Connection Relations in Mereotopology

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Abstract. We provide a model-theoretic framework for investigating and comparing a variety of mereotopological theories with respect to (i) the intended interpretation of their connection primitives, and (ii) the composition of their intended domains (e.g., whether or not they allow for boundary elements).

1. INTRODUCTION

In recent years there has been an outgrowth of theories for representing space and time in a qualitative way based on a primitive notion of topological connection [32]. Most of these theories have been influenced by the work of Clarke [10, 11], which in turn was inspired by Whitehead [35]. However, various other theories have been developed on independent grounds [29], and it is interesting to see how topology has itself become a point of connection among previously unrelated research areas.

Unfortunately, this variety of outlooks corresponds to a variety of theories that are not always in agreement on the basic terms. In some cases there is genuine philosophical dissension. In other cases the disagreement reflects the applicative agenda. In other cases still, the differences are due simply to a different understanding of the basic connection primitives. This is not surprising, since the ordinary set-theoretic account of topological connection rests on the distinction between open and closed entities (sets), and this is a problematic distinction in the domain of spatio-temporal entities [34, 35]. Indeed, the difficulty in applying standard point-set topology to ordinary space is one main reason behind the development of many connection-based theories. As mereology (the theory of parthood) was initially developed as an alternative to set theory (the theory of membership) in the constructional analysis of the common-sense world, viewed as likewise the theory of connection may be an alternative to point-set topology. The resultant theories are sometimes called, quite aptly, *mereotopologies*. And the lack of a unified framework bears witness to the difficulty of the task.

Our aim in this paper is to go some way in the direction of such a framework. We shall attempt to delineate the logical space of mereotopological theories based on an account of their intended models. The frame of reference used for this purpose will, in fact, be ordinary point-set topology. But this is not in contrast with the nature of mereotopology. For although the purpose of most mereotopological theories is to go beyond set theory, the latter still provides a general apparatus in terms of which the intended interpretation of the connection relations axiomatized by those theories can be expressed in precise terms. For instance, some theories explicitly interpret 'x is connected with y' as 'x and y share a common point' [1, 10], though points are then excluded from the domain of quantification. Other theories suggest the renderings 'there is no distance between x and y' as well as 'the closures of x and y share a common point' [24]; the former is the favored interpretation, but the latter makes it possible to compare such theories with others.

Our approach thus is essentially model-theoretic in nature. Given a topological space $\mathcal{T}$, we want to compare the models of different mereotopological theories when their variables range over elements of $\mathcal{T}$. More specifically, we want to compare theories with respect to (i) the intended interpretation of their connection primitives and (ii) the composition of their intended domains, i.e., which elements of $\mathcal{T}$ are included in their domain of quantification. This is one major source of disagreement among the theories available in the literature. Here we shall be interested in comparing them particularly with regard to whether or not their domains include unextended boundary elements (sets with empty interiors, such as points, lines, surfaces).

Note that although our examples will focus primarily on spatial domains, our results apply to domains of arbitrary dimensions, of which space and time may be seen as special cases. Moreover, we are only interested in mereotopological theories insofar as they account for the connection relation, ignoring other important topological notions such as compactness. Likewise, we shall ignore here the question of how a mereotopology can be combined with a theory of location to account for the relationship between an entity and the region (e.g., spatial region) where it is located. This is an important issue, but it lies beyond the scope of the present study [7, 8].

2. DEFINITION SCHEMAS

A pair (A, T(c)) is a *topological space* iff A is a non-empty set and T(c)={x A: c(x)=x}, where c is a closure operator axiomatised in the usual Kuratowski style:

= c( ) (A0)

x c(x) (A1)

c(c(x))   c(x) (A2)

c(x)   c(y) = c(x y) (A3)

The set T(c) is called the *topology* on A associated with c, and its elements are the *closed* sets determined by c. It follows from the axioms that T(c) is closed under intersection and finite union, and that the *closure* of a set x, c(x), is the smallest closed set including x. Likewise, let O(c)={x A: A–x T(c)} be the family of *open* sets determined by c. Then it follows that O(c) is closed under union and finite intersection, and one can define the *interior* of a set x, i(x), to be the greatest open set included in x.

