The Geometry of Negation

Achille C. Varzi

Department of Philosophy, Columbia University, New York, New York (USA)

Massimo Warglien

Department of Business Economics, Università Ca' Foscari, Venice (Italy)

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Abstract. We consider two ways of thinking about negation: (i) as a form of complementation (the negation of a proposition p holds exactly in those situations in which p fails), and (ii) as an operation of reversal, or inversion (to deny that p is to say that things are *the other way around*). These two conceptions are significantly different. But whereas a variety of techniques exist to model the intuition behind conception (i)—from Euler and Venn diagrams to Boolean algebras—conception (ii) has not been given comparable attention. In this note we outline a simple, geometric proposal. In fact, conception (ii) can be modeled in different ways depending on whether one understands the geometric metaphor of an inversion as involving a *rotation* or a *reflection*. These two options are equivalent in classical two-valued logic, but they differ significantly in many-valued logics. Here we show that they correspond to two basic sorts of negation operators—familiar from the works of Post and Kleene, respectively—and we provide a simple group-theoretic argument demonstrating their generative power.

1. Introduction

There are, naturally, two ways of thinking about ordinary sentential negation. On one conception, negation is a form of *complementation*. To deny a proposition p is to make an assertion that holds exactly in those situations in which p fails. More generally, if a proposition p is identified or otherwise correlated with a class of worlds (situations, contexts) W, then the negation of p is identified or correlated with the relative complement $\overline{W}$ of W, and to deny that p is to assert that the actual world is a member of $\overline{W}$. Thus, this sort of operation presupposes a previous classification, or partition, of the set of all possible worlds into those that verify p and those that falsify p. The other way of thinking about sentential negation is well exemplified by Ramsey's [14] suggestion that the negation of a proposition p be represented by writing p upside down (or by mirror reflecting p

about its vertical axis, as one could also suggest). In this sense negation is a form of *reversal* or *inversion*: to deny a given proposition is to say that things are *the other way around*. Semantically, the same intuition can be found also in Behmann’s typographical convention of writing ‘$\top$’ for truth and ‘ ’ for falsehood [2], or in Hintikka’s game-theoretic characterization of negation as swapping of roles between players [7], or even in Peirce’s rule for “negativing” a graph by cutting through the Phemic sheet and turning over the excised piece [12]. Classification does not play a significant role in such inversions; for example, Ramsey’s suggestion does not presuppose that one knows in what circumstances the negated proposition *p* is true, and the convention of writing ‘$\top$’ and ‘ ’ does not presuppose any particular reading of these labels. We could as well agree to understand ‘$\top$’ as falsehood and ‘ ’ as truth. More simply, on the inversion conception negation is an operator that treats its argument (the statement or proposition to be negated, or the corresponding truth-value) as a sort of black box. Negation is a way of manipulating a (syntactic or semantic) object.

Extensionally, of course, these two conceptions of negation are equivalent, in that both yield the same truth-functions.[1] This equivalence finds its most obvious representation in the familiar apparatus of truth-tables (Figure 1), where each row corresponds to a class into which the relevant states of affairs can be partitioned, as the case may be.

p	not *p*
$\top$	$\bot$
$\bot$	$\top$

Figure 1

Intensionally, however, the two conceptions are quite distinct. But whereas a variety of techniques exist to model the intuition behind negation as complementation (from Euler and Venn diagrams to Boolean algebras), the modeling of negation as inversion has not been given comparable attention, apart from Peirce’s idiosyncratic graph-theoretic account and Ramsey’s typographical (and occasional) intuition. Our purpose in this note is to outline a simple, geometric proposal. In

[1] And there are conceptions of negation that are not extensional at all—e.g., negation as *refutation* (intuitionistic logic), as *incompatibility* (quantum logic), or as *failure* (logic programming). (See [6, 18] for a general map and [8] for linguistic ramifications.) Here we are concerned exclusively with conceptions that admit of a truth-functional account.

fact, two different strategies can be considered depending on whether one understands the geometric metaphor of an inversion as involving a *rotation* or a *reflection.* These two strategies turn out to be equivalent in classical two-valued logic, but they differ significantly in many-valued logic. Here we show that they correspond to two basic sorts of negation operators—familiar from the works of Post and Kleene, respectively—and we provide a simple group-theoretic argument demonstrating the generative power of such operators.

