Adding Convexity to Mereotopology

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Abstract. Convexity predicates and the convex hull operator continue to play an important role in theories of spatial representation and reasoning, yet their first-order axiomatization is still a matter of controversy. In this paper, we present a new approach to adding convexity to a mereotopological theory with boundary elements by specifying first-order axioms for a binary segment operator s. We show that our axioms yield a convex hull operator h that supports, not only the basic properties of convex regions, but also complex properties concerning region alignment. We also argue that h is stronger than convex hull operators from existing axiomatizations and show how to derive the latter from our axioms for s.

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Introduction

The notion of convexity plays an important role in various areas of common-sense spatial thinking. It has been studied, for instance, in the context of qualitative shape representation as an instrument for representing containment relations (see e.g. [1–4]) and it has been used to model betweenness and visibility relations [5]. The usefulness of such a notion in qualitative spatial calculi comes as no surprise. As convexity is, first and foremost, a characteristic of shape, its introduction allows for the discrimination of a variety of shape properties that cannot be distinguished in weaker theories such as standard mereology and mereotopology [6, 7]. What is more, the distinction between convex and concave regions seems to have high cognitive salience. Thus, already Gibson's visual arrays [8] call, albeit implicitly, for a formalization in terms of convexity.

At the same time, adding convexity to a qualitative theory of spatial representation has proven notoriously difficult to handle. Authors often insist that it is not clear to what extent current axiomatizations of the convexity predicate, or of the convex hull operator, do indeed capture the usual intended model of convex sets in two- or three-dimensional (Euclidean) vectorial space [7, 9, 10]. In particular, it remains to be seen whether the standard way of extending mereotopology with a convex hull operator and axiomatizing it along the lines of [7, 10, 11] gives rise to a sufficiently strong theory.

In this paper, we present a new approach to adding convexity to a mereotopology with boundary elements by introducing a primitive binary functional operator $s(x, y)$, whose intended interpretation is that of a straight line segment between points x and y in a vectorial space. On the basis of the axioms for this operator, we introduce a convexity predicate and a corresponding convex hull operator h and show that (i) h satisfies the standard characteristics of convex hulls and (ii) it is strong enough to sustain complex results about the alignment of extended regions. The novelty of the approach is not, of course, in the idea of defining convexity in terms of segments, which is customary in point-set geometry [12], but in its treatment within the limited resources of a qualitative mereotopological framework. Finally, we give some reasons for regarding h as a stronger operator than the main one used so far and show that all axioms for the latter can be recovered as theorems of our segment-based theory.

1. Axioms for Segments

1.1. Mereotopology

We shall be working within *KGEMT*, the Kuratowski extension of General Extensional Mereotopology [13, 14]. *KGEMT* is based on the two primitives P (parthood) and C (connection) and differs from purely C-based mereotopologies such as *RCC*, the Region Connection Calculus of [15], in that its models allow for entities of different dimensions. In particular, boundary elements such as points and lines enter the domain of quantification of *KGEMT* via a supplementation axiom for P that always guarantees the existence of something making up for the difference between a closed region and its open interior. (For a comparison between mereotopologies with and without boundaries, see [16].)

In *KGEMT*, two regions are connected iff the closure of one overlaps the other, or vice versa. (Thus, open regions are connected iff they overlap; similarly for closed regions.) Boundary items can be characterized with the help of auxiliary definitions [17]:

(D1) $O(x,y) =_{df} \exists z(P(z,x) \wedge P(z,y))$
x overlaps y iff x and y have a part in common.

(D2) $EC(x,y) =_{df} C(x,y) \wedge \neg O(x,y)$
x is externally connected to y iff x is connected to, but does not overlap, y.

(D3) $TP(x,y) =_{df} P(x,y) \wedge \exists z(C(x,z) \wedge EC(z,y))$
x is a tangential part of y iff x is part of, and connected to something externally connected to, y.

(D4) $BP(x,y) =_{df} \forall z(P(z,x) \rightarrow TP(z,y))$
x is a boundary part of y iff every part of x is a tangential part of y.

In particular, we identify points with mereologically atomic boundary elements, which in *KGEMT* turn out to be topologically closed:

(D5) $PP(x,y) =_{df} P(x,y) \wedge x \neq y$
x is a proper part of y iff x is part of, but not equal to, y.

(D6) $Pt(x) =_{df} \exists y BP(x,y) \wedge \neg \exists y PP(y,x)$
x is a point iff x is a boundary part (of something) with no proper parts of its own.

(T1) $Pt(x) \rightarrow Cl(x)$
If x is a point, then x is closed.[1]

[1]For theorems stated without proof, we refer to [18] and references therein.

Here '*Cl*' is defined in a way akin to the standard topological characterization, according to which an entity is closed iff it coincides with its own closure. For convenient reference, we list the relevant definitions, along with others that will be needed in the following.

(D7) $\sigma x \phi x =_{df} \iota y \forall z(O(y,z) \leftrightarrow \exists x(\phi x \wedge O(z,x)))$
The fusion of the ϕers is the unique thing that overlaps all and only those things that overlap some ϕer.[2]

(D8) $\pi x \phi x =_{df} \sigma y \forall x(\phi x \rightarrow P(y,x)))$
The nucleus of the ϕers is the fusion of their common parts.