Now let $\mathcal{T}$ = (A, T(c)) be any topological space. We shall focus on the following three ways of characterizing a relation of connection between subsets of A (Figure 1):

$C_1(x, y) \Leftrightarrow x \cap y \neq$

$C_2(x, y) \Leftrightarrow x \cap c(y) \neq$ or $c(x) \cap y \neq$

$C_3(x, y) \Leftrightarrow c(x) \cap c(y) \neq$

These three notions correspond—or can be made to correspond—to the main variants found in the literature. However, to get a proper picture of the alternatives offered by these options, two

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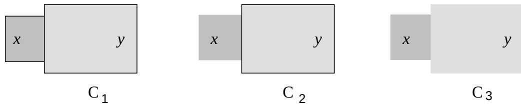

Figure 1: The three C relations (limit cases); a solid line indicates closure.

more parameters must be considered, corresponding to the ways in which the relation of parthood and the operation of fusion can be characterized in terms of connection:

$$P_k(x, y) =: \quad z(C_k(z, x) \rightarrow C_k(z, y)) \qquad (1 \le k \le 3)$$
$$\sigma_k x\ \varphi x =: \iota z \quad y(C_k(y, z) \leftrightarrow \quad x(\varphi x \quad C_k(y, x))) \qquad (1 \le k \le 3)$$

(Here 'φx' stands for any well-formed expression in which 'x' occurs free.) Most theories define parthood or fusion (typically both) in terms of the connection relation that is assumed as a topological primitive. However, this need not be the case, and in fact an important family of theories stems precisely from the intuition that parthood and connection cannot be defined in terms of each other. This effectively amounts to using two distinct primitives: two notions of connection (one of which is used in defining parthood), or a notion of connection and an independent notion of parthood. Accordingly, and more generally, we shall consider the entire space of theories that result from the options determined by the above definitions. That is to say, we shall work with a language in which all three connection predicates are available as primitives, treating theories in which some such predicates are defined in terms of others as taking place within a proper fragment of the language.

To do so in a systematic manner, let us call a triple $\tau = \ i,j,k$ (where $1 \le i,j,k \le 3$) a *type*: The first coordinate of a type, i, indicates a corresponding relation of connection, while j and k will indicate corresponding choices for the definition of parthood and fusion, respectively. For example, $2,1,3$ is the type associated with C_2 as a primitive for topological connection, P_1 as a parthood predicate (defined in terms of the primitive C_1), and σ_3 as a fusion operator (defined in terms of C_3). If the three coordinates of a type are all equal, then the type is *uniform* and corresponds to the case in which the primitive for topological connection is the only primitive used in defining all other mereotopological notions.

Using the notion of a type, the following notation provides a convenient generalization of the notation introduced above:

$$C_{i,j,k}(x, y) =: C_i(x, y)$$
$$P_{i,j,k}(x, y) =: P_j(x, y)$$
$$\sigma_{i,j,k}\, x\ \varphi x =: \sigma_k x\ \varphi x$$

We can then define customary mereotopological notions by relativizing them to types. (To simplify notation, we shall assume variables to range exclusively over non-empty sets, as virtually every account in the literature makes this assumption.)

$$O_\tau(x, y) =: \quad z(P_\tau(z, x) \quad P_\tau(z, y)) \qquad \tau\text{-overlap}$$
$$A_\tau(x, y) =: C_\tau(x, y) \quad \neg O_\tau(x, y) \qquad \tau\text{-abutting}$$
$$E_\tau(x, y) =: P_\tau(x, y) \quad P_\tau(y, x) \qquad \tau\text{-equality}$$
$$PP_\tau(x, y) =: P_\tau(x, y) \quad \neg P_\tau(y, x) \qquad \text{proper } \tau\text{-part}$$
$$TP_\tau(x, y) =: P_\tau(x, y) \quad z(A_\tau(z, x) \quad A_\tau(z, y)) \qquad \text{tangential } \tau\text{-part}$$
$$IP_\tau(x, y) =: P_\tau(x, y) \quad \neg TP_\tau(x, y) \qquad \text{interior } \tau\text{-part}$$
$$BP_\tau(x, y) =: \quad z(P_\tau(z, x) \rightarrow TP_\tau(z, y)) \qquad \text{boundary } \tau\text{-part}$$
$$PO_\tau(x, y) =: O_\tau(x, y) \quad \neg P_\tau(x, y) \quad \neg P_\tau(y, x) \qquad \text{proper } \tau\text{-overlap}$$
$$TO_\tau(x, y) =: \quad z(BP_\tau(z, x) \quad BP_\tau(z, y)) \qquad \text{tangential } \tau\text{-overlap}$$
$$IO_\tau(x, y) =: \quad z(IP_\tau(z, x) \quad IP_\tau(z, y)) \qquad \text{internal } \tau\text{-overlap}$$
$$BO_\tau(x, y) =: O_\tau(x, y) \quad \neg IO_\tau(x, y) \qquad \text{boundary } \tau\text{-overlap}$$