2. Cyclic Inversions

Consider first the following way of representing a Ramsey-style inversion in classical, two-valued logic. (Figure 2). Here the line connecting ⊤ and indicates that there is no intermediate truth-value, and the arrow indicates the result of the inversion: the truth-value set is flipped upside down, i.e., rotated 180 degrees. Thus truth turns to falsehood and falsehood to truth. Of course, if we flip the truth-value set twice, we get back to the original configuration—a fact that reflects the classic principle of double negation.

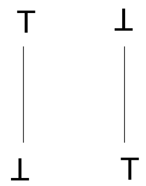

Figure 2

If we now consider the possibility that statements may take a third truth-value, say ‘I’ (for “intermediate” or “indeterminate”), one natural way of generalizing this representation is illustrated in Figure 3:

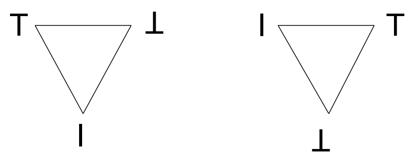

Figure 3

Here, again, the solid lines indicate the existence of a direct link between the truth-values and the arrow represents the outcome of the manipulation. Since there are three values, the structure is only rotated 120 degrees and it takes three negations rather than two to get back to the original configuration. Double negation fails and triple negation—as we may say—holds instead.

In general, for any number n of distinct truth-values t_1, ..., t_n we can imagine a corresponding polygonal representation, with the relevant transformations represented graphically in the obvious way as the result of a rotation of 360/n degrees (Figure 4). Here the intuition is that the negation of each value is the next (clockwise) value, and since the polygon is closed the negation of the last value t_n (intuitively: falsehood) returns the initial value t_1 (intuitively: truth). The principle of double negation is then replaced by a corresponding principle of n-fold negation: denying a proposition n times is tantamount to asserting the proposition. Intuitively: after a complete cycle we are back to the starting point.

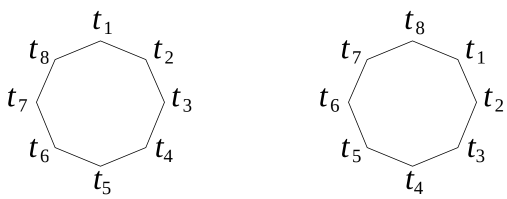

Figure 4

Regardless of the number of truth-values, in each case the representation relies on the ordinary assumption that the set of truth-values forms a single, connected, topologically rigid structure. This structure may be called, quite naturally, a *truth-polygon* (or *truth-segment* for n=2). And it is precisely the connectedness of polygons that makes them available for manipulation.

3. Symmetric Inversions

The cyclic nature of negation illustrated above represents one natural way of interpreting and generalizing Ramsey's suggestion that negation is a way of turning things around. Mathematically, the upshot corresponds to the negation operator as characterized in the many-valued logic of Post [13]. Negation turns each value t_i to t_{i+1}, mod n+1.