(D9) $x + y =_{df} \sigma z(P(z,x) \vee P(z,y))$
The sum of x and y is the fusion of those things that are parts of x or y.

(D10) $x \times y =_{df} \sigma z(P(z,x) \wedge P(z,y))$
The product of x and y is the fusion of those things that are parts of x and y.

(D11) $x - y =_{df} \sigma z(P(z,x) \wedge \neg O(z,y))$
The difference between x and y is the fusion of those parts of x that do not overlap y.

(D12) $\sim x =_{df} \sigma y \neg O(y,x)$
The complement of x is the fusion of those things that do not overlap x.

(D13) $i(x) =_{df} \sigma y(P(y,x) \wedge \neg TP(y,x))$
The interior of x is the fusion of those parts of x that are not tangential.

(D14) $c(x) =_{df} \sim i(\sim x)$
The closure of x is the complement of the interior of the complement of x.

(D15) $b(x) =_{df} c(x) - i(x)$
The boundary of x is the difference between the closure and the interior of x.

(D16) $Op(x) =_{df} x = i(x)$
x is open iff x equals its own interior.

(D17) $Cl(x) =_{df} x = c(x)$
x is closed iff x equals its own closure.

1.2. Adding Segments

We extend *KGEMT* to a mereotopology with segments, *KGEMTS*, by adding a primitive binary functional operator $s(x,y)$, to be axiomatized as standing for the straight-line "segment" between x and y. In terms of this operator, convexity is then defined in a way that parallels the usual definition of convex sets in $\mathbb{R}^2$:

(D18) $Cv(x) =_{df} \forall y \forall z((Pt(y) \wedge Pt(z) \wedge P(y,x) \wedge P(z,x)) \rightarrow P(s(y,z),x))$
x is convex iff all segments between points in x are themselves part of x.

Our axiomatization of s is based on Coppel's axioms for standard point-set topology [19,20].[3] As will follow from the axioms, s takes points as arguments and always yields a self-connected segment.[4]

[2] We assume the definite descriptor 'ι' to be handled *à la* Russell, as in [13,14].

[3] Alternative axiomatic approaches may be found in [21,22]. Axioms for segments go back to the early work of Hilbert [23] and Tarski [24] on axiomatic treatments of geometry.

[4] Here and in the following we simplify notation by dropping all initial universal quantifiers. All axioms and theorems are to be understood as universally closed.

(S1) $Pt(x) \wedge Pt(y) \leftrightarrow \exists z(P(x,z) \wedge P(y,z) \wedge z = s(x,y))$
Any two points x and y determine a segment $\overline{xy}$ to which they belong.

(S2) $Pt(x) \rightarrow s(x,x) = x$
If x is a point, the segment $\overline{xx}$ is equal to x.

(S3) $Pt(u) \wedge P(u, s(v,x)) \wedge P(v, s(u,y)) \wedge u \neq v \rightarrow P(u, s(x,y))$
If u and v are distinct points on $\overline{vx}$ and $\overline{uy}$, respectively, then u is a point on $\overline{xy}$ (as is v).

(S4) $Pt(z) \wedge P(z, s(x,y)) \rightarrow s(x,y) = s(x,z) + s(z,y)$
If z is a point on $\overline{xy}$, then this segment equals the sum of $\overline{xz}$ and $\overline{zy}$.

(S5) $Pt(x) \wedge Pt(y) \wedge x \neq y \rightarrow \exists z(P(z, s(x,y) - (x+y)))$
If x and y are distinct points, then there is a point z that is part of the segment $\overline{xy}$ minus its endpoints.

(S6) $Pt(x) \wedge Pt(y) \wedge x \neq y \rightarrow \exists z(P(x, s(y,z) - (y+z)))$
If x and y are distinct points, then there is a point z such that x is part of the segment $\overline{yz}$ minus its endpoints.

(S7) $x \neq s(x,y) \times s(x,z) \rightarrow \exists w(Pt(w) \wedge P(y, s(x,w)) \wedge P(z, s(x,w)))$
If $\overline{xy}$ and $\overline{xz}$ overlap at more than just point x, then both y and z are part of a segment $\overline{xw}$ for some point w.

(S8) $Pt(z_1) \wedge Pt(z_2) \wedge P(z_1, s(x,y_1)) \wedge P(z_2, s(x,y_2)) \rightarrow O(s(y_1,z_2), s(y_2,z_1))$
If z_1 and z_2 are points on $\overline{xy_1}$ and $\overline{xy_2}$, respectively, then $\overline{y_1z_2}$ and $\overline{y_2z_1}$ overlap.

(S9) $Pt(z_1) \wedge Pt(z_2) \wedge Pt(z) \wedge P(z_1, s(x,y_1)) \wedge P(z_2, s(x,y_2)) \wedge P(z, s(z_1,z_2)) \rightarrow$
$\exists y(Pt(y) \wedge P(y, s(y_1,y_2)) \wedge P(z, s(x,y)))$
If z_1, z_2, and z are points on $\overline{xy_1}$, $\overline{xy_2}$, and $\overline{z_1z_2}$, respectively, then there is a point y on $\overline{y_1y_2}$ such that z is a point on $\overline{xy}$.