$$x +_\tau y =: \sigma_\tau z\,(P_\tau(z, x) \quad P_\tau(z, y)) \qquad \tau\text{-sum}$$
$$x \times_\tau y =: \sigma_\tau z\,(P_\tau(z, x) \quad P_\tau(z, y)) \qquad \tau\text{-product}$$
$$x -_\tau y =: \sigma_\tau z\,(P_\tau(z, x) \quad \neg O_\tau(z, y)) \qquad \tau\text{-difference}$$
$$k_\tau(x) =: \sigma_\tau z\, \neg O_\tau(z, x) \qquad \tau\text{-complement}$$
$$i_\tau(x) =: \sigma_\tau z\ IP_\tau(z, x) \qquad \tau\text{-interior}$$
$$e_\tau(x) =: i_\tau(k_\tau(x)) \qquad \tau\text{-exterior}$$
$$c_\tau(x) =: k_\tau(e_\tau(x)) \qquad \tau\text{-closure}$$
$$b_\tau(x) =: c_\tau(x) -_\tau i_\tau(x) \qquad \tau\text{-boundary}$$
$$U_\tau =: \sigma_\tau z\, O_\tau(z, z) \qquad \tau\text{-universe}$$
$$Rg_\tau(x) =: \quad z\ IP_\tau(z, x) \qquad \tau\text{-region}$$
$$Op_\tau(x) =: E_\tau(x, i_\tau(x)) \qquad \tau\text{-open}$$
$$Cl_\tau(x) =: E_\tau(x, c_\tau(x)) \qquad \tau\text{-closed}$$
$$Cn_\tau(x) =: \quad y \quad z(E_\tau(x, y +_\tau z) \rightarrow C_\tau(y, z)) \qquad \tau\text{-connected}$$

Depending on the structure of τ, the notions thus defined may receive different interpretations, hence the glosses on the right should not be taken too strictly. One intended interpretation of the binary relations relative to the Euclidean plane R^2—the interpretation that justifies the glosses—is illustrated in Figure 2. We shall call it the *standard interpretation* (see §4). However, the exact meaning of these definitions may change radically from one framework to another, depending on the type τ and on the relevant constraints in the model theory—e.g., on special provisions on the underlying topology *T*(c) or on which subsets of *A* should be included in the domain of quantification Our concern here is precisely with this variety of interpretations.

Note incidentally that the possibility arises of extending the set of defined mereotopological predicates and operators by relying on higher-order notions of connection. For instance, using parthood and fusion we can define overlap and closure; but then we could use these notions to introduce a corresponding variety of connection predicates, which in turn can be used to define corresponding notions of parthood and fusion, and so on. We may therefore amend our notion of type by including a 4th coordinate, indicating the level at which the predicate is defined:

if $1 \le i, j, k \le 3$, then $\ i, j, k, 0\ $ is a type;
if τ is a type and $1 \le i, j, k \le 3$, then $\ i, j, k, \tau\ $ is a type;
nothing else is a type.

The basic types give us the same as above:

$$C_{i,j,k,0}(x, y) =: C_i(x, y),$$
$$P_{i,j,k,0}(x, y) =: P_j(x, y),$$
$$\sigma_{i,j,k,0}\ \varphi x =: \sigma_k x\ \varphi x.$$

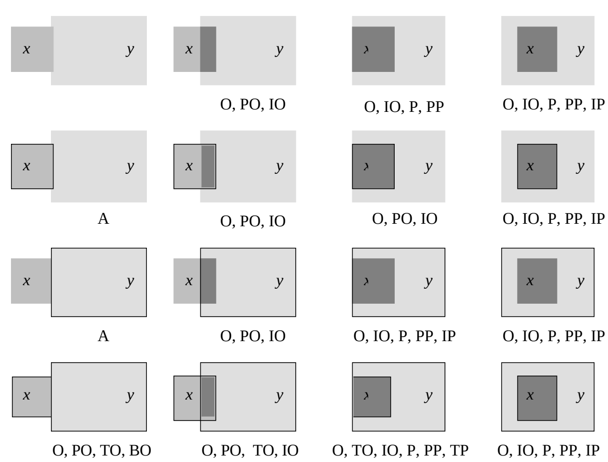

Figure 2: Standard interpretation of the mereotopological relations on R^2. (The labels indicate which relations hold between *x* and *y*, in this order.)