Let us consider a different, perhaps more intuitive way of representing negation-as-inversion, based on an alternative interpretation of Ramsey's suggestion. Take again the classic case depicted in Figure 2. We have described the arrow as indicating the result of a *rotation*. But one could describe it equally well as indicating the result of a symmetric *reflection*: rather than yielding a 180 degree shift in the truth-values, negation flips them upside down—it mirror-reflects the truth-segment along the horizontal axis. Equivalently, if we represent the truth-segment as a horizontal structure, then again the first interpretation corresponds to a half rotation but the alternative interpretation corresponds to a reflection about the vertical axis:

Figure 5

If this alternative interpretation is exploited, then the generalization to the *n*-valued case is different from the one considered above and corresponds to the negation operator as characterized in different systems of many-valued logics.
For reasons of graphic convenience, let us focus on horizontal mirror reflections (reflections along the vertical axis, as in Figure 5). The case n=3 can be represented as in Fugure 6. Mathematically, this corresponds to negation as treated in the “strong” version of Kleene’s three-valued logic [9], or in the system of Lukasiewicz [10]: truth and falsehood are opposite and the third value, indeterminacy, is self-opposite (the intuition being that the denial of an indeterminate statement is just as indeterminate as the statement itself). There is, accordingly, no cyclic process involved in iterating negation, and the principle of triple negation does not hold. What holds, rather, is a straigthforward extension of the principle of double negation of classical logic: *not-not*-⊤=⊤, *not-not*- = , and *not-not*-I=I.

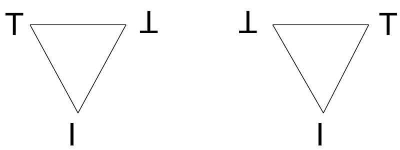

Figure 6

One could also consider different accounts, corresponding to different ways of choosing an axis of symmetry, or different ways of structuring the truth-polygon. Such variants, however, can all be regarded as species of the same genus, which can be represented abstractly as in Figure 7. Depending on how the three values are intuitively interpreted, the resulting operator will correspond to a different account of negation in three-valued logic. In no way, however, can the negation of Post's three-valued logic be recovered as a special case of this pattern, since here one of the values (t_3) is not affected by the inversion.

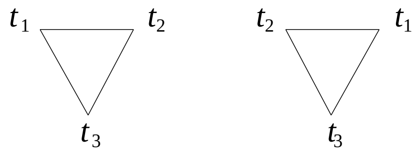

Figure 7

The generalization of this model to arbitrary n-valued truth-polygons is now obvious. For example, when the value of n is even we have the pattern in Figure 8. Again, it is immediately seen that this pattern satisfies a simple principle of double negation (rather than the principle of n-fold negation delivered by cyclic inversions).

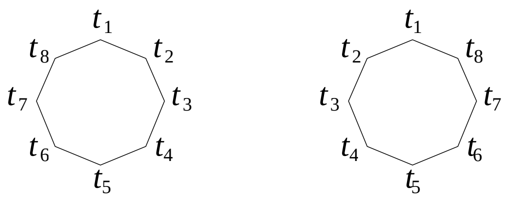

Figure 8

An interesting special case is the one obtained by setting $n = 4$ and interpreting t_4 as truth, t_2 as falsehood, t_3 as indeterminacy (neither true not false), and t_1 as overdeterminacy (both true and false, written 'O'). Then the pattern in Figure 8 corresponds to the negation connective in the four-valued logic of Belnap [1], which in turn is sometimes described as the obvious four-valued extension of Kleene's three-valued logic (see e.g. [5]): the two classical values are opposite and the two extra values are self-opposite. (Figure 9)

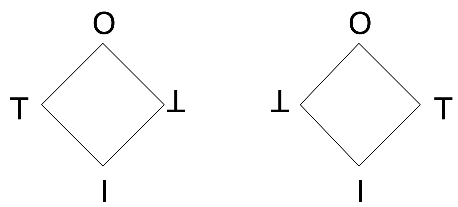

Figure 9

4. Manipulating Truth-Polygons: The Dihedral Group

Post negations and Kleene-like negations correspond to the operations of cyclic rotation and (horizontal) symmetric reflection, respectively—two natural ways of manipulating a truth-polygon. Are there others? There are. But it is easy to see that they are not as basic.

