On the Meaning of Complementary Systems

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1. Complementary systems. We say that two proof systems are complementary iff what can be derived in one system is precisely that which cannot be derived in the other, and vice versa [11]. For instance, classical propositional logic and Łukasiewicz's rejection calculus [5] are complementary. If we focus on decidable logics, given one system there exist of course many algorithms to compute what is not provable in it, e.g., semantic tableaux or refutation trees. Even so, a formulation in terms of axioms and rules of inference is usually not a trivial task, as these may be of a most non-standard sort. For instance, in complementary systems for classical logic standard substitution rules cannot hold, since non-tautologies may become tautologous upon substitution. Moreover, such systems must be paraconsistent, i.e., such as to allow non-trivial derivation of mutually inconsistent sentences.

The study of complementary systems of this sort—we argue—presents several points of interest. For one thing, they yield an exhaustive characterization of a logical theory in purely syntactic terms. This weakens, if one will, the view that some amount of semantic reasoning is required in order to fully dominate a logic. Secondly, the combination of complementary notions of derivability discloses interesting perspectives for metamathematics, for instance in connection with stronger forms of decidability of the sort hitherto investigated under the rubric of Ł-decidability. Thirdly, these systems have important bearing on certain fundamental issues in computer science, such as non-monotonic reasoning, the semantics of programming languages, and the logical characterization of complexity classes.

2. Basic methods and developments. Łukasiewicz's system was based on the idea that non-derivability is essentially a refutation procedure, hence a form of reverse derivability. Thus, a system for classical non-tautologies can be obtained simply by taking falsehood ($\bot$) as an axiom and reverse substitution plus reverse modus ponens as rules of inference:

[R*s*] If $\nvdash \alpha$, and α is a substitution instance of β, then $\nvdash \beta$

[R*mp*] If $\vdash (\alpha \rightarrow \beta)$ and $\nvdash \beta$; then $\nvdash \alpha$

This strategy has been extensively studied [9] and has been exploited also to provide complementary systems for intuitionistic logic [3, 6] and for various modal logics [4, 8]. However, systems of this sort do not do full justice to non-derivability, as they characterize it in terms of the complementary notion of derivability (as in [R*mp*]). Extant Hilbert-style axiomatizations of classical non-tautologies [2, 12] or Gentzen-style systems for non-derivability in the first order logic of finite structures [10] also share this feature to some degree.

This limitation can nonetheless be overcome, at least in some cases. As an illustration, let an anti-sequent $\Gamma \nvdash \Delta$ count as an axiom iff Γ and Δ are disjoint sets of atoms; then we can show that the following rules [1] define a sound and complete (cut free) Gentzen-style complementary system for classical propositional logic:

[¬ r]	[¬ l]	[→r]	[→l_1]	[→l_2]
$\dfrac{\Gamma, \alpha \nvdash \Delta}{\Gamma \nvdash \Delta, \neg\alpha}$	$\dfrac{\Gamma \nvdash \Delta,\alpha}{\Gamma,\neg\alpha \nvdash \Delta}$	$\dfrac{\Gamma,\alpha \nvdash \Delta,\beta}{\Gamma \nvdash \Delta,\alpha\rightarrow\beta}$	$\dfrac{\Gamma \nvdash \Delta,\alpha}{\Gamma,\alpha\rightarrow\beta \nvdash \Delta}$	$\dfrac{\Gamma,\beta \nvdash \Delta}{\Gamma,\alpha\rightarrow\beta \nvdash \Delta}$

Many properties of the standard sequent calculus are preserved in this system. For example, rules are perfectly symmetric and satisfy the subformula property. On the other hand, note that while the structure of classical rules with one premiss is preserved, the others rules (in this case, [→l]) split into pairs with one premiss each. This reflects the fundamental intuition that the exhaustive search of the classical sequent calculus becomes nondeterministic in its complementary version—a result that can be viewed as a logical account of the distinction between the computational complexity classes NP and co-NP.

3. Semantic implications. Most of the aforementioned systems for non-derivability can naturally be regarded as doing a semantic job: an anti-proof is essentially the construction of a counterexample. For instance, in the Gentzen-style system above, the unique axiom $\Gamma \nvdash \Delta$ of an anti-proof corresponds to a partial interpretation where the atoms in Γ are all true, those in Δ all false, and all the others undefined; such an interpretation gives a counterexample to the sequent $\Gamma' \vdash \Delta'$, where $\Gamma' \nvdash \Delta'$ is the conclusion of the anti-proof. This is especially significant for logical formalisms for which no "natural" model theory is available [7]. Interesting examples can be found in theoretical computer science, where the semantics of imperative programming languages is typically given by means of Gentzen systems that have no model-theoretic counterpart. In this perspective, two intereresting lines for further research are (i) the characterization of complementary systems for a wider range of formalisms, and (ii) the study of general, uniform methods for transforming a given system (e.g., Hilbert or Gentzen-style) into an elegant complementary counterpart of the same type.

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