But the inductive types allow us to introduce higher-order connection relations:

$C_{1,j,k,\tau}(x, y) =: O_\tau(x, y)$
$C_{2,j,k,\tau}(x, y) =: O_\tau(x, c_\tau(y)) \quad O_\tau(c_\tau(x), y)$
$C_{3,j,k,\tau}(x, y) =: O_\tau(c_\tau(x), c_\tau(y))$

Using these notions, the list of definitions given above can be iterated, yielding further mereotopological predicates and operators. Some of these will collapse, but not necessarily all.

3. GENERAL FACTS

Before proceeding to a comparative analysis of mereotopological theories, we note here some general facts.

First of all, let us be more explicit about the formal setting. We assume afirst-order language with identity L={C_1, C_2, C_3} whose non-logical vocabulary consists of the three connection predicates. The model theory follows a standard first-order presentation. Only notice that we are interested in models that are based on some topological space $\mathcal{T}=(A, T(c))$, i.e., models $M=(U, f)$ whose domain U is a non-empty subset of (A)– and whose interpretation function f treats each connection predicate 'C_k' as indicated in §2 (relative to $\mathcal{T}$). Such models are called *canonical*.

It is easy to see that the following are universally satisfied in every canonical model for each i,j (1 ≤ i, j ≤ 3):

$C_i(x, x)$ $(C1_i)$
$P_j(x, x)$ $(P1_j)$
$C_i(x, y) \rightarrow C_i(y, x).$ $(C2_i)$
$P_j(x, y) \quad P_j(y, z) \rightarrow P_j(x, z).$ $(P2_j)$

In other words, each connection relation is *reflexive* and *symmetric* and each parthood relation is *reflexive* and *transitive*. Another property often associated with P_j is *antisymmetry*:

$P_j(x, y) \quad P_j(y, x) \rightarrow x = y.$ $(P3_j)$

However, this may fail in some models. For instance, if the domain includes only two sets with a non-empty intersection (but does not include the intersection itself), then $(P3_j)$ is false for each j. Indeed, requiring j-parthood to be antisymmetric amounts to treating i,j,k,0 -equality as identity, which in turns is logically equivalent to requiring j-connection to be *extensional*:

$z(C_j(z, x) \leftrightarrow C_j(z, y)) \rightarrow x = y.$ $(C3_j)$

Whether this holds depends crucially on the closure operator c and on which subsets of A are included in the domain. Figure 3 shows that there are models satisfying or falsifying any combination of the three instances of $(C3_j)$, thus showing the relative independence of the three sorts of extensionality. This represents a significant parameter in comparing competing theories.

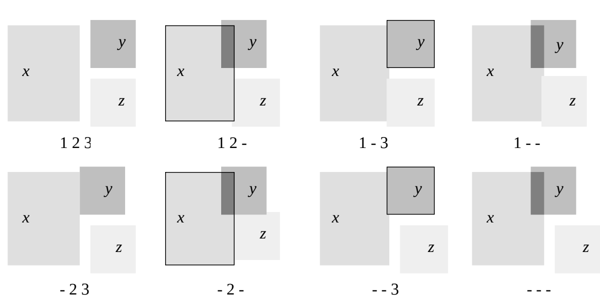

Figure 3. Independence of the extensionality axioms: each pattern shows a model that is j-extensional (1 ≤ j ≤ 3) iff the value j is displayed underneath.