(S10) $Cv(u) \wedge PP(u, s(x,y)) \wedge PP(x,u) \wedge PP(y, s(x,y) - u) \rightarrow \exists z(Pt(z) \wedge P(z, s(x,y)) \wedge$
$P(s(x,z) - z, u) \wedge P(s(z,y) - z, {\sim}u))$
If u is a convex proper part of $\overline{xy}$ such that x is a proper part of u and y a proper part of the remainder, then there is a point z in $\overline{xy}$ such that the segment $\overline{xz}$ minus its endpoint z is part of u and the segment $\overline{zy}$ minus its endpoint z is part of u's complement.[5]

These axioms are for the most part straightforward and intuitive. We only note that the more complex **(S8)** and **(S9)** correspond to Pasch's axiom and Euclid's axiom, respectively, both of which were used in Tarski's elementary geometry [24], whereas **(S10)** reflects the Dedekind cut property for $\mathbb{R}$. Instead of going through each axiom in detail, we now show how they can be used to establish a powerful notion of convexity along with a correspondingly strong convex hull operator, which is defined as mapping every region x to the smallest convex region containing x:

(D19) $h(x) =_{df} \pi y(Cv(y) \wedge P(x,y))$
The convex hull of x is the nucleus of all convex entities of which x is a part.

[5] A simple rendering of Coppel's formulation of this axiom would have '$\neg P(y,u)$' in the antecedent in place of '$PP(y, s(x,y) - u)$'. The present formulation is due to the fact that, in the consequent, $s(z,y) - z$ is defined only for $u \neq s(x,y) - y$, as $KGEMT$ has no room for the empty region. Thanks to Stefano Borgo for helpful discussion on this point.

1.3. Convexity and Convex Hulls

Some basic properties of the convexity predicate and the convex hull operator follow directly from definitions **(D18)–(D19)**, while others depend crucially on axioms **(S1)–(S10)**. Among the former, we can for instance prove that convexity can be characterized in terms of convex hulls, that the product of overlapping convex regions is always convex, and that h is a well-behaved closure operator (extensive, idempotent, non-decreasing):

(T2) $Cv(x) \leftrightarrow x = h(x)$
A region is convex iff it coincides with its own convex hull.

(T3) $(Cv(x) \wedge Cv(y) \wedge O(x,y)) \rightarrow Cv(x \times y)$
The product of two overlapping convex regions is itself convex.

(T4) $P(x, h(x))$
Every region is part of its own convex hull.

(T5) $h(h(x)) = h(x)$
The convex hull of the convex hull of a region x is equal to the convex hull of x.

(T6) $P(x,y) \rightarrow P(h(x), h(y))$
If x is part of y, then the convex hull of x is part of the convex hull of y.

On the other hand, among the properties that follow from the axioms we can immediately prove e.g. that s is commutative. More importantly, we can prove that all segments are convex, that any point on a segment is the meet of two sub-segments, and that the sum of any two segments with more than one point in common is again a segment:

(T7) $Cv(s(x,y))$
Every segment is convex.

(T8) $Pt(z) \wedge P(z, s(x,y)) \rightarrow s(x,z) \times s(z,y) = z$
If z is a point on $\overline{xy}$, then the product of $\overline{xz}$ and $\overline{zy}$ is exactly z.

(T9) $\exists x(Pt(x) \wedge PP(x, s(y_1,z_1) \times s(y_2,z_2))) \rightarrow \exists u \exists v(s(y_1,z_1) + s(y_2,z_2) = s(u,v))$
If a point x is a proper part of the product of two segments $\overline{y_1 z_1}$ and $\overline{y_2 z_2}$, then the sum of these segments is itself a segment.

The full strength of the axioms, however, becomes visible through the more complex properties they support. To begin with, **(T9)** can be extended to a general characterization of convex hulls in terms of fusions of segments:

(T10) $h(x+y) = \sigma z \exists u \exists v(Pt(u) \wedge Pt(v) \wedge P(u, h(x)) \wedge P(v, h(y)) \wedge z = s(u,v))$
The convex hull of the sum of two regions x and y is the fusion of all segments determined by pairs of points in the convex hull of x and the convex hull of y, respectively.

As a special case, we have:

(T11) $Pt(x) \wedge Pt(y) \rightarrow h(x+y) = s(x,y)$
When both x and y are points, the convex hull of their sum is just the segment $\overline{xy}$.

Furthermore, the axioms provide the grounds for proving the counterpart of an important result in point-set topology, to the effect that every set has the same "extreme" points as its convex hull. (The importance of this result will become clear in section 2.) Standardly, given a topological space X and a set $A \subseteq X$, an extreme subset of A is any $E \subseteq A$ such that, for all $B \subseteq A$, the intersection of E with the convex hull of B is included in the convex hull of the intersection of E and B (i.e., intuitively, no point of E lies in the interior of a segment whose endpoints are in A but not in E). To reproduce the result in question within $KGEMTS$, we thus begin by defining an analogous mereotopological predicate for extreme parthood, and then define extreme points as extreme parts that are points:

(D20) $EP(x,y) =_{df} P(x,y) \wedge \forall z(P(z,y) \rightarrow P(x \times h(z), h(x \times z)))$
x is an extreme part of y iff x is a part of y whose product with the convex hull of any part z of y is part of the convex hull of the product of x and z.