For example, consider Fitting's [4] operation of *convolution*, which is defined with reference to Belnap's four-valued logic. Convolution has the effect of flipping I and O (indeterminacy and overdeterminacy), while leaving T and fixed. It is a kind of "vertical" symmetric reflection, and it cannot be obtained by any number of repeated applications of either type of negation. Nonetheless it is easy to see that convolution can be generated via a suitable composition of the two negations, specifically by applying Kleene reflection once and Post rotation twice. (Figure 10)

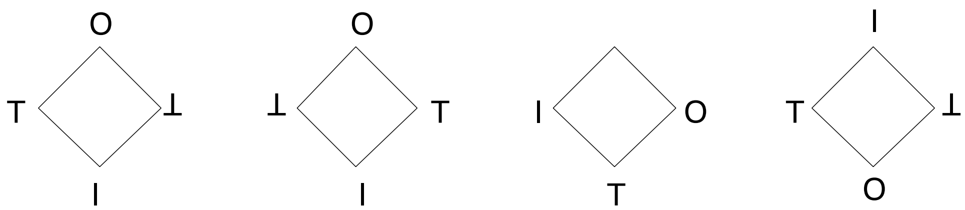

Figure 10

The idea of composition suggests an algebraic generalization of this example that preserves its geometric inspiration. Given any $n > 2$, the rotations and reflections of an n-sided regular polygon P_n define a familiar group structure, called the dihedral group Δ_n [11]. This is the group of *all* symmetric operations on the polygon P_n that leave its shape unaltered, without breaking it or deforming it—i.e., those operations that are topologically rigid. (The number of such operations is the order of the group, which is $2n$.) Now, it turns out that regardless of the value of n, the dihedral group Δ_n can be generated using just rotation and reflection, together with a set of identity relations. Hence, just as convolution can be defined in terms of Post negation and Kleene negation, so can every other symmetric operation on the 4-sided truth-polygon. More generally, each one of the $2n$ symmetric operations on an n-sided truth-polygon can be generated from the corresponding n-valued operations of Post and Kleene negation.

Formally, if we use the symbols ‘ ’ and ‘¬’ for these two n-valued negation operations (respectively) and ‘i’ for the identity operation, then the corresponding dihedral group is defined by the following relations:

(a) $\neg^2 = i$

(b) $\;^n = i$

(c) $(\neg\;)^2 = i$

The first two equalities are straightforward. Equality (a) says that for any n-sided polygon ($n \geq 2$), only two mirror reflections are needed to produce a configuration equivalent to the starting one (this fact corresponds to the double negation property). Equality (b) says that n rotations are needed to get back to a configuration in space equivalent to the starting one (this is Post's n-fold negation property). Finally, equality (c) concerns the interaction between ¬ and . It says that their composition always yields the double negation property, regardless of the number n of truth-values.[2]

When truth-values are restricted to two, the corresponding dihedral group is the classic tetragroup, Δ_2, representing symmetries of a 2-gon. A 2-gon can be thought of as a plane figure with two sides constituted by two edges. (Figure 11) In this case, Post and Kleene negations collapse onto classic negation, and their behavior is indistinguishable, since in Δ_2 the equality in (b) is just double negation. From this perspective, too, classic negation can be considered as a special case of both generalized Kleene and Post negations.[3]

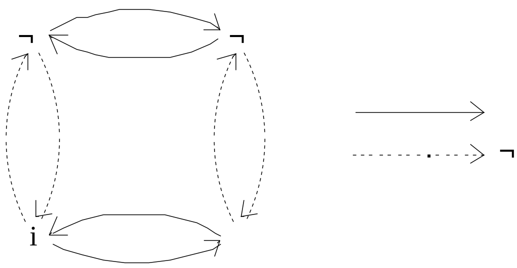

Figure 11

[2] The standard presentation of (c) in group theory (e.g. [10]) is slightly different:

(c') $\neg\; = \;^{n-1}\neg$.

The formulation we use, more expressive for our aims, can be derived from this one by applying in sequence equalities (c'), (a) and (b):

(c) $(\neg\;)^2 = \neg\;\neg\; = \;^{n-1}\neg\neg\; = \;^{n-1}\neg^2\; = \;^{n-1}i\; = \;^{n-1}\; = \;^{n} = i$.