Another important question is how the various C_τ, P_τ, and σ_τ are related. The relationship among the three connection predicates is easily stated—they are ordered by increasing strength:

$C_1(x, y) \rightarrow C_2(x, y)$ $(C4_{12})$
$C_2(x, y) \rightarrow C_3(x, y).$ $(C4_{23})$

That every canonical model satisfies these conditionals follows from (A1). For if something belongs to $x \cap y$, then *a fortiori* it belongs both to $x \cap c(y)$ and to $c(x) \cap y$ (making $(C4_{12})$ true), and if something belongs either to $x \cap c(y)$ or to $c(x) \cap y$ then it belongs to $c(x) \cap c(y)$ (making $(C4_{23})$ true). On the other hand, the converse conditionals may fail: if the domain contains two disjoint sets x and y, with $x \cap c(y) \neq$ and y $O(c)$, then x and y are connected in the sense of C_2 and C_3 but not of C_1; and if x, y $O(c)$ and $c(x) \cap c(y) \neq$, then x and y are connected in the sense of C_3 but not of C_1 or C_2. (See the limit cases of Figure 1.)

The three parthood predicates are not, in general, related in a similar fashion. In fact, no instance of the following *inclusion* schema (with 1 ≤ j, j' ≤ 3) is generally true:

$P_j(x, y) \rightarrow P_{j'}(y, x).$ $(P4_{j,j'})$

A glimpse at Figure 3 is sufficient to see that there are models satisfying or falsifying any combination of the three parthood relations, thus showing their relative independence. (Each pattern illustrates a model that satisfies $P_j(x, y)$ iff it is not j-extensional.) However, all these models are, in a sense, non-intended, and one might want to rule them out precisely by assuming instances of $(P4_{j,j'})$ along with some form of extensionality. For example, any model in which parthood is set inclusion satisfies $(P4_{12})$ besides the 1-extensionality principle $(P3_1)$.

With the fusion operator the situation is more complex. Say that a model is *k-fused for* φ, where φ is any formula, iff it satisfies the following axiom:

$z\varphi z \rightarrow \quad x \quad y(C_k(y, x) \leftrightarrow \quad z(\varphi z \quad C_k(y, z))).$ $(C4_k)$

Then the determination of the necessary and sufficient conditions that a model must satisfy in order for it to be k-fused for a given formula φ is, as far as we can see, an open problem.

4. BOUNDARY-TOLERANT THEORIES

We now proceed to examine in some detail the logical space of the theories that result from the options discussed above. Let τ= i,j,k,τ' be a type. A theory which formalizes topological connection by C_τ, parthood by P_τ, and fusion by σ_τ we call a τ-*theory*. There are of course many distinct τ-theories, depending on how the basic predicates are axiomatized. Here we consider only some indicative examples, confining ourselves to the case $\tau'=0$ (zero-order theories). We begin in this section with boundary-tolerant τ-theories, i.e., theories whose models may include boundary elements; in the next section we shall move on to boundary-free theories.

Consider first the case where τ is uniform (i=j=k). In this case a τ-theory could be formulated within a proper fragment of the language L, namely the fragment L_i={C_i}, and with $\tau'=0$ one can distinguish three main options (1 ≤ i ≤ 3). However, we may note immediately that none of these options is viable.

(a) The option i=1 yields implausible topologies in which the boundary of a region is never connected to the region's interior (since they never share any points).

(b) The option i=2 yields implausible mereologies in which every boundary is part of its own complement (since anything connected to the former is connected to the latter).

(c) The option i=3 yields implausible mereotopologies in which the interior of a region is always connected to its exterior (so that boundaries make no difference) and in which the closure of a region is always part of the region's interior.

There is also a sense in which these theories trivialize all mereotopological distinctions in the presence of boundaries. For (a)–(c) imply that if τ is uniform, any canonical model that includes the boundaries of its elements satisfies the conditional:

$$C_\tau(x, y) \rightarrow O_\tau(x, y).$$

(This is obvious for i=1. For i=2 or 3, it follows from (b) and (c), which imply that every object overlaps its complement.) Hence, in every such model the τ-abut predicate A_τ defines the empty relation, and so do the predicates of tangential and boundary parthood (TP_τ, BP_τ) and tangential and boundary overlap (TO_τ, BO_τ). We take these results to show that if boundaries are admitted in the domain, uniformly typed theories are inadequate. In fact, this applies not only to uniform types, but to all types where i=j. (See [3] and [24] for related material.)

Moving on to non-uniform types, we may note that some have been proposed in the literature, specifically for the case τ= 2,1,1,0 : an early example is to be found in [5], though the topological primitive there is Op_τ rather than C_τ. (One gets a definitionally equivalent characterization of C_τ via the definitions of §2. A similar warning applies to some other theories discussed below.) Other examples may be found in [19, 20, 27, 31, 33]. Since parthood P_τ is not defined in terms of the connection primitive C_τ, these theories need at least two distinct primitives (corresponding to the parameters 1 and 2 in the type); but since fusion σ_τ is typically understood using the same primitive as parthood, a third primitive is not needed (whence the equality of the second and third coordinates in the type).