(D21) $EPt(x,y) =_{def} Pt(x) \wedge EP(x,y)$
x is an extreme point of y iff x is both a point and an extreme part of y.

(T12) $EPt(x,y) \leftrightarrow P(x,y) \wedge \neg P(x, h(y-x))$
x is an extreme point of y iff x is part of y but not of the convex hull of $y-x$.

On this basis, we can then prove that the extreme points of a region add up to the nucleus of a distinguished set of parts, from which the desired result follows:

(T13) $\sigma v EPt(v,x) = \pi w(P(w,x) \wedge h(x) = h(w))$
The fusion of the extreme points of a region x equals the nucleus of all parts of x having the same convex hull as x.

(T14) $EPt(x,y) \leftrightarrow EPt(x, h(y))$
x is an extreme point of y iff x is an extreme point of y's convex hull.

Another important result worth mentioning, secured by **(S4)** together with **(T8)**, is that all segments, and *a fortiori* all convex regions, are self-connected, as one would expect:

(D22) $Cn(x) =_{df} \forall y \forall z(x = y+z \rightarrow C(y,z))$
x is self-connected iff x cannot be decomposed into two disconnected parts.

(T15) $Cv(x) \rightarrow Cn(x)$
Every convex region is self-connected.

Finally, the axioms for segments warrant a number of complex quasi-geometrical properties concerning convex hulls. With the help of two key theorems, it can in fact be shown that h is sufficient to express a relation of *alignment* for extended regions. The first theorem states the conditions under which three disconnected convex regions are aligned with one another, in the sense that segments connecting points in two of them always pass through the third region. More precisely, if we suppose that x, y, z have pairwise disconnected convex hulls and satisfy the condition $h(x+y+z) = h(x+y) + h(x+z)$, then there are exactly three possibilities, depending on whether one of the two convex hulls on the right side of the equality is part of the other or neither is. The theorem says that in this last case, the alignment between x, y, and z can be characterized in terms of the segments between points in y and z: each of them must overlap $h(x)$.

(T16) $h(x+y+z) = h(x+y) + h(x+z) \wedge \neg C(h(x), h(y)) \wedge \neg C(h(y), h(z)) \wedge \neg C(h(x), h(z)) \wedge \neg P(h(x+y), h(x+z)) \wedge \neg P(h(x+z), h(x+y)) \wedge Pt(b) \wedge P(b, h(y)) \wedge Pt(c) \wedge P(c, h(z)) \rightarrow O(s(b,c), h(x))$
If x, y, and z satisfy the given conditions and b and c are points in the convex hulls of y and z, respectively, then the segment $\overline{bc}$ overlaps the convex hull of x.

The proof of **(T16)** is lengthy and tedious and we refrain from reproducing it here, but here is the main idea. Considering all possible configurations for a segment $s(b,c)$ such that $P(b,h(y))$ and $P(c,h(z))$, and repeatedly exploiting axioms **(S8)** and **(S9)**, one can show that under the constraints expressed in the antecedent of the theorem all configurations but one lead to contradiction. The only viable option is for there to be a point a in $h(x)$ such that $s(b,c) = s(b,a) + s(a,c)$. If we suppose that there are segments $s(b,c)$ that are part of a segment $s(b,a_1)$ or of a segment $s(c,a_2)$, for some points a_1 and a_2 in $h(x)$, or if we suppose that there is a segment $s(b,c)$ that is not colinear with any point in $h(x)$, we will find that either at least two of $h(x)$, $h(y)$, and $h(z)$ overlap, contrary to our hypothesis, or that either $h(y)$ or $h(z)$ turns out to be part of $h(x+z)$ or $h(x+y)$, respectively, which again conflicts with the assumptions of the theorem. Figure 1 illustrates a configuration for x, y, and z that fulfills these assumptions. □

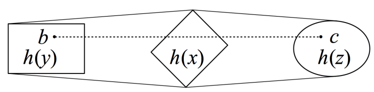

Figure 1. Every segment $\overline{bc}$ overlaps $h(x)$, with x, y, and z satisfying the assumptions of **(T16)**.

From **(T16)** we can derive the second theorem mentioned above, which provides a more precise picture of the alignment of $h(x)$, $h(y)$, and $h(z)$ under the same conditions:

(T17) $h(x+y+z) = h(x+y) + h(x+z) \wedge \neg C(h(x),h(y)) \wedge \neg C(h(y),h(z)) \wedge \neg C(h(x), h(z)) \wedge \neg P(h(x+y),h(x+z)) \wedge \neg P(h(x+z),h(x+y)) \rightarrow h(x+y) \times h(x+z) = h(x)$
If x, y, and z satisfy the given conditions, then the product of the convex hulls of $x+y$ and $x+z$ is exactly the convex hull of x.

