_{\delta,c}$, only one of the two regions can have multiple components. For $C_{\delta,d}$ we show two alternative ways of achieving the relationship, but note that the left-hand configuration is not extendible futher, while the right-hand configuration can be extended by adding further lighter shaded components.[9] Also notice that, in the figure, all individual connections are of the same type; multiple connections of varying types are of course possible, but we will not analyse this further here.

These definitions are certainly all reflexive and we have been careful to define these relations so that they are symmetric as well. It is also easy to show that they are ordered in terms of strength, though we do not have the space to show this formally here.

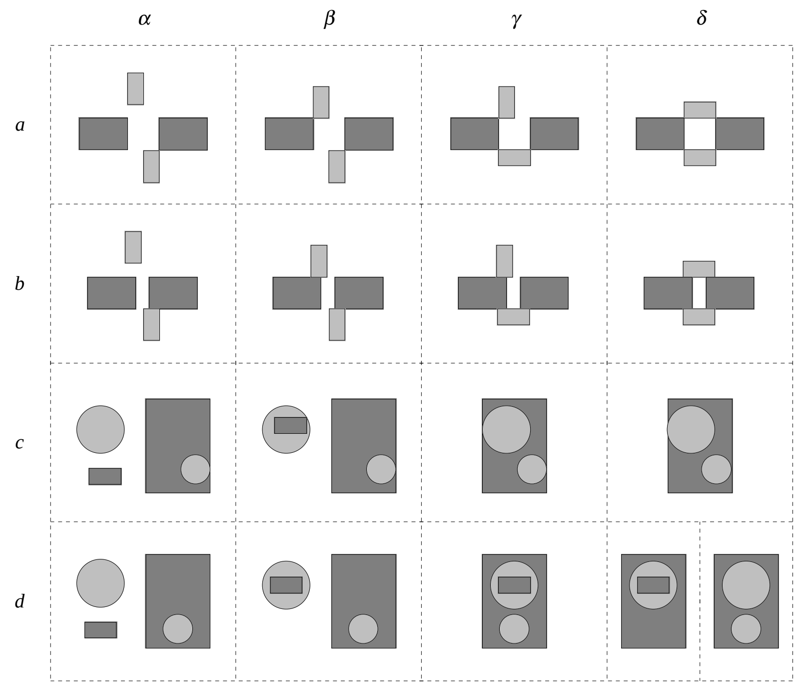

Fig. 8. Varieties of multiple connection

8 Related Work

The title of this paper is based on a paper by Galton "Modes of Overlap" [21] in which he analyses a variety of ways in which two regions can partially overlap

[9] It is interesting to note that with the conduit-based approach to defining C_c, there is no corresponding configuration to the left-hand configuration for $C_{\delta,d}$, whilst for the boundary-based approach there would have been.

each other. In most previous work (an exception is [13]), partial overlap has always been taken to be a single relation (usually denoted PO(x,y)), just as connection itself is usually taken to be a single relation. Whilst recognising that there are potentially infinitely many varieties of partial overlap relation, Galton parameterised these using a matrix notation:

$$\begin{pmatrix} x & a \\ b & o \end{pmatrix}$$

where x, a, b and o are the numbers of connected components of $x \cap y$, $x \backslash y$, $y \backslash x$, $compl(x \cup y)$. He investigates all matrices with numbers no greater than two; of the 54 theoretical possibilities, just 23 are physically realisable. However although two of these cases are in fact relations of external connection and one is a relation of disconnection (rather than partial overlap), the calculus does not in fact allow the distinctions to be made that we address in this paper.

It is also worth pointing to the work of Egenhofer and Franzosa [16], who present a calculus which allows, at the cost of arbitrary complexity of course, the possibility of distinguishing any topological distinct situations. (Compare [9] for a related proposal.)

9 Further Work

The analysis of connection presented here and in [14] certainly does not exhaust the possibilities. For example, we have not explicitly investigated irregular regions or regions of higher order topological genus (i.e., regions with holes) and this may bear explicit investigation. Equally, the analysis could be extended to connection relations between spatial entities of differing dimensions (cf. [23, 20, 10, 17]).

Another possibility is to extend the analysis to cover for instance the notion of "weak connection" defined in [1], the Brentanian notion of connection [6], or the notion of connection through "fiat" boundaries [30]. Yet another notion of connection is given by a pair of linked (interlocked) tori.

Finally, we intend to complete work on the framework outlined in [14] integrating all the dimensions, exploring the various elements of the framework, and placing other existing work in the literature into the framework. We also intend to consider further the possibility of applying the framework to the geographic domain, in the spirit of our illustrative examples of section 4.

10 Final Comments

We have unashamedly eschewed computational issues in this paper – although such issues are important, this work is directed toward providing a semantical framework in which extant and future theories can be placed and compared. The importance of secure ontological foundations for knowledge representation is now widely recognised e.g. [25, 18] and this work can be regarded as forming part of this enterprise.

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