These theories typically represent an attempt to reconstruct ordinary topological intuitions on top of a mereological basis. In fact, it is immediate from the definition that in this case C_τ corresponds to the notion of connection of ordinary point-set topology: two regions are connected if the closure of one intersects the other, or vice versa. Moreover, P_τ is typically assumed to satisfy the relevant extensionality and inclusion principles. Thus, a minimal theory of this kind is typically axiomatized using ($C1_2$), ($C2_2$), ($P1_1$), ($P2_1$), ($P3_1$), ($P4_{12}$). If we also add the fusion principle ($C4_1$), the result is a mereotopology subsuming what is known as classical extensional mereology [25], in which P_τ defines a complete Boolean algebra with the null element deleted. And if we add the following:

$$P_\tau(x, c_\tau(x)) \quad \text{(K1)}$$
$$P_\tau(c_\tau(c_\tau(x)), c_\tau(x)) \quad \text{(K2)}$$
$$E_\tau(c_\tau(x) +_\tau c_\tau(y), c_\tau(x +_\tau y)) \quad \text{(K3)}$$

the result is what may be called a full mereotopology, in which c_τ behaves as the standard Kuratowski closure operator. (A0 has no analogue due to the lack of a null element.) This corresponds to the "standard interpretation" of Figure 2.

All of these theories, of course, must account in some way for the intuitive difficulties that arise out of the notion of a boundary, and correspondingly of the distinction between open and closed entities. For instance, [27] considers various ways of supplementing a full mereotopology with a rendering of the intuition that boundaries are ontologically dependent entities, i.e., can only exist as boundaries of some open entity (contrary to the ordinary set-theoretic conception). In our notation the simplest formulation of this intuition is given by the axiom:

$$y BP_\tau(x, y) \rightarrow \quad z(Op_\tau(z) \quad BP_\tau(x, c_\tau(z))). \quad \text{(B1)}$$

Further proposals along these lines exploit a distinction between "fiat" and "bona fide" boundaries [26, 30]. At present, however, the metatheory of such theories is still unexplored.

It is also noteworthy that all theories of this sort have type 2,1,1,0 . We conjecture that this is indeed the only viable option. For instance, it is easy to see that a 1,2,k,0 -theory would run into the troubles mentioned in (a)-(b) above.

5. BOUNDARY-FREE THEORIES

Though the idea of a uniform type appears to founder in the case of boundary-tolerant theories, it has been taken very seriously in the context of boundary-free theories, i.e., theories that leave out boundaries from the universe of discourse in the intended models. Theories of this sort are rooted in [13, 36] and have recently become popular under the impact of Clarke's formulation in [10, 11] (see also [15]). Clarke's own is a 1,1,1,0 -theory, and some later authors followed this account (e.g. [1, 2, 22, 23]). However, one also finds examples of theories of type 2,2,2,0 (e.g. in [17, 21]) as well as of type 3,3,3,0 (especially in the work Cohn *et al.* [12, 16, 24], which has led to a rather extended body of results and applications in the area of spatial reasoning; see [14] for an independent example). Indeed, most if not all boundary-free theories in the literature are uniformly typed: this is remarkable but not surprising, since the main difficulties in reducing mereology to topology lies precisely in the presence of boundaries.

Note that, by definition, a boundary-free τ-theory admits no boundary elements. This is typically accomplished by adding as an axiom the requirement that everything has an interior part:

$$x Rg_\tau(x), \quad \text{(R)}$$

which immediately implies the emptiness of the relations BP_τ and BO_τ. However, let us emphasize that even in a boundary-free theory boundaries may be included, not in a model's domain, but in the topological space relative to which the model is defined.