To prove this, one can reason by contradiction as follows. From the supposition that there is a point p in $h(x+y) \times h(x+z)$ that is not part of $h(x)$, one can infer by **(T10)** that there exist segments $s(a_1,b_1)$ and $s(a_2,c_2)$, respectively part of $h(x+y)$ and $h(x+z)$, that overlap exactly in p. By **(T16)**, every segment $s(b,c)$, with $P(b,h(y))$ and $P(c,h(z))$, is a sum of segments $s(b,a)$ and $s(a,c)$ for some a in $h(x)$. As a result, there exists a point a in $h(x)$ such that $P(a,s(b_1,c_2))$. Consider $s(a,p)$. This segment defines a line l all of whose points are *colinear* with a and p, and we can prove that l overlaps $s(a_1,a_2)$. For l is the fusion of all those points c such that $P(s(p,c),s(a,c))$ or $P(s(a,c),s(c,p))$ or $P(c,s(a,p))$, and it can be shown that all (sufficiently long) segments $s(p,c)$ such that $P(s(p,c),s(a,c))$ overlap one of the segments $s(a_2,b_1)$, $s(b_1,c_2)$, $s(c_2,a_2)$ and $s(a_2,a_1)$. It then suffices to repeatedly apply axiom **(S8)** to prove that $s(p,c)$ cannot overlap segments $s(a_2,b_1)$, $s(b_1,c_2)$, and $s(c_2,a_2)$. By exclusion, it follows that these segments overlap $s(a_1,a_2)$, which entails that p is part of $h(x)$—a contradiction. □

2. Cohn *et al.*'s Convex Hull Axioms

How does *KGEMTS* compare with the more popular accounts of convexity mentioned in the introduction? In the mereotopological literature, the most prominent treatment comes from Cohn *et al.*'s axioms for the convex hull operator [7, 11], whose earliest formulation goes back to [1]. Such axioms have been widely employed in applications of qualitative spatial reasoning (see e.g. [4, 9, 10]), though not without qualms regarding their adequacy. In this section we argue that the properties of the operator h defined above are significantly stronger, and more adequate, and we show that the h-counterparts of Cohn *et al.*'s axioms are derivable in $KGEMTS$ (modulo minor qualifications).

It should be noted that the axioms in question are based, not on $KGEMT$, but on RCC. These mereotopological theories differ in two important respects. First, while it makes room for a substantive distinction between tangential and non-tangential parts, and consequently between open and closed regions, RCC restricts its domain of quantification to extended regions: lower-dimensional boundary elements are ruled out explicitly by an anti-atomistic axiom that forces every region to have non-tangential proper parts, and the intended interpretation of 'C' corresponds to a relation of connection that holds between two regions iff their closures intersect. (Thus, two regions may count as connected even if they are both open, or both closed.) Second, in RCC the mereological predicate 'P' is not treated as a primitive but is defined in terms of the topological predicate 'C', following [26], yielding a parthood relation that holds between two regions, x and y, iff whatever is connected to x is connected to y. Obviously, these differences are important, and we refer to [14, 16] for detailed comparisons. In spite of this, here we shall continue to work within the framework of $KGEMT$, that is, we shall look at the behavior of Cohn *et al.*'s convex hull operator—which we denote by 'ch'—within the same mereotopological framework we have been assuming for h. While this will perforce involve some discrepancies, it will nonetheless provide a faithful enough reconstruction to support an informative comparison between the two accounts.

Let us begin, then, by reproducing the axioms for ch, which we take from [25]. For ease of comparison, the following standard definition will be used in addition to those already recalled in Section 1.1:

(D23) $IP(x,y) =_{df} PP(x,y) \wedge \neg TP(x,y)$
x is an internal part of y iff x is a proper, non-tangential part of y.

The axioms, which treat ch as an undefined primitive, can then be formulated as follows. (We omit the obvious informal glosses.)

(H1) $ch(ch(x)) = ch(x)$

(H2) $TP(x, ch(x))$

(H3) $P(x,y) \to P(ch(x), ch(y))$

(H4) $P(ch(x) + ch(y), ch(x+y))$

(H5) $ch(x) = ch(y) \to C(x,y)$

(H6) $ch(x) \times ch(y) = ch(ch(x) \times ch(y))$

(H7) $\neg C(x,y) \to ch(x+y) \neq x+y$

(H8) $IP(x,y) \to ch(y-x) \neq y-x$

(H9) $EC(x,y) \wedge ch(x+y) = x+y \wedge EC(y,z) \wedge ch(y+z) = y+z \wedge \neg C(x,z) \to ch(y) = y$

Our first point concerns the relative inadequacy of these axioms. Consider, for example, the important property of h expressed by the conditional in **(T17)**. It is not clear to us whether the ch-counterpart of this theorem is provable from **(H1)**–**(H9)** even within the full strength of $KGEMT$. However, it can be shown that **(T17)** crucially hinges on axiom **(H5)**. Absent this axiom, countermodels arise in which the conditions in the antecedent

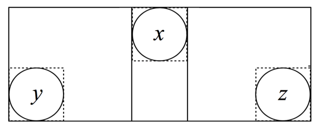

Figure 2. A counterexample to the *ch*-counterpart of **(T17)**. The dashed lines indicate $ch(x)$, $ch(y)$, and $ch(z)$; the two larger (overlapping) drawn-out rectangles indicate $ch(x+y)$ and $ch(x+z)$. Upon replacing '*ch*' for '*h*' in **(T17)**, the antecedent holds but the consequent fails, as $ch(x)$ is a proper part of $ch(x+y) \times ch(x+z)$.

of the theorem hold and yet the consequent fails. One such countermodel can be obtained by interpreting '*ch*' as an operator that maps each region to the smallest circumscribing rectangle (relative to a fixed orientation). It is easy to verify that such an interpretation satisfies all axioms except for **(H5)**. Yet it does not validate the *ch*-counterpart of **(T17)**, as illustrated in Figure 2.