[3] Alternatively, we could say that when n=2 the corresponding dihedral group is Δ_1, representing symmetries of a segment. In this case, however, Post negation would collapse to identity, since the equality in (b) would reduce to = i. (By contrast, Kleene negation would reduce to classic negation, as expected.)

5. Extensions

The generative power of the (generalized) Kleene and Post negations can be assessed by looking at the order of the Δ_n group. Thus, for *n* truth-values, these two negations will suffice to generate a set of 2*n* unary connectives. Not *all* unary connectives can be generated this way, though. (The total number of *n*-valued unary connectives is n^n.) Not even all "normal" negation-like connectives can be generated, counting as normal those connectives that assign distinct values to T and , i.e., more generally, to the designated and antidesignated values [15]. Among those normal connectives that cannot be generated are, for example, those corresponding to transformations that violate shape invariance, e.g., connectives that "mash" the truth-polygon. Bochvar's [3] "external negation" for three-valued logic (also known as Kleene's "strong negation"[4]) is one such example. It sends both and I to the same value T, hence it has the effect of reducing a three-valued truth-polygon to a classic, two-valued truth-segment (Figure 12). But this limitation of the generative power of the generalized Kleene and Post negations is not accidental. Bochvar-like negations, it could be argued, are of a significantly different sort precisely *because* they violate the constraint of shape invariance.

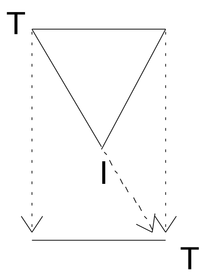

Figure 12

Furthermore, for $n < 4$ the dihedral group represents all possible permutations of truth-values. For example, for $n = 3$ there are six possible permutations of the truth-values, and they can all be generated by relevant Kleene and Post negations. However, things get more complex with $n \geq 4$. Consider the simplest case, $n = 4$. The corresponding truth-polygon is the one shown on the left of Figure 9. The reader can check that there is no way to generate by rotation and

[4] Or as Von Wright's "weak negation" [17]. As it turns out, the relevant terminology is not very consistent.

reflection the truth-polygon in Figure 13, left. To generate this as well as the other topologically non-rigid permutations one need resort to operations involving one more dimension, considering the rotations and symmetric reflections of what may be called, by extenting our terminology, a *truth-polyhedron*—in this case a truth-cube (Figure 13, right). Such an extension, however, would once again correspond to a major conceptual jump.

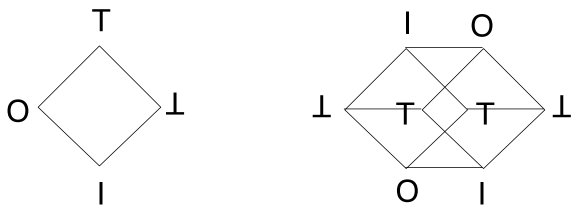

Figure 13

6. Final Remarks

We have argued that the inversion-based conception of negation leads to a natural geometric semantic modeling based on rotation and reflection of truth-polygons. These transformations correspond to two basic ways of manipulating topologically rigid objects and their generative power—it turns out—leverages on a powerful group structure, the dihedral group.[5] As our last example suggests, broader combinatorial possibilities could now be achieved by allowing for other sorts of manipulations, e.g., manipulations involving the gluing of truth-simplexes (polygons) in higher-dimensional complexes (polyhedrons). Bochvar's "mashing" of truth-values goes just the other way around, reducing dimensionality through topological identification. The conceptual and mathematical complications introduced by such manipulations promises to be a rewarding subject for further explorations along the lines illustrated in this note.[6]

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[5] We are thinking here of physical manipulation, but perhaps this is too crude. See [16] for (Kleene-style) symbol manipulation without inscription manipulation.

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