Moreover, note that an axiom such as (R) gives us a way of studying the spectrum of boundary-free theories in terms of their boundary-tolerant counterparts. To this end, define the τ-*region relativization* of a formula φ, written φ^{Rg_τ}, in the obvious way:

$$C_\tau(y, x)^{Rg_\tau} = C_\tau(y, x)$$
$$(\neg\varphi)^{Rg_\tau} = \neg(\varphi^{Rg_\tau})$$
$$(\varphi \quad \psi)^{Rg_\tau} = \psi^{Rg_\tau} \quad \varphi^{Rg_\tau}$$
$$(\; x\varphi)^{Rg_\tau} = \quad x(Rg_\tau(x) \quad \varphi^{Rg_\tau})$$
$$(\; x\varphi)^{Rg_\tau} = \quad x(Rg_\tau(x) \rightarrow \varphi^{Rg_\tau})$$

Clearly, the following is true in every (canonical) model, as one can prove by ordinary induction on φ:

$$x Rg_\tau(x) \rightarrow (\varphi \leftrightarrow \varphi^{Rg_\tau}).$$

Hence, it follows that in general a formula φ is a theorem of a boundary-free τ-theory iff its relativization φ^{Rg_τ} is a theorem of the boundary-tolerant theory obtained by dropping (R).

More specifically, consider now the three main options mentioned above, where τ is a basic uniform type i,i,i,0 . Unlike their boundary-tolerant counterparts, none of these options yields a collapse of the basic mereotopological distinctions between tangential and interior parthood (TP_τ, IP_τ) or between tangential and interior overlap (TO_τ, IO_τ). However, the options diverge noticeably with regard to the distinction between open and closed regions (Op_τ, Cl_τ). The general picture is as follows.

(a) The case i=1 allows for the open/closed distinction, yielding theories in which the relation of abutting (A_τ) is a prerogative of closed regions (open regions abut nothing). As a corollary, such theories determine non-standard mereologies that violate the supplementation principle:

$$z(P_\tau(z, x) \rightarrow O_\tau(z, y)) \rightarrow P_\tau(x, y) \quad \text{(S)}$$

(It is enough to take y open and x equal to the closure of y.) This is so even if the theory includes the extensionality axioms ($P3_1$) or ($C3_1$). For although extensionality guarantees that a closed region x is never part of its own interior y, this is due to a mereological difference (a boundary) which cannot be found in the domain of regions. This is a feature that some authors have found unpalatable: as [25] put it, one can discriminate regions that differ by as little as a point, but one cannot discriminate the point. There are also some topological peculiarities that follow from the choice of C_1 as a connection relation. For instance, it follows immediately that no region is connected to its complement, hence that the universe is bound to be disconnected. This was noted in [1, 11], where the suggestion is made that self-connectedness should be redefined accordingly:

$$Cn'_\tau(x) =: \quad y \quad z(E_\tau(x, y +_\tau z) \rightarrow C_\tau(c_\tau(y), c_\tau(z))).$$

(b) The case i=2 also allows for the open/closed distinction, but yields theories in which the relation of abutting may hold between two regions when one is open and the other closed in the relevant contact area. This yields a rather standard topological apparatus, modulo the absence of boundary elements. However, also in this case the mereology is bound to violate (S).

(c) The case i=3 is the only one that does not allow for the open/closed distinction: in this case every region turns out to be τ-equal to its interior as well as to its τ-closure. This means that τ-theories of this type cannot be extensional—in fact, they yield highly non-standard mereologies. However, this is coherent with the fundamental idea of a boundary-free approach. For one of the main motivations for going boundary-free is precisely to avoid the puzzles that arise from the open/closed distinction [16]. In addition, and for this very same reason, such theories can validate (S), thereby eschewing the problem mentioned in (a)–(b).

We are not aware of any non-uniformly typed boundary-free theories, but it would certainly be interesting to pursue some abstract study in this direction. We hope our framework may constitute a first step towards this possibility.

6. CONCLUDING REMARKS

The paper lays the foundation for a systematic comparison and analysis of mereotopological theories, but further work is in order. For one thing, a systematic taxonomy of theories would require a completeness proof for each of them. There are not many results of this kind in the literature, and we hope the apparatus developed here may be of use for this purpose. Secondly, we have not investigated any higher level theories (theories of type i,j,k,τ for $\tau \neq 0$), or the axiomatic treatment needed to block higher-level ramifications of more familiar theories. Finally, although we have illustrated some points in the space exposed by our framework with extant theories and shown that others are not sensible, there are other existing theories and notions that need to be analysed and placed into the framework, for instance the notions of *weak connection* [1, 2] and *strong connection* [4], or the coincidence-based account of [9, 28].

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