Why is this important? Our countermodel merely shows that **(H5)** is crucial for a proper interpretation of '*ch*', on pain of forgoing **(T17)**. But there is more. For, as it stands, **(H5)** is too strong. As already noted in [9] and [11], and hinted at in [7], this axiom may fail if x is allowed to range over unbounded regions. Indeed, it is easy to find actual counterexamples. A case in point is illustrated in Figure 3, where x is the fusion of the circular regions and y the fusion of the square regions. If the sequence extends to infinity in both directions, then x and y have the same convex hull even though they are not connected to each other, contrary to **(H5)**. Unfortunately, finitude is not first-order characterizable, so this sort of counterexample cannot be blocked within the limited resources of a mereotopological theory. The axiomatization of '*ch*' is left wanting.

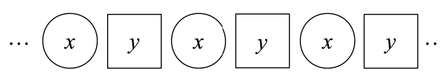

Figure 3. A counterexample to **(H5)**.

That being said, we now proceed to showing that—minor qualifications aside—the properties expressed by the other eight axioms are all derivable as theorems of $KGEMTS$ upon replacing '*h*' for '*ch*'. Indeed, some cases are straightforward: the *h*-counterparts of **(H1)** and **(H3)** correspond to **(T5)** and **(T6)**, respectively; that of **(H4)** follows from **(T6)** via **(D9)**, which entails both $P(x, x+y)$ and $P(y, x+y)$; **(H6)** can be recovered immediately from **(T2)** and **(T3)**; and the *h*-counterpart of **(H7)** is a simple consequence of the self-connectedness of convex entities, i.e. **(T15)**, via **(D22)**. The case for the remaining three axioms, however, requires more work.

We begin with **(H9)**, whose *h*-counterpart can be proved in the following simplified form, owing to **(T2)**:[6]

(T18) $EC(x,y) \wedge EC(y,z) \wedge Cv(x+y) \wedge Cv(y+z) \wedge \neg C(x,z) \rightarrow Cv(y)$
If x is externally connected to y and y to z, both $x+y$ and $y+z$ are convex, and x and z are not connected, then y is convex.

To prove this, we first apply **(T2)** and **(T10)** to the hypothesis that $x+y$ and $y+z$ are convex to derive that (i) $x+y = \sigma w \exists u \exists v (Pt(u) \wedge Pt(v) \wedge P(u, h(x)) \wedge P(v, h(y)) \wedge w =$

[6] Actually, in [25] the biconditional in **(T2)** corresponds to a definition of '*Cv*' in terms of '*ch*' and, as such, it is used in the original formulation of **(H9)**.

$s(u,v))$ and (ii) $y+z=\sigma w\exists u\exists v(Pt(u)\wedge Pt(v)\wedge P(u,h(y))\wedge P(v,h(z))\wedge w=s(u,v))$. We note that, for every pair of points a, b that are part of y, the segment $s(a,b)$ is part of the convex sums on the left side of the equalities in (i) and (ii), hence of $(x+y)\times(y+z)$. When x and z are disconnected, this product equals y. Therefore, every segment $s(a,b)$ for a and b as specified must be part of y. By **(D18)**, this means that y is convex. □

Note that this proof also reveals the first two conjuncts in the antecedent of **(H9)** to be redundant, as they are not required to derive $Cv(y)$. Let us now consider **(H8)**. Here we should note that, again, this axiom is strictly speaking false. Suppose, for instance, that y is the universal region. If x is y minus some convex region somewhere at the "center", as in Figure 4, left, then x counts as an internal part of y (since nothing is externally connected to the universe) with $ch(x)=y$; yet $ch(y-x)=y-x$. Similarly, if x is one of the two parts obtained by splitting y with a straight line, as in Figure 4, right, then again x is an internal part of y, but since x and $y-x$ are both convex, we have $ch(y-x)=y-x$.

Figure 4. Two counterexamples to **(H8)**.

Unlike **(H5)**, however, such counterexamples can easily be blocked, at least insofar as **(H8)** is meant to capture the idea that shapes with "interior holes" cannot be convex (see [25], §5). Thus, consider the following definition (adapting from [14], §2.5.3):

> **(D24)** $PE(x,y)=_{df}\exists z(P(z,y)\wedge P(x,z)\wedge Cv(z)\wedge\neg Cn(y-z)\wedge\neg Cn(z-x))$
> x is properly enclosed in y iff x is a bisecting part of a convex bisecting part of y (i.e., intuitively, iff the removal of x leaves a topological annulus).

Then a natural suggestion is to strengthen the antecedent of **(H8)** by replacing '$IP(x,y)$' by '$PE(x,y)$'. Thus amended, the axiom can be recovered in $KGEMTS$ in the following form (again, via **(T2)**):[7]

> **(T19)** $PE(x,y)\rightarrow\neg Cv(y-x)$
> If x is properly enclosed in y, then the difference between y and x is not convex.

For a proof, assume the antecedent holds. By assumption, z is a convex part of y, thus z contains all segments connecting points that are part of z. Still by assumption, x is a part of z and $z-x$ is not self-connected. By **(T.15)**, $z-x$ is not convex. Therefore, there exist points z_1 and z_2, part of $z-x$, such that the segment $s(z_1,z_2)$ is not part of $z-x$. By convexity of z, it is part of z, thus it overlaps x. Since z is part of y, both z_1 and z_2 are part of y. There exists thus a segment, $s(z_1,z_2)$, between points in y which overlaps x. Hence $y-x$ is not convex. □

Finally, let us consider **(H2)**, focusing on the non-trivial case where x is not convex and does not, therefore, coincide with its own convex hull. In this case, despite its intuitiveness and *prima facie* simplicity, the property expressed by **(H2)** turns out to be surprisingly difficult to recover on our segment-based approach, so it is convenient to

[7] As it turns out, the operator denoted by '$-$' behaves differently in RCC and $KGEMT$, owing to the underlying disagreement concerning the status of boundary elements. As far as we can see, however, this complication has no impact on the intended meaning of axiom when this is translated from one theory into the other.

proceed in steps. First of all, we note that the proof is rather straightforward for those special regions x that have extreme points, for in that case we only need to show that such points are among x's tangential parts: **(T14)** will then guarantee that the same points—and therefore x itself—are also tangential parts of x's convex hull.

(T20) $EPt(y,x) \rightarrow TP(y,x)$
If y is an extreme point of x, it is also a tangential part of x.

(T21) $\exists y EPt(y,x) \rightarrow TP(x,h(x))$
Any region with extreme points is a tangential part of its convex hull.

To prove **(T20)**, note that if y is an extreme point of x, then there exist no points x_1 and x_2, both parts of x, such that $P(y, s(x_1,x_2) - (x_1 + x_2))$. This follows immediately by **(T12)**. Also, note that if y is a point that counts as an internal part of x, then there must exist x_1 and x_2, both parts of x, such that $P(y, s(x_1,x_2) - (x_1 + x_2))$. This follows from a reasoning similar to the one in the proof of **(T19)**. Putting these two facts together, it follows immediately that no extreme point of x can be an internal part of x. Given **(T20)**, **(T21)** follows immediately by **(T14)**. □

As a second step, we generalize **(T21)** to open regions, which typically do not have extreme points. To this end, we need a better picture of the interaction between c and the topological open/closed-distinction. Here we limit ourselves to registering the following two facts, which can be established with the help of theorems **(T13)**, **(T14)**, and **(T17)**:

(T22) $Cv(x) \rightarrow Cv(i(x)) \wedge Cv(c(x))$
The interior and the closure of a convex region are themselves convex.

(T23) $\exists y EPt(y,c(x)) \rightarrow h(c(x)) = c(h(x))$
Whenever the closure of a region has extreme points, the convex hull of the closure and the closure of the convex hull coincide.

On these grounds, **(T21)** can be extended to cover the case of any region *whose closure* has some extreme point:

(T21′) $\exists y EPt(y,c(x)) \rightarrow TP(x,h(x))$
Any region whose closure has extreme points is a tangential part of its convex hull.

Proof: If x is a closed or bounded region, it is sure to have extreme points and thus **(T21′)** reduces to **(T21)**. Suppose, then, that x is not closed and assume $c(x)$ has extreme points (i.e., x is not the universe). In this case, x is a tangential part of $h(x)$ iff $b(x)$ and $b(h(x))$ overlap. Suppose, by reductio, that such boundaries do not overlap. Then it follows from the *KGEMT*-theorem $P(c(x), c(h(x)))$ that $P(c(x), i(h(x)))$. Via **(T3)** and the convexity of $i(h(x))$, i.e. **(T22)**, we get $P(h(c(x)), i(h(x)))$. Yet $c(x)$ has extreme points by assumption, and therefore we get $h(c(x)) = c(h(x))$ via **(T23)**. Hence, $P(c(h(x)), i(h(x)))$ which is absurd, since $c(y) \neq i(y)$ unless y is the universal region (which doesn't have extreme points). Therefore, x is a tangential part of $h(x)$. □

Can we go beyond **(T21′)** and recover **(H2)** in its full generality? Effectively, this would amount to dropping the antecedent altogether and, unfortunately, there appear to be some situations in which *KGEMTS* doesn't seem to fully settle the issue. The reason is that, in general, the relationship between x and $h(x)$ depends crucially on how exactly the open/closed distinction is handled in limit cases. Thus, consider the real vector space $\mathbb{R}^2$ with its standard topology and let $A = \{(x,y) | -tanh(x) \leq y \leq tanh(x)\}$ (where *tanh* is the hyperbolic tangent function). This set is closed and has no extreme points, hence its closure has no extreme points, either. Moreover, the convex hull of A is the open tube $h(A) = \{(x,y) | -\infty \leq x \leq \infty, -1 < y < 1\}$. Clearly, whether or not A is a tangential part of its convex hull depends on the behavior of C, and in some versions of $KGEMT$ the answer is indeed in the affirmative. However, if $KGEMT$ is strengthened so as to include the following Fusion axiom, as suggested in [14], then A is *not* a tangential part of $c(A)$.

(F) $C(x, \sigma y \phi y) \to \exists y(\phi y \wedge C(x,y))$
x is connected to the fusion of all ϕers only if x is connected to some ϕer.

At the moment, it is an open question whether $KGEMT$ should be strengthened in this or similar ways.[8] Even if it is not, however, such limit situations constitute a difficulty *vis à vis* **(H2)** and there seems to be no general way to recover this axiom unrestrictedly.

This completes our discussion of Cohn *et al.*'s axioms for the convex hull operator. We have shown that, with the exception **(H5)**, which is not generally true, and **(H2)**, which may not be fully recoverable in the case of open unbounded regions, all other axioms can be (fixed and) derived as theorems of $KGEMTS$.

3. Concluding Remarks

$KGEMTS$ provides us with a powerful tool for investigating the properties of convex regions, allowing us to go some way towards a better understanding of the currently available first-order axiomatizations. At the same time, the axioms for segments on which the theory is based inscribe themselves in a long-standing tradition of axiomatic geometries, going back to Hilbert's foundations of geometry [23] and Tarski's elementary geometry [24]. Two final comments are therefore in order.

First, the account we have presented depends substantively on the availability of boundary elements in the domain of quantification, including points. There is, however, a long-standing tradition (from [27, 28] to [29, 30]) claiming that qualitative reasoning theories should better resist points and deal exclusively with extended regions, as in *RCC*. Indeed, this negative attitude towards points often stems from two distinct sorts of worry: that points are "abstract" entities that cannot be perceptually discriminated, and that points are ontologically "suspect" entities created by mere mathematical abstraction. Similar worries have been raised with regard to any kind of boundaries, including lines and surfaces (see [17] for a review).

In response to this line of reasoning, it must be stressed that the notion of point countenanced by $KGEMTS$ is mereotopological, not metrical: a point is, in essence, a topologically closed mereological atom (see again **(D6)** and **(T1)**). On the most natural interpretation, such atoms qualify as ordinary 0-dimensional elements given the mereotopological background theory $KGEMT$. However, it is conceivable to reread all segment axioms **(S1)–(S10)** by interpreting points as *extended* regions, i.e., as spatially extended mereological simples (a notion that has received much attention in recent philosophical literature, beginning with [31]). On such a reading, the axioms do not suffice

[8] One reason to resist **(F)** in its full generality is that it rules out infinitary models one might want to retain. Consider, for instance, the one-dimensional space $\mathbb{R}$ and the family of closed intervals $A_n = [\frac{1}{n}, 1 - \frac{1}{n}]$, for all $n > 1$. The fusion of this family is the open interval $]0,1[$, and this is connected to $[-1,0]$ even though none of the A_n's is.

to capture straight-line segments in real vectorial spaces; yet they determine an operator that behaves in a way sufficiently close to straight-line segments to yield a satisfactory notion of convexity. In addition, and independently, it is a well-known fact that in boundary-free mereotopologies it is still possible to mimic the behavior of points and other lower-dimensional entities by retrieving them as higher-order entities, viz. as equivalence classes of convergent series of nested extended regions (as in Whitehead's method of *extensive abstraction* [28]; see e.g. [32,33]). Accordingly, our characterization of convexity in terms of the segment axioms **(S1)–(S10)** may be turned into a sustainable theory even if one sympathizes with the anti-atomistic line of reasoning mentioned above: one just needs to reinterpret the predicate '*Pt*' as standing for equivalence classes of regions that "converge" to a point. Indeed, with '*Pt*' understood this way, one may consider adding **(S1)–(S10)** directly to *RCC*, the theory presupposed by Cohn *et al.*'s axiomatization of *ch*.

The second comment is this. Some authors, such as Pratt [34] and, to some extent, Borgo and Masolo [6], have doubted that any qualitative account of convexity is possible. The underlying idea is that a mereotopology endowed with convexity has virtually the same expressive power as affine geometry. Since in affine geometry we can define (nonmetrical) coordinate systems—via the use of intersection and parallelism of lines [34]—no such theory should therefore count as "qualitative" in the sense in which this term is used in the spatial reasoning literature.

In a way, this claim can hardly be resisted. If the term "qualitative" is only meant to designate theories with limited expressive power—but interesting enough to describe common-sense patterns of reasoning—then $KGEMTS$ falls outside its scope. On the other hand, this way of understanding the term is grounded on the fact that theories with richer expressive power are typically *too* rich from the perspective of common-sense reasoning. As such, the understanding may be extensionally adequate as a matter of contingent fact, but it need not be intensionally appropriate. And there is, we claim, an important sense in which $KGEMTS$ runs afoul of it. For this theory is still based on a purely first-order axiomatization, and in such a way as to sustain complex properties of the convexity predicate and the convex hull operator that reflect very closely their intuitive understanding. This is not to say that $KGEMTS$ completely characterizes convexity in real vectorial spaces: obviously, axioms **(S1)–(S10)** are bound to have various "unintended" models. Yet such axioms are strong enough to fix some key features of the notion of segment and similar notions that are at play in ordinary spatial reasoning and talk. In this sense, we submit, $KGEMTS$ is as qualitative as any good theory need